Chapter 1

Historical Definitions and Basic Properties

We begin with Hochschild's historical definition of homology and cohomology for algebras [**114**], working in the general setting of algebras over commutative rings as in the book of Cartan and Eilenberg [**48**]. We then present some additional structure found by Gerstenhaber [**82**] under which we now say that Hochschild cohomology is a Gerstenhaber algebra. These early developments were largely based on one choice of chain complex known as the bar complex, and we focus in this chapter particularly on the many properties and structural results that can be derived from this complex. For example, we discuss here the meaning of Hochschild homology and cohomology in low degrees where connections to derivations and deformations appear. We present the cap product that pairs Hochschild homology and cohomology, and the shuffle product on Hochschild homology of a commutative algebra. We discuss Harrison cohomology and the Hodge decomposition arising from a symmetric group action on the bar complex. Other work invokes other complexes, depending on the setting, and we begin to include in this chapter some examples and discussion of the structure of Hochschild cohomology that takes advantage of other such chain complexes. We expand on this discussion of other complexes in later chapters.

1.1. Definitions of Hochschild homology and cohomology

For now, let k be a commutative associative ring (with 1), and let A be a k-algebra. That is, A is an associative ring (with multiplicative identity) that is also a k-module for which multiplication is a k-bilinear map. Denote the multiplicative identity of A also by 1, identified with the multiplicative identity of k via the unit map $k \to A$ given by $c \mapsto c \cdot 1$ for all $c \in k$. Denote by A^{op} the *opposite algebra* of A; this is A as a module over k, with multiplication $a \cdot_{\mathrm{op}} b = ba$ for all $a, b \in A$. Tensor products will often be taken over k, and unless otherwise indicated, $\otimes = \otimes_k$. Let $A^e = A \otimes A^{\mathrm{op}}$, with the tensor product multiplication:

$$(a_1 \otimes b_1) \cdot (a_2 \otimes b_2) = a_1 a_2 \otimes b_2 b_1$$

for all $a_1, a_2, b_1, b_2 \in A$. (Technically, we are really taking b_1, b_2 to be elements of A^{op}, but since the underlying k-modules are the same, we write $b_1, b_2 \in A$ where convenient.) We call A^e the *enveloping algebra* of A.

By an *A-bimodule*, we mean a k-module M that is both a left and a right A-module for which $(a_1 m)a_2 = a_1(m a_2)$ for all $a_1, a_2 \in A$ and $m \in M$, and the left and right actions of k induced by the unit map $k \to A$ agree. Thus an A-bimodule M is equivalent to a left A^e-module, where we define

$$(a \otimes b) \cdot m = amb$$

for all $a, b \in A$ and $m \in M$. It is also equivalent to a right A^e-module where the action is defined by $m \cdot (a \otimes b) = bma$ for all $a, b \in A$ and $m \in M$. We will use both structures in the sequel, but for simplicity, when we refer to a module we generally mean a left module unless otherwise specified.

Note that the algebra A is itself an A^e-module (equivalently, an A-bimodule) under left and right multiplication: $(a \otimes b) \cdot c = acb$ for all $a, b, c \in A$. More generally, let $A^{\otimes n} = A \otimes \cdots \otimes A$ (n factors of A), where $n \geq 1$. This tensor power of A is an A^e-module (equivalently, an A-bimodule) by letting

$$(a \otimes b) \cdot (c_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n) = ac_1 \otimes c_2 \otimes \cdots \otimes c_{n-1} \otimes c_n b$$

for all $a, b, c_1, \ldots, c_n \in A$.

Consider the following sequence of A-bimodules:

$$\cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \xrightarrow{\pi} A \to 0, \tag{1.1.1}$$

where π is the multiplication map, that is, $\pi(a \otimes b) = ab$ for all $a, b \in A$, and $d_1(a \otimes b \otimes c) = ab \otimes c - a \otimes bc$ for all $a, b, c \in A$, and in general

$$d_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1} \tag{1.1.2}$$

for all $a_0, \ldots, a_{n+1} \in A$. One may check directly that (1.1.1) is a complex, that is, $d_{n-1} d_n = 0$ for all n (see Section A.1). Moreover, it is exact, as a consequence of the existence of the following contracting homotopy (see Section A.1). Let s_n be the k-linear map defined by

$$s_n(a_0 \otimes \cdots \otimes a_{n+1}) = 1 \otimes a_0 \otimes \cdots \otimes a_{n+1} \tag{1.1.3}$$

for all $n \geq -1$ and all $a_0, \ldots, a_{n+1} \in A$. A calculation shows that indeed $s_\bullet$ is a contracting homotopy, and so the complex (1.1.1) is exact. We write $B_n(A) = A^{\otimes(n+2)}$ for $n \geq 0$ and often consider the truncated complex associated to (1.1.1):

$$(1.1.4) \qquad B(A): \quad \cdots \xrightarrow{d_3} A^{\otimes 4} \xrightarrow{d_2} A^{\otimes 3} \xrightarrow{d_1} A \otimes A \to 0.$$

This is the *bar complex* of the A^e-module A. As a complex, its homology is concentrated in degree 0, where it is simply A, as a consequence of exactness of (1.1.1). Sometimes we use the isomorphism of left A^e-modules

$$(1.1.5) \qquad A^{\otimes(n+2)} \cong A^e \otimes A^{\otimes n}$$

given by $a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1} \mapsto (a_0 \otimes a_{n+1}) \otimes (a_1 \otimes \cdots \otimes a_n)$ for all $a_0, \ldots, a_{n+1} \in A$. (The action of A^e on $A^e \otimes A^{\otimes n}$ is multiplication on the leftmost factor A^e.) If A is free as a k-module, we see in this way that the terms in the bar complex (1.1.4) are free A^e-modules: $A^e \otimes A^{\otimes n} \cong \bigoplus_{i \in I} A^e(1 \otimes 1 \otimes \alpha_i)$, where $\{\alpha_i \mid i \in I\}$, for an indexing set I, is a basis of $A^{\otimes n}$ as a free k-module. In this case, $B(A)$ is a free resolution of the A^e-module A, also called the *bar resolution* (or *standard resolution*). See [**48**, Section IX.6].

Remark 1.1.6. The term *bar* complex arose historically due to an abbreviation of a tensor product $a_1 \otimes \cdots \otimes a_n$ as $[a_1 | \cdots | a_n]$. This convention was begun by Eilenberg and Mac Lane.

Let M be an A-bimodule. Take the tensor product of the bar complex (1.1.4) with M, writing

$$(1.1.7) \qquad C_*(A, M) = \bigoplus_{n \geq 0} M \otimes_{A^e} B_n(A),$$

a complex of k-modules with differentials $1_M \otimes d_n$, where d_n is defined by equation (1.1.2) and 1_M is the identity map on M. Call $C_*(A, M)$ the k-module of *Hochschild chains with coefficients in* M. There is a k-module isomorphism

$$(1.1.8) \qquad M \otimes_{A^e} B_n(A) \xrightarrow{\sim} M \otimes A^{\otimes n}$$

given by

$$m \otimes_{A^e} (a_0 \otimes \cdots \otimes a_{n+1}) \mapsto a_{n+1} m a_0 \otimes a_1 \otimes \cdots \otimes a_n$$

for all $m \in M$ and $a_0, \ldots, a_{n+1} \in A$. (The inverse isomorphism is given by $m \otimes a_1 \otimes \cdots \otimes a_n \mapsto m \otimes_{A^e} (1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1)$; recall the left action of A^e on $A^{\otimes(n+2)}$ involves the first and last tensor factors only, and the right action of A^e on M involves both the left and right actions of A on M.) Therefore there is a k-module isomorphism

$$C_n(A, M) \cong M \otimes A^{\otimes n}$$

for each n. Combined with the isomorphism (1.1.5), we find that the induced differential on the complex $M \otimes A^{\otimes *}$, corresponding to the map $1_M \otimes d_n$ on $M \otimes_{A^e} A^{\otimes(n+2)}$ for each $n > 0$, is the map $(d_n)_* : M \otimes A^{\otimes n} \to M \otimes A^{\otimes(n-1)}$ given by

$$\begin{aligned}
&(d_n)_*(m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_n) \\
&\quad = \; ma_1 \otimes a_2 \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
&\qquad\qquad + (-1)^n a_n m \otimes a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}
\end{aligned}$$

for all $m \in M$ and $a_1, \ldots, a_n \in A$. We define Hochschild homology to be the homology of this complex.

Definition 1.1.9. The *Hochschild homology* $\mathrm{HH}_*(A, M)$ of A with coefficients in an A-bimodule M is the homology of the complex (1.1.7), equivalently

$$\mathrm{HH}_n(A, M) = \mathrm{H}_n(M \otimes A^{\otimes *}),$$

that is, $\mathrm{HH}_n(A, M) = \mathrm{Ker}((d_n)_*)/\mathrm{Im}((d_{n+1})_*)$ for all $n \geq 0$, taking $(d_0)_*$ to be the zero map, and differentials $(d_n)_*$ are as given above for $n > 0$. Elements in $\mathrm{Ker}((d_n)_*)$ are *Hochschild n-cycles* and those in $\mathrm{Im}((d_{n+1})_*)$ are *Hochschild n-boundaries.* Let

$$\mathrm{HH}_*(A, M) = \bigoplus_{n \geq 0} \mathrm{HH}_n(A, M),$$

an $\mathbb{N}$-graded k-module.

We have chosen the common notation HH to denote Hochschild homology (and cohomology below) so as to distinguish it from other versions of cohomology that we will use later. It is instead denoted with a single letter H in some of the literature.

Next we will apply $\mathrm{Hom}_{A^e}(-, M)$ to the bar complex (1.1.4): Let

$$C^*(A, M) = \bigoplus_{n \geq 0} \mathrm{Hom}_{A^e}(B_n(A), M), \tag{1.1.10}$$

a complex of k-modules with differentials d_n^*, where $d_n^*(f) = f d_n$ for all functions f in $\mathrm{Hom}_{A^e}(A^{\otimes(n+1)}, M)$. (For simplicity here and throughout the book, we denote such a function composition by the concatenation $f d_n$ rather than $f \circ d_n$.) Call $C^*(A, M)$ the k-module of *Hochschild cochains with coefficients in M*. There is a k-module isomorphism

$$\mathrm{Hom}_{A^e}(B_n(A), M) \xrightarrow{\sim} \mathrm{Hom}_k(A^{\otimes n}, M) \tag{1.1.11}$$

given by $g \mapsto (a_1 \otimes \cdots \otimes a_n \mapsto g(1 \otimes a_1 \otimes \cdots \otimes a_n \otimes 1))$ for all $g \in \mathrm{Hom}_{A^e}(B(A)_n, M)$ and $a_1, \ldots, a_n \in A$. (If $n = 0$ this is $g \mapsto (1 \mapsto g(1 \otimes 1))$.)

The inverse isomorphism is $g' \mapsto (a_0 \otimes \cdots \otimes a_{n+1} \mapsto a_0 g'(a_1 \otimes \cdots \otimes a_n)a_{n+1})$. We thus have an isomorphism of complexes,

$$C^*(A, M) \cong \bigoplus_{n \geq 0} \mathrm{Hom}_k(A^{\otimes n}, M), \tag{1.1.12}$$

with differential $d_n^* : \mathrm{Hom}_k(A^{\otimes(n-1)}, M) \to \mathrm{Hom}_k(A^{\otimes n}, M)$ for each $n > 0$ given by

$$\begin{aligned} d_n^*(h)(a_1 \otimes \cdots \otimes a_n) &= a_1 h(a_2 \otimes \cdots \otimes a_n) \\ &\quad + \sum_{i=1}^{n-1} (-1)^i h(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n) \\ &\quad + (-1)^n h(a_1 \otimes \cdots \otimes a_{n-1}) a_n \end{aligned}$$

for all $h \in \mathrm{Hom}_k(A^{\otimes(n-1)}, M)$ and $a_1, \ldots, a_n \in A$. In this expression and others, we interpret an empty tensor product to be the element 1 in k. We define Hochschild cohomology to be the cohomology of this complex.

Definition 1.1.13. The *Hochschild cohomology* $\mathrm{HH}^*(A, M)$ of A with coefficients in an A-bimodule M is the cohomology of the complex (1.1.10), equivalently

$$\mathrm{HH}^n(A, M) = \mathrm{H}^n(\mathrm{Hom}_k(A^{\otimes *}, M)),$$

that is, $\mathrm{HH}^n(A, M) = \mathrm{Ker}(d_{n+1}^*)/\,\mathrm{Im}(d_n^*)$ for all $n \geq 0$, where d_0^* is taken to be the zero map, and differentials d_n^* are as given above for $n > 0$. Elements in $\mathrm{Ker}(d_{n+1}^*)$ are *Hochschild n-cocycles* and those in $\mathrm{Im}(d_n^*)$ are *Hochschild n-coboundaries.* Let

$$\mathrm{HH}^*(A, M) = \bigoplus_{n \geq 0} \mathrm{HH}^n(A, M),$$

an $\mathbb{N}$-graded k-module.

As a special case, consider $M = A$ to be an A-bimodule under left and right multiplication. The resulting Hochschild homology and cohomology k-modules are sometimes abbreviated

$$\mathrm{HH}_n(A) = \mathrm{HH}_n(A, A) \quad \text{and} \quad \mathrm{HH}^n(A) = \mathrm{HH}^n(A, A),$$

and so we write $\mathrm{HH}_*(A) = \bigoplus_{n \geq 0} \mathrm{HH}_n(A)$ and $\mathrm{HH}^*(A) = \bigoplus_{n \geq 0} \mathrm{HH}^n(A)$. A disadvantage of this abbreviated notation in the case of cohomology is that it appears to indicate a functor, however HH^* is not a functor: we will see next that $\mathrm{HH}^*(A, M)$ can alternatively be defined via the bifunctor $\mathrm{Ext}^*_{A^e}(-, -)$, and this bifunctor is covariant in the second argument and contravariant in the first argument. Another disadvantage is that some authors use this notation to refer to Hochschild homology and cohomology with coefficients in a dual module to A, in which case these are functors related to cyclic homology and cohomology [**146**, §1.5.5]. For simplicity, we adopt the common abbreviated notation above in spite of these disadvantages.

Remarks 1.1.14. (i) In the case that k is a field (the case of focus for most of this book), A is free as a k-module and so each A^e-module $A^{\otimes(n+2)}$ ($n \geq 0$) is free, as noted earlier. It follows that the bar complex $B(A)$ given by (1.1.4) is a free left A^e-module resolution of A, called the *bar resolution.* Thus

$$\text{(1.1.15)} \qquad \mathrm{HH}_n(A, M) \cong \mathrm{Tor}_n^{A^e}(M, A) \quad \text{and} \quad \mathrm{HH}^n(A, M) \cong \mathrm{Ext}^n_{A^e}(A, M)$$

for all $n \geq 0$. (See Section A.3.) More generally, as long as A is flat over the commutative ring k, the first isomorphism holds, and as long as A is projective over k, the second holds.

(ii) Even more generally, it follows from the definitions that Hochschild homology and cohomology may be realized as relative Tor and relative Ext [**223**, Lemma 9.1.3]. See also [**115**]. We will not use relative Tor or relative Ext in this book.

We will use the equivalent definitions of Hochschild homology and cohomology given by the isomorphisms (1.1.15) in the case that k is a field. An advantage is that we may thus choose any flat (respectively, projective) resolution of A as an A^e-module to define Hochschild homology (respectively, cohomology). Depending on the algebra A, there may be more convenient resolutions than the bar resolution; the latter is quite large and not conducive to explicit computation. The bar resolution may also obscure important information that stands out in other resolutions which are tailored more closely to the shapes of specific algebras. However, the bar resolution is very useful theoretically, as we will see.

Also useful is the following variant of the bar resolution. Identify k with the k-submodule $k \cdot 1_A$ of the k-algebra A, and write $\overline{A} = A/k$, the quotient k-module. For each nonnegative integer n, let

$$\text{(1.1.16)} \qquad \overline{B}_n(A) = A \otimes \overline{A}^{\otimes n} \otimes A.$$

Let $p_n : B_n(A) \to \overline{B}_n(A)$ be the corresponding quotient map. A calculation shows that the kernels of the maps p_n form a subcomplex of $B(A)$, and thus the quotients $\overline{B}_n(A)$ constitute a complex $\overline{B}(A)$. The contracting homotopy (1.1.3) can be shown to factor through this quotient, implying that $\overline{B}(A)$ is a free resolution of the A^e-module A in case A is free as a k-module. We give this resolution a name.

Definition 1.1.17. Assume that A is free as a k-module. The *reduced bar resolution* (or *normalized bar resolution*) is $\overline{B}(A)$ given in each degree by equation (1.1.16), a free resolution of A as an A^e-module.

See also [**85**, §13.5] for more details on the reduced bar resolution.

Hochschild cohomology modules hold substantial information about the algebra A and its modules, some of which we will find in this chapter by harnessing the bar complex, and some in later chapters via a wider range of complexes. Hochschild cohomology is invariant under some standard equivalences on rings: in case k is a field, invariance under Morita equivalence is automatic since this is by definition an equivalence of module categories and Hochschild cohomology is given by Ext in this case (see Remark 1.1.14(i)). For more details and related and more general statements, see, e.g., [**22**, Theorem 2.11.1], [**146**, Theorem 1.2.7], or the articles [**37**, **56**, **169**]. For Morita invariance of Hochschild homology, see, e.g., [**223**, §9.5]. For tilting and derived category equivalence, see, e.g., [**105**, Theorem 4.2] and [**9**, **185**, **220**], and for derived invariance of higher structures on the Hochschild complex itself, see [**127**].

We end this section with some examples that apply Remark 1.1.14(i), taking advantage of resolutions smaller than the bar resolution.

Example 1.1.18. Let k be a field and $A = k[x]$. Consider the following sequence:

$$0 \longrightarrow k[x] \otimes k[x] \xrightarrow{(x\otimes 1 - 1\otimes x)\cdot} k[x] \otimes k[x] \xrightarrow{\pi} k[x] \longrightarrow 0, \tag{1.1.19}$$

where π is multiplication and the map $(x \otimes 1 - 1 \otimes x)\cdot$ is multiplication by the element $x \otimes 1 - 1 \otimes x$. Since $k[x]$ is commutative, this sequence is a complex. It is in fact exact, as can be shown directly via a calculation. Alternatively, exactness can be shown by exhibiting a contracting homotopy (see Section A.1): Let $s_{-1}(x^i) = x^i \otimes 1$ and

$$s_0(x^i \otimes x^j) = -\sum_{l=1}^{j} x^{i+j-l} \otimes x^{l-1}$$

for all i, j. (We interpret an empty sum to be 0, thus $s_0(x^i \otimes 1) = 0$ for all i.) A calculation shows that $s_\bullet$ is a contracting homotopy for the above sequence. Note that for each i, the map s_i is left (but not right) $k[x]$-linear. The terms in nonnegative degrees are visibly free as A^e-modules, and so the sequence (1.1.19) is a free resolution of the A^e-module A. Now apply $\mathrm{Hom}_{k[x]^e}(-, k[x])$ to the truncation of sequence (1.1.19) given by deleting the term $k[x]$. Identify $\mathrm{Hom}_{k[x]^e}(k[x] \otimes k[x], k[x])$ with $k[x]$ under the isomorphism in which a function f is sent to $f(1 \otimes 1)$. The resulting complex, with arrows reversed, becomes

$$0 \longleftarrow k[x] \longleftarrow k[x] \longleftarrow 0. \tag{1.1.20}$$

There is only one map to compute, namely composition with $(x\otimes 1 - 1\otimes x)\cdot$. Let $a \in k[x]$, identified with the function f_a in $\mathrm{Hom}_{k[x]^e}(k[x]\otimes k[x], k[x])$ that takes $1 \otimes 1$ to a. Composing with the differential, since f_a is a $k[x]^e$-module homomorphism,

$$f_a((x \otimes 1 - 1 \otimes x) \cdot (1 \otimes 1)) = x f_a(1 \otimes 1) - f_a(1 \otimes 1)x = xa - ax = 0,$$

since $k[x]$ is commutative. Therefore all maps in complex (1.1.20) are 0, and the homology of the complex in each degree is just the term in the complex. We thus find that $\mathrm{HH}^0(k[x]) \cong k[x]$, that $\mathrm{HH}^1(k[x]) \cong k[x]$, and $\mathrm{HH}^n(k[x]) = 0$ for $n \geq 2$. A similar argument yields Hochschild homology $\mathrm{HH}_n(k[x])$ by first applying $k[x] \otimes_{k[x]^e} -$ to the truncation of the sequence (1.1.19) and identifying $k[x] \otimes_{k[x]^e} (k[x] \otimes k[x])$ with $k[x]$.

Example 1.1.21. Let k be a field, $n \geq 2$, and $A = k[x]/(x^n)$, called a *truncated polynomial ring*. Consider the following sequence:

$$\cdots \xrightarrow{v\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{v\cdot} A^e \xrightarrow{u\cdot} A^e \xrightarrow{\pi} A \longrightarrow 0, \tag{1.1.22}$$

where $u = x \otimes 1 - 1 \otimes x$, $v = x^{n-1} \otimes 1 + x^{n-2} \otimes x + \cdots + 1 \otimes x^{n-1}$, and π is multiplication. This sequence is exact, as can be shown directly. Alternatively, the following is a contracting homotopy (see Section A.1). For each i, define a left A-linear map s_i by $s_{-1}(1) = 1 \otimes 1$ and for all $m \geq 0$,

$$s_{2m}(1 \otimes x^j) = -\sum_{l=1}^{j} x^{j-l} \otimes x^{l-1} \quad \text{and} \quad s_{2m+1}(1 \otimes x^j) = \delta_{j,n-1} \otimes 1$$

for all j, where $\delta_{j,n-1}$ is the Kronecker delta (that is, $\delta_{j,n-1} = 1$ if $j = n-1$ and $\delta_{j,n-1} = 0$ otherwise). The terms in nonnegative degrees are visibly free as A^e-modules, and so the sequence (1.1.22) is a free resolution of the A^e-module A.

Apply $\mathrm{Hom}_{A^e}(-, A)$ to (1.1.22) after truncating by deleting A, and identify each term $\mathrm{Hom}_{A^e}(A \otimes A, A)$ with A. The resulting sequence may be viewed as:

$$\cdots \xleftarrow{nx^{n-1}\cdot} A \xleftarrow{0} A \xleftarrow{nx^{n-1}\cdot} A \xleftarrow{0} A \longleftarrow 0.$$

Thus we see that if n is divisible by the characteristic of k, then $\mathrm{HH}^i(A) \cong A$ for all i. If n is not divisible by the characteristic of k, then $\mathrm{HH}^0(A) \cong A$, $\mathrm{HH}^{2m+1}(A) \cong (x)$ (the ideal generated by x) for all $m \geq 0$, and $\mathrm{HH}^{2m}(A) \cong A/(x^{n-1})$ for all $m \geq 1$.

Exercise 1.1.23. Verify the claims that formulas given for the following differentials are induced by the differential (1.1.2) on the bar complex (1.1.4):

(a) $(d_n)_*$ right before Definition 1.1.9.

(b) d_n^* right before Definition 1.1.13.

Exercise 1.1.24. Show that the following maps constitute contracting homotopies as claimed:

(a) s_n defined in (1.1.3).

(b) s_{-1}, s_0 defined in Example 1.1.18.

(c) s_i defined in Example 1.1.21.

Exercise 1.1.25. Finish Example 1.1.18 by finding Hochschild homology $\mathrm{HH}_n(k[x])$ for each n.

Exercise 1.1.26. Assume that k is a field and let $0 \to U \to V \to W \to 0$ be an exact sequence of A-bimodules. Use Remark 1.1.14(i) and Theorem A.4.7 (second long exact sequence for Tor) to show that there is a long exact sequence for Hochschild homology,

$$\cdots \to \mathrm{HH}_n(A, U) \to \mathrm{HH}_n(A, V) \to \mathrm{HH}_n(A, W) \to \mathrm{HH}_{n-1}(A, U) \to \cdots .$$

Derive a similar long exact sequence for Hochschild cohomology using Theorem A.4.4 (first long exact sequence for Ext).

Exercise 1.1.27. Assume that A is free as a k-module. Check that $\overline{B}(A)$ as in Definition 1.1.17 is indeed a free resolution of A as an A^e-module by verifying the claim that the kernels of the maps p_n form a subcomplex of $B(A)$ and that the contracting homotopy (1.1.3) for the bar complex (1.1.4) factors through $\overline{B}(A)$.

1.2. Interpretation in low degrees

The historical Definitions 1.1.9 and 1.1.13 of Hochschild homology $\mathrm{HH}_n(A)$ and cohomology $\mathrm{HH}^n(A)$ lead directly to specific information encoded by these k-modules when n is small. For example, in the nth Hochschild cohomology k-module $\mathrm{HH}^n(A)$ we can see the center of A in degree $n = 0$, derivations on A in degree 1, and infinitesimal deformations together with obstructions to lifting these to formal deformations in degrees 2 and 3. We make some of these observations in this section, including more general statements for A-bimodules M as well as analogous statements for Hochschild homology. More detailed discussion of deformations and obstructions is in Chapters 5 and 7.

In what follows, we will frequently identify spaces of chains under isomorphism (1.1.8) and spaces of cochains under isomorphism (1.1.11). We will abuse notation, using the same for differentials on such spaces when they have been so identified. Thus, for example, we write d_n^* for the map from $\mathrm{Hom}_{A^e}(A^{\otimes(n-1)}, M)$ to $\mathrm{Hom}_{A^e}(A^{\otimes n}, M)$ given by

$$\begin{aligned}
&d_n^*(h)(a_0 \otimes a_1 \otimes \cdots \otimes a_n \otimes a_{n+1}) \\
&\quad = \sum_{i=0}^{n} (-1)^i h(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}) \\
&\quad = a_0 a_1 h(1 \otimes a_2 \otimes \cdots \otimes a_n \otimes 1) a_{n+1} \\
&\qquad + \sum_{i=1}^{n-1} (-1)^i a_0 h(1 \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \otimes 1) a_{n+1} \\
&\qquad + (-1)^n a_0 h(1 \otimes a_1 \otimes \cdots \otimes a_{n-1} \otimes 1) a_n a_{n+1}
\end{aligned}$$

for all $h \in \mathrm{Hom}_{A^e}(A^{\otimes(n-1)}, M)$ and $a_0, \ldots, a_{n+1} \in A$.

Degree 0. Let M be an A-bimodule. In the notation of Definition 1.1.13, $\mathrm{HH}^0(A, M) = \mathrm{Ker}(d_1^*)$. We determine necessary and sufficient conditions for a function $f \in \mathrm{Hom}_{A^e}(A \otimes A, M)$ to be in $\mathrm{Ker}(d_1^*)$. First assume that $d_1^*(f) = 0$, that is, for all $a \in A$,

$$\begin{aligned}
0 = d_1^*(f)(1 \otimes a \otimes 1) &= f(d_1(1 \otimes a \otimes 1)) \\
&= f(a \otimes 1 - 1 \otimes a) = af(1 \otimes 1) - f(1 \otimes 1)a.
\end{aligned}$$

Then $f(1 \otimes 1)$ is equal to an element m of M for which $am = ma$ for all $a \in A$, and f is in fact determined by this element m:

$$f(b \otimes c) = bf(1 \otimes 1)c = bmc$$

for all $b, c \in A$. Conversely, any such element of M defines a function in $\mathrm{Ker}(d_1^*)$, that is, given $m \in M$ for which $am = ma$ for all $a \in A$, let $f_m \in \mathrm{Hom}_{A^e}(A \otimes A, M)$ be the function given by $f_m(b \otimes c) = bmc$ for all $b, c \in A$. Then $d_1^*(f_m) = 0$. Thus as a k-module,

$$\mathrm{HH}^0(A, M) \cong \{m \in M \mid am = ma \text{ for all } a \in A\}.$$

In the special case $M = A$, it follows that $\mathrm{HH}^0(A, A) \cong Z(A)$, the center of the algebra A.

Similarly, Hochschild homology in degree 0 is

$$\mathrm{HH}_0(A, M) \cong M/\, \mathrm{Span}_k\{am - ma \mid a \in A,\ m \in M\}. \tag{1.2.1}$$

Degree 1. By Definition 1.1.13, $\mathrm{HH}^1(A, M) = \mathrm{Ker}(d_2^*)/\, \mathrm{Im}(d_1^*)$. Let $f \in \mathrm{Ker}(d_2^*)$, that is, $f \in \mathrm{Hom}_{A^e}(A^{\otimes 3}, M)$ and fd_2 is the zero map on $A^{\otimes 4}$. This condition is equivalent to

$$\begin{aligned} 0 &= d_2^*(f)(1 \otimes a \otimes b \otimes 1) \\ &= f(d_2(1 \otimes a \otimes b \otimes 1)) \\ &= f(a \otimes b \otimes 1 - 1 \otimes ab \otimes 1 + 1 \otimes a \otimes b) \\ &= af(1 \otimes b \otimes 1) - f(1 \otimes ab \otimes 1) + f(1 \otimes a \otimes 1)b \end{aligned}$$

for all $a, b \in A$. By abuse of notation, we identify f with a function in $\mathrm{Hom}_k(A, M)$ under the isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes 3}, M) \cong \mathrm{Hom}_k(A, M)$ of (1.1.11), and the above equation becomes $0 = af(b) - f(ab) + f(a)b$, or

$$f(ab) = af(b) + f(a)b$$

for all $a, b \in A$. This is precisely the definition of a *k-derivation* from A to M (also called more simply a *derivation* from A to M when k is understood). The k-module of all k-derivations from A to M is denoted

$$\mathrm{Der}(A, M).$$

We also write $\mathrm{Der}_k(A, M)$ when we want to emphasize dependence on the ground ring k. Suppose in addition that $f \in \mathrm{Im}(d_1^*)$, that is, $f = d_1^*(g)$ for some g in $\mathrm{Hom}_{A^e}(A^{\otimes 2}, M)$. The function g is determined by its value on $1 \otimes 1$, say m. Then

$$\begin{aligned} d_1^*(g)(1 \otimes a \otimes 1) &= g(d_1(1 \otimes a \otimes 1)) \\ &= g(a \otimes 1 - 1 \otimes a) \\ &= ag(1 \otimes 1) - g(1 \otimes 1)a \quad = \quad am - ma. \end{aligned}$$

That is, $d_1^*(g)$ corresponds to the *inner k-derivation* from A to M defined by the element m. Conversely, any inner k-derivation will determine an element of $\mathrm{Im}(d_1^*)$. The k-module of all inner k-derivations from A to M is denoted

$$\mathrm{InnDer}(A, M).$$

We have shown that

$$\mathrm{HH}^1(A, M) \cong \mathrm{Der}(A, M)/\,\mathrm{InnDer}(A, M).$$

In particular, if $M = A$, then $\mathrm{HH}^1(A)$ is isomorphic to the k-module of derivations of A modulo inner derivations, also called *outer derivations*. If A is commutative, the zero function is the only inner derivation, so in this case, the k-module of outer derivations, $\mathrm{HH}^1(A)$, is simply the k-module of derivations of A. Some results on when $\mathrm{HH}^1(A)$ vanishes and further discussion are in a paper of Buchweitz and Liu [**40**].

It can be shown that Hochschild homology in degree 1, $\mathrm{HH}_1(A, M)$, is isomorphic to the kernel of the canonical map $I \otimes_{A^e} M \to IM$, where I is the kernel of multiplication $\pi : A \otimes A \to A$. See Exercise 1.2.8 in case k is a field. For more details in the general case, see [**223**, §9.2], where a connection to Kähler differentials of commutative algebras is also given. In Section 4.2, we will identify I with the space $\Omega^1_{nc}A$ of noncommutative Kähler differentials.

Degree 2. By Definition 1.1.13, $\mathrm{HH}^2(A,M) = \mathrm{Ker}(d_3^*)/\,\mathrm{Im}(d_2^*)$. Let $f \in \mathrm{Hom}_{A^e}(A^{\otimes 4}, M)$. Then f is in $\mathrm{Ker}(d_3^*)$ if and only if for all $a,b,c \in A$,

$$\begin{aligned} 0 &= d_3^*(f)(1\otimes a\otimes b\otimes c\otimes 1) \\ &= f(d_3(1\otimes a\otimes b\otimes c\otimes 1)) \\ &= f(a\otimes b\otimes c\otimes 1 - 1\otimes ab\otimes c\otimes 1 + 1\otimes a\otimes bc\otimes 1 - 1\otimes a\otimes b\otimes c) \\ &= af(1\otimes b\otimes c\otimes 1) - f(1\otimes ab\otimes c\otimes 1) + f(1\otimes a\otimes bc\otimes 1) \\ &\quad -f(1\otimes a\otimes b\otimes 1)c. \end{aligned}$$

Identifying f with a function in $\mathrm{Hom}_k(A^{\otimes 2}, M)$ under the isomorphism $\mathrm{Hom}_{A^e}(A^{\otimes 4}, M) \cong \mathrm{Hom}_k(A^{\otimes 2}, M)$ of (1.1.11), we find that $f \in \mathrm{Ker}(d_3^*)$ if and only if

$$af(b\otimes c) + f(a\otimes bc) = f(ab\otimes c) + f(a\otimes b)c \tag{1.2.2}$$

for all $a,b,c \in A$. A calculation shows that the image of d_2^* may be identified with the k-module of all functions f in $\mathrm{Hom}_k(A\otimes A, M)$ given by

$$f(a\otimes b) = ag(b) - g(ab) + g(a)b \tag{1.2.3}$$

for some $g \in \mathrm{Hom}_k(A, M)$.

We will see in Section 4.2 that Hochschild 2-cocycles, that is, functions satisfying equation (1.2.2), define algebra structures on the A-bimodule $A \oplus M$. These algebras are called square-zero extensions, and they arise in connection with smoothness of algebras. In the case that $M = A$, equation (1.2.2) gives rise to infinitesimal deformations of A as discussed in Chapter 5. There we will develop algebraic deformation theory and we will see that obstructions to lifting a Hochschild 2-cocycle to a formal deformation lie in $\mathrm{HH}^3(A)$. We will also see that functions f satisfying (1.2.3) for some g, that is, Hochschild 2-coboundaries, correspond to deformations isomorphic to the original algebra. In this way, each formal deformation, up to isomorphism, will have associated to it an element of $\mathrm{HH}^2(A)$.

Actions of the center of A. In the interpretation of $\mathrm{HH}^0(A)$ as the center $Z(A)$ of A, some actions of $Z(A)$ take on a broader meaning. We present these actions of $Z(A)$ here, and extend them to actions of $\mathrm{HH}^*(A)$ in Sections 1.3 and 2.5. First note that $Z(A)$ acts on $\mathrm{Hom}_{A^e}(U,V)$ for any two A^e-modules U,V by

$$(a\cdot f)(u) = af(u) \tag{1.2.4}$$

for all $a \in Z(A)$, $u \in U$, and $f \in \mathrm{Hom}_{A^e}(U,V)$. Taking $U = A^{\otimes(n+2)}$ and $V = A$, this action commutes with the differentials on the bar complex, inducing an action of $Z(A)$ on $\mathrm{HH}^n(A)$ under which $\mathrm{HH}^n(A)$ becomes a $Z(A)$-module. Identifying $Z(A)$ with $\mathrm{HH}^0(A)$ as described above, this is an action of $\mathrm{HH}^0(A)$ on $\mathrm{HH}^n(A)$ for each n, thus on $\mathrm{HH}^*(A)$. In the next section, this action will be extended to a graded product on $\mathrm{HH}^*(A)$. In Section 2.5, the action of $Z(A)$ on $\mathrm{Hom}_{A^e}(U,V)$ will be extended to an action of $\mathrm{HH}^*(A)$ on $\mathrm{Ext}^*_{A^e}(U,V)$.

The action of $Z(A)$ on $\mathrm{HH}^*(A)$ described above has some useful consequences. For example, let $1 = e_1 + \cdots + e_i$ be an expansion of the multiplicative identity of A as the sum of a set of orthogonal central idempotents $e_1, \ldots, e_i$. (That is, each e_j is central in A and $e_j e_l = \delta_{j,l} e_j$ for all j, l, where $\delta_{j,l}$ is the Kronecker delta.) Then $A = \bigoplus_{j=1}^i A e_j$, a direct sum of ideals $Ae_j = e_jA = e_jAe_j$ of A. This leads to a similar decomposition $\mathrm{Hom}_{A^e}(U,A) = \bigoplus_{j=1}^i \mathrm{Hom}_{A^e}(U, e_jA)$ for any A^e-module U, and further,

$$\mathrm{HH}^*(A) \cong \bigoplus_{j=1}^{i} \mathrm{HH}^*(e_jA) \cong \bigoplus_{j=1}^{i} e_j\, \mathrm{HH}^*(A). \tag{1.2.5}$$

Here we view the ideal e_jA of A itself as an algebra with multiplicative identity e_j, and we may identify $e_j\,\mathrm{HH}^*(A)$ with $\mathrm{HH}^*(e_jA)$ in this expansion, under the action of $Z(A)$ on $\mathrm{HH}^*(A)$.

Exercise 1.2.6. Verify the isomorphism (1.2.1).

Exercise 1.2.7. Describe all k-derivations of $k[x]$ with the help of Example 1.1.18 and the connection to Hochschild cohomology explained in this section.

Exercise 1.2.8. Let k be a field, and let I be the kernel of the multiplication map $\pi : A \otimes A \to A$. Show that $\mathrm{HH}_1(A, M)$ is isomorphic to the kernel of the map $I \otimes_{A^e} M \to IM$ given by $\sum_i (a_i \otimes b_i) \otimes_{A^e} m \mapsto \sum_i a_i m b_i$. (*Hint*: Consider the short exact sequence $0 \to I \to A^e \xrightarrow{\pi} A \to 0$ and use Exercise 1.1.26, noting that $\mathrm{HH}_1(A, A^e) = 0$ since A^e is flat as an A^e-module.)

Exercise 1.2.9. Find the action of $Z(A)$ on $\mathrm{HH}^*(A)$ in each case:

(a) $A = k[x]$ as in Example 1.1.18.

(b) $A = k[x]/(x^n)$ as in Example 1.1.21.

1.3. Cup product

Hochschild cohomology $\mathrm{HH}^*(A)$ is a graded k-module by its definition. (That is, it is graded by $\mathbb{N}$, which we understand to include 0.) We will see next that it has an associative product making it into a graded commutative algebra, that is, homogeneous elements commute up to a sign determined by homological degrees. (See Theorem 1.4.6 below.) We define this product at the chain level for functions on the bar complex (1.1.4) in Definition 1.3.1 below. In fact the cup product is the unique associative product on $\mathrm{HH}^*(A)$ satisfying some basic conditions (see Sanada [**188**]). There are many equivalent definitions of this associative product on $\mathrm{HH}^*(A)$, particularly in case k is a field, making it very versatile. We give some of these other definitions in Chapter 2.

We again use the isomorphism (1.1.11) to identify $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, M)$ with $\mathrm{Hom}_k(A^{\otimes n}, M)$ as a k-module. As in (1.1.12),

$$C^*(A, M) \cong \bigoplus_{n\geq 0} \mathrm{Hom}_k(A^{\otimes n}, M),$$

the k-module of Hochschild cochains on A with coefficients in M. In what follows we will take $M = A$.

Definition 1.3.1. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. The *cup product* $f \smile g$ is the element of $\mathrm{Hom}_k(A^{\otimes(m+n)}, A)$ defined by

(1.3.2)

$$(f \smile g)(a_1 \otimes \cdots \otimes a_{m+n}) = (-1)^{mn} f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n})$$

for all $a_1, \ldots, a_{m+n} \in A$. If $m = 0$, we interpret this formula to be

$$(f \smile g)(a_1 \otimes \cdots \otimes a_n) = f(1) g(a_1 \otimes \cdots \otimes a_n),$$

and similarly if $n = 0$.

Remark 1.3.3. The historical definition of cup product, used by many authors, does not include the factor $(-1)^{mn}$. Our choice here will be more convenient with respect to our sign conventions, and also agrees with much of the literature. The difference is not so essential, since at the level of cohomology, this product is graded commutative, as we will see.

By its definition, the cup product is associative. A sign modification makes the k-module $C^*(A, A)$ of Hochschild cochains a *differential graded algebra*, that is, a graded algebra (i.e., $C^i(A, A) \smile C^j(A, A) \subseteq C^{i+j}(A, A)$) with a graded derivation of degree 1 and square 0. Precisely, for an m-cochain f, let $\partial(f) = (-1)^m d^*_{m+1}(f)$ and similarly for g, $f \smile g$. Then

$$\partial(f \smile g) = (\partial(f)) \smile g + (-1)^m f \smile (\partial(g)). \tag{1.3.4}$$

(If we omit the factor $(-1)^{mn}$ in the definition (1.3.2) of cup product, then $C^*(A, A)$ is a differential graded algebra without modifying the sign of the differential. Note that the cohomology is the same in either case.) A consequence of equation (1.3.4) is that this cup product $\smile$ induces a well-defined graded associative product on Hochschild cohomology, which we denote by the same symbol:

$$\smile : \ \mathrm{HH}^m(A) \times \mathrm{HH}^n(A) \to \mathrm{HH}^{m+n}(A).$$

Remark 1.3.5. More generally, if B is an A-bimodule that is also an algebra for which $a(bb') = (ab)b'$, $(bb')a = b(b'a)$, and $(ba)b' = b(ab')$ for all $a \in A$ and $b, b' \in B$, a calculation shows that a formula analogous to (1.3.2) induces a product on $\mathrm{HH}^*(A, B)$. This condition is satisfied, for example, if A is a subalgebra of B and the A-bimodule structure of B is given by left and right multiplication.

We have seen that in degree 0, the Hochschild cohomology k-module $\mathrm{HH}^0(A)$ is isomorphic to $Z(A)$, the center of the algebra A. As a consequence of the definition, the cup product of two elements in degree 0 is precisely the product of the corresponding elements in $Z(A)$, and the cup product of an element in arbitrary degree n with a degree 0 element corresponds to multiplying the values of a representative function by an element in $Z(A)$. This agrees with the $\mathrm{HH}^0(A)$-module structure on Hochschild cohomology $\mathrm{HH}^*(A)$ given by equation (1.2.4). Sometimes this is enough information to determine the full structure of $\mathrm{HH}^*(A)$ as a ring under cup product, such as in the following example.

Example 1.3.6. We return to Example 1.1.18, letting k be a field and $A = k[x]$. We describe the cup product using only general properties and the graded vector space structure. We found that $\mathrm{HH}^0(k[x]) \cong k[x]$, $\mathrm{HH}^1(k[x]) \cong k[x]$, and $\mathrm{HH}^n(k[x]) = 0$ for $n > 1$, and so as a graded vector space,

$$\mathrm{HH}^*(k[x]) \cong k[x] \oplus k[x], \tag{1.3.7}$$

with the first copy of $k[x]$ in degree 0 and the second in degree 1. We will describe cup products on $\mathrm{HH}^*(k[x])$ in view of this expression. In degree 0, the cup product is simply multiplication on $k[x]$. Likewise, the product of an element in degree 0 with an element in degree 1 corresponds to multiplication on $k[x]$, with the result having degree 1 (see Exercise 1.2.9(a)). Since $\mathrm{HH}^2(k[x]) = 0$, the product of two elements in degree 1 is 0. Thus we know all cup products on $\mathrm{HH}^*(k[x])$. To express this algebraic structure more compactly, denote by y the copy of the multiplicative identity of $k[x]$ that is in the degree 1 component in expression (1.3.7). We may then rewrite (1.3.7) as

$$\mathrm{HH}^*(k[x]) \cong k[x] \oplus k[x]y$$

with the first summand $k[x]$ the degree 0 component and the second summand $k[x]y$ the degree 1 component (treating y as a place holder). By our above description of products, we now see that $y^2 = 0$ and

$$\mathrm{HH}^*(k[x]) \cong k[x,y]/(y^2)$$

as a k-algebra, where $|x| = 0$ and $|y| = 1$. (The notation $|x|, |y|$ here refers to their homological degrees.)

We have noted that the cup product is associative as a direct consequence of formula (1.3.2). Associativity can also be deduced readily from each of the equivalent definitions of cup product that will be given in Chapter 2.

We will next turn to our claim that the cup product is graded commutative. This may be shown in many different ways. It may be proven by induction, as in [**188**]. Another proof uses two of the equivalent definitions of the product, namely the Yoneda product of Section 2.2 and the tensor product of Section 2.3, and the observation that the latter is an algebra homomorphism over the former (see [**211**]). Yet another proof uses tensor products of generalized extensions and an argument similar to the proof of Theorem 2.5.5 below (see [**201**, Theorem 1.1] and some discussion in Section 2.4). Our proof in the next section takes the more concrete historical approach of Gerstenhaber [**82**]. This proof is connected to the first appearance of the graded Lie bracket on Hochschild cohomology, also defined in the next section.

Exercise 1.3.8. Verify formula (1.3.4). For comparison, derive a similar formula for $d^*_{m+n+1}(f \smile g)$.

Exercise 1.3.9. Let B be an algebra and A a subalgebra of B. Consider B to be an A-bimodule under left and right multiplication. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$, $g \in \mathrm{Hom}_k(A^{\otimes n}, B)$. Define $f \smile g \in \mathrm{Hom}_k(A^{\otimes(m+n)}, B)$ by a formula analogous to (1.3.2). Show that this induces a well-defined product on $\mathrm{HH}^*(A, B)$. (More generally, see Remark 1.3.5.)

Exercise 1.3.10. Let M be an A-bimodule. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, M)$. Define $f \cdot g \in \mathrm{Hom}_k(A^{\otimes(m+n)}, M)$ by

$$(f \cdot g)(a_1 \otimes \cdots \otimes a_{m+n}) = (-1)^{mn} f(a_1 \otimes \cdots \otimes a_m) g(a_{m+1} \otimes \cdots \otimes a_{m+n})$$

for all $a_1, \ldots, a_{m+n} \in A$. Show that this induces a well-defined action of $\mathrm{HH}^*(A)$ on $\mathrm{HH}^*(A, M)$ (cf. Section 2.5 below).

1.4. Gerstenhaber bracket

Together with addition and cup product, Hochschild cohomology $\mathrm{HH}^*(A)$ has another binary operation. We define this operation at the chain level on the bar complex (1.1.4) in this section, allowing k to be an arbitrary commutative ring. In Chapter 6, under the assumption that k is a field, we will examine equivalent definitions by way of other projective resolutions and exact sequences.

Definition 1.4.1. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. Their *Gerstenhaber bracket* $[f, g]$ is defined at the chain level to be the element of $\mathrm{Hom}_k(A^{\otimes(m+n-1)}, A)$ given by

$$[f, g] = f \circ g - (-1)^{(m-1)(n-1)} g \circ f,$$

where the *circle product* $f \circ g$ generalizes composition of functions and is defined by

$$(f \circ g)(a_1 \otimes \cdots \otimes a_{m+n-1})$$
$$= \sum_{i=1}^{m} (-1)^u f(a_1 \otimes \cdots \otimes a_{i-1} \otimes g(a_i \otimes \cdots \otimes a_{i+n-1}) \otimes a_{i+n} \otimes \cdots \otimes a_{m+n-1}),$$

in which $u = (n-1)(i-1)$, and similarly for $g \circ f$. If $m = 0$, then $f \circ g = 0$ (as indicated by the empty sum), while if $n = 0$, then the formula should be interpreted as taking the value $g(1)$ in place of $g(a_i \otimes \cdots \otimes a_{i+n-1})$ (as indicated by the empty tensor product).

Related to the Gerstenhaber bracket is the divided square operation.

Definition 1.4.2. Let m be a nonnegative integer and $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$. Assume that the characteristic of k is 2 or m is even. The *divided square* $\mathrm{Sq}(f)$ of f is defined by

$$\mathrm{Sq}(f) = f \circ f,$$

where the circle product is as in Definition 1.4.1.

The following lemma may be proven by direct computation on the bar complex (1.1.4) as in [**82**]. For ease of notation, as is common, we leave out subscripts on the differentials.

Lemma 1.4.3. *Let* $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$, $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$, *and* $h \in \mathrm{Hom}_k(A^{\otimes p}, A)$. *Then*

(i) $[f, g] = -(-1)^{(m-1)(n-1)}[g, f]$,

(ii) $(-1)^{(m-1)(p-1)}[f, [g, h]] + (-1)^{(n-1)(m-1)}[g, [h, f]]$
$+ (-1)^{(p-1)(n-1)}[h, [f, g]] = 0$,

(iii) $d^*([f, g]) = (-1)^{n-1}[d^*(f), g] + [f, d^*(g)]$.

Property (i) is graded anticommutativity of the bracket, where we shift the homological degrees of f, g by -1. Property (ii) is the *graded Jacobi identity*. From these properties and Definition 1.4.1, we see that the k-module of Hochschild cochains $C^*(A, A) = \bigoplus_{n \geq 0} \mathrm{Hom}_k(A^{\otimes n}, A)$ is a *graded Lie algebra*. Property (iii) and a sign modification further make it a *differential graded Lie algebra*, that is, a graded Lie algebra with a graded derivation δ of degree 1 and square 0: letting $\delta(f) = (-1)^{m-1}d^*(f)$ for all $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$, by Lemma 1.4.3(iii) we have

$$\delta([f,g]) = [\delta(f), g] + (-1)^{m-1}[f, \delta(g)].$$

It follows from Lemma 1.4.3 that Hochschild cohomology $\mathrm{HH}^*(A)$ is a graded Lie algebra. Moreover, $\mathrm{HH}^1(A)$ is a Lie algebra over which $\mathrm{HH}^*(A)$ is a module.

Remark 1.4.4. We emphasize that the degree of a cochain here is shifted by 1 from its homological degree, so that the cochain f in $\mathrm{Hom}_k(A^{\otimes m}, A)$ has degree $m-1$ when considering the Lie structure. Some authors choose notation to clarify this distinction, introducing a shift operator that shifts degree when needed. We will instead point out whenever we are using this shifted degree for functions f and corresponding elements of Hochschild cohomology, and will always denote this shifted degree by $|f|-1$, reserving the notation $|f|$ exclusively for the homological degree m of the corresponding element of $\mathrm{HH}^m(A)$.

Gerstenhaber [**82**] more generally developed the notion of a pre-Lie algebra for handling the circle product and bracket operations and proving results about the Lie structure. The details are very informative and may be found in his paper.

Recall that $\pi : A \otimes A \to A$ denotes the multiplication map. The following lemma may be proven by tedious direct computation. See also [**82**], taking care to convert the cup product there to our cup product as in Definition 1.3.1 (see Remark 1.3.3).

Lemma 1.4.5. *Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$. Then*

(i) $(-1)^{mn} f \smile g - g \smile f$
$= (d^*(g)) \circ f + (-1)^m d^*(g \circ f) + (-1)^{m-1} g \circ (d^*(f))$,

(ii) $[f, \pi] = -d^*(f)$.

The following theorem is a consequence of Lemma 1.4.5(i).

Theorem 1.4.6. *Let A be an associative algebra over the commutative ring k. The cup product on $\mathrm{HH}^*(A)$ is graded commutative, that is, for all $\alpha \in \mathrm{HH}^m(A)$ and $\beta \in \mathrm{HH}^n(A)$,*

$$\alpha \smile \beta = (-1)^{mn} \beta \smile \alpha.$$

Proof. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$ be cocycles, that is, $d^*(f) = 0$ and $d^*(g) = 0$. By Lemma 1.4.5(i),

$$(-1)^{mn} f \smile g = g \smile f + (-1)^m d^*(g \circ f).$$

The images α and β in $\mathrm{HH}^*(A)$ of f and g thus satisfy

$$\alpha \smile \beta = (-1)^{mn} \beta \smile \alpha.$$

□

As we have seen, a consequence of Lemma 1.4.3(iii) is that the bracket $[\ ,\ ]$, as defined at the cochain level, induces a well-defined operation on $\mathrm{HH}^*(A)$. Now we note that a consequence of Lemma 1.4.5(i) is that the divided square operation Sq, as defined at the cochain level, induces a well-defined operation on $\mathrm{HH}^*(A)$ in case $\mathrm{char}(k) = 2$ and on its subalgebra generated by homogeneous even degree elements in case $\mathrm{char}(k) \neq 2$. Next we state a further property satisfied by the bracket on Hochschild cohomology. For a proof, see [**82**, Corollary 1 of Theorem 5], where at the cochain level, the difference of the left and right sides of the stated equation in the lemma is shown to be a specific coboundary.

Lemma 1.4.7. *Let* $\alpha \in \mathrm{HH}^m(A)$, $\beta \in \mathrm{HH}^n(A)$, *and* $\gamma \in \mathrm{HH}^p(A)$. *Then*

$$[\gamma, \alpha \smile \beta] = [\gamma, \alpha] \smile \beta + (-1)^{m(p-1)} \alpha \smile [\gamma, \beta].$$

As a consequence of the lemma, for each γ, the operation $[\gamma, -]$ is a graded derivation with respect to cup product. Moreover, Hochschild cohomology is a Gerstenhaber algebra (sometimes also called a G-algebra), as we define next.

Definition 1.4.8. A *Gerstenhaber algebra* $(H, \smile, [\ ,\ ])$ is a free $\mathbb{Z}$-graded k-module H for which $(H, \smile)$ is a graded commutative associative algebra, $(H, [\ ,\ ])$ is a graded Lie algebra with bracket $[\ ,\ ]$ of degree -1 and corresponding degree shift by -1 on elements, and

$$[\gamma, \alpha \smile \beta] = [\gamma, \alpha] \smile \beta + (-1)^{|\alpha|(|\gamma|-1)} \alpha \smile [\gamma, \beta]$$

for all homogeneous α, β, γ in H.

Theorem 1.4.9. *Hochschild cohomology* $\mathrm{HH}^*(A)$ *is a Gerstenhaber algebra.*

Proof. The main properties to prove are dealt with in Theorem 1.4.6 and Lemmas 1.4.3 and 1.4.7. □

In Chapter 6 we will examine the Gerstenhaber bracket in more detail, including ways to define it for functions on an arbitrary resolution independent of the bar resolution, and on generalized extensions.

Exercise 1.4.10. Show that if $\mathrm{char}(k) \neq 2$ and f is a Hochschild cocycle of homogeneous even degree, then $\mathrm{Sq}(f) = \frac{1}{2}[f, f]$.

Exercise 1.4.11. Verify Lemma 1.4.3(i) and (ii) by direct computation.

Exercise 1.4.12. Verify that the bracket [,], as defined at the cochain level, induces a well-defined operation on $\mathrm{HH}^*(A)$ by Lemma 1.4.3(iii). Similarly verify that the divided square operation Sq is well-defined on cohomology.

Exercise 1.4.13. Verify all other properties of the Gerstenhaber bracket stated in this section either by direct computation or by reading the verifications in [**82**], taking care to convert the cup product there to our cup product as in Definition 1.3.1 (see Remark 1.3.3).

1.5. Cap product and shuffle product

The cap product is an action of $\mathrm{HH}^*(A)$ on $\mathrm{HH}_*(A)$. The shuffle product is an associative and graded commutative product on $\mathrm{HH}_*(A)$ for commutative algebras A. We define these products in this section.

The cap product is a specific pairing between Hochschild homology and cohomology modules, that is, a function

$$\mathrm{HH}_n(A) \otimes \mathrm{HH}^m(A) \xrightarrow{\frown} \mathrm{HH}_{n-m}(A),$$

defined at the chain level as follows. (We set $\mathrm{HH}_i(A)$ equal to 0 for all $i < 0$.) Identify $A \otimes_{A^e} A^{\otimes(n+2)}$ with $A \otimes A^{\otimes n}$ via the isomorphism (1.1.8) with $M = A$, symbolically representing an element of $\mathrm{HH}_n(A)$ at the chain level by a sum of elements of the form $a_0 \otimes \cdots \otimes a_n$ in $A \otimes A^{\otimes n}$.

Definition 1.5.1. Let $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ be a function representing an element of $\mathrm{HH}^m(A)$. The *cap product* of f with $a_0 \otimes \cdots \otimes a_n$ is defined by

$$(a_0 \otimes a_1 \otimes \cdots \otimes a_n) \frown f = (-1)^{m(n-m)} a_0 f(a_1 \otimes \cdots \otimes a_m) \otimes a_{m+1} \otimes \cdots \otimes a_n.$$

This formula induces a well-defined function from $\mathrm{HH}_n(A) \otimes \mathrm{HH}^m(A)$ to $\mathrm{HH}_{n-m}(A)$ as claimed: assuming that $\sum_i a_0^i \otimes \cdots \otimes a_n^i$ is a cycle for some $a_j^i \in A$, and that $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ is a cocycle, one sees that their cap product is a cycle by rewriting the image of the differential on $\sum_i a_0^i \otimes \cdots \otimes a_n^i$ in such a way as to take advantage of the relation $d_{m+1}^*(f) = 0$. Thus the cap product of a cocycle with a cycle is a cycle. Similarly one sees that the cap product of a coboundary with a cycle, or of a cocycle with a boundary, is a boundary. By its definition, the cap product gives $\mathrm{HH}_*(A)$ the structure of a right $\mathrm{HH}^*(A)$-module.

Example 1.5.2. Let k be a field and $A = k[x]$, as in Example 1.1.18. By our work in that example, we see that $\mathrm{HH}_*(A)$ is a free A-module. In degree 0, $\mathrm{HH}^0(A) \cong A$ acts on this free A-module in the canonical way. Let V be the one-dimensional vector space with basis x. We identify $\mathrm{HH}^1(A)$ with $A \otimes V^*$, where $V^* = \mathrm{Hom}_k(V, k)$ is the dual vector space, and $\mathrm{HH}_1(A)$ with $A \otimes V$ (see Exercise 1.1.25). The resolution (1.1.19) may be embedded in the bar resolution (1.1.4) by identifying the two resolutions in degree 0 and

Chapter 4

Smooth Algebras and Van den Bergh Duality

In this chapter we look at noncommutative analogs of some commutative concepts, beginning with dimension, smoothness, differential forms, and square-zero extensions. These are defined using Hochschild cohomology. We present Van den Bergh's algebraic version of Poincaré duality, that is, a duality between Hochschild homology and cohomology for some types of smooth algebras. We define Calabi-Yau algebras and take a closer look at skew group algebras in this light. For Calabi-Yau algebras and for symmetric algebras we further define Batalin-Vilkovisky structures on Hochschild cohomology using different types of duality.

Throughout, k will be a field and A will be a k-algebra.

4.1. Dimension and smoothness

We first use the notion of projective dimension (see Section A.2) in a definition of dimension of the algebra A.

Definition 4.1.1. The *Hochschild dimension* of A is its projective dimension as an A^e-module:

$$\dim(A) = \operatorname{pdim}_{A^e}(A).$$

Some authors simply refer to this as the *dimension* of A. There are however many other types of dimension for algebras, depending on context, such as global dimension, Krull dimension, Gelfand-Kirillov dimension, or vector space dimension. See, for example, [**158**] for a discussion of dimension for noncommutative rings. Note that the global dimension of A is always less than or equal to its Hochschild dimension, since any A^e-projective resolution of A may be tensored over A with any module M to yield an A-projective resolution of M, as we saw in Section 2.5. For some algebras, more is known, for example if A is $\mathbb{N}$-graded and connected (i.e., $A_0 = k$), the Hochschild dimension is equal to the (left or right) global dimension, and both are equal to the projective dimension of the A-module k given by projection onto A_0 [**26**].

Example 4.1.2. By our work in Example 3.1.3, $\dim(k[x_1, \ldots, x_m]) = m$. Specifically, we found there a projective resolution of A as an A^e-module of length m. There cannot exist a shorter resolution since $\mathrm{HH}^m(A) \neq 0$.

Our first result describes a relationship among the Hochschild dimensions of two algebras and that of their tensor product algebra, or of their twisted tensor product algebra when the twisting is given by a bicharacter as defined in Section 3.2.

Theorem 4.1.3. *Let A and B be two algebras. Then*

(i) $\dim(A \otimes B) \leq \dim(A) + \dim(B)$*, and more generally*

(ii) *if A, B are graded by abelian groups Γ, Γ' and $t : \Gamma \times \Gamma' \to k^{\times}$ is a bicharacter, then* $\dim(A \otimes^t B) \leq \dim(A) + \dim(B)$*, where $A \otimes^t B$ is the twisted tensor product defined in Section* 3.2.

Proof. (i) Let $P_\bullet$ (respectively, $Q_\bullet$) be a projective resolution of A as an A^e-module (respectively, of B as a B^e-module). In Section 3.1 we showed that the total complex of the tensor product complex $P_\bullet \otimes Q_\bullet$ is a projective resolution of $A \otimes B$ as an $(A \otimes B)^e$-module. The length of this total complex is the sum of the lengths of $P_\bullet$ and $Q_\bullet$. Therefore there is a projective resolution of $A \otimes B$ as an $(A \otimes B)^e$-module of length $\dim(A) + \dim(B)$. It follows that $\dim(A \otimes B)$ is at most this number.

(ii) The proof of (i) applies more generally to the twisted tensor product algebra $A \otimes^t B$, using Theorem 3.2.3. □

By contrast, there is no such theorem for the more general twisted tensor product algebra $A \otimes_\tau B$, arising from a twisting map $\tau : B \otimes A \to A \otimes B$, defined at the end of Section 3.2. There are conditions under which resolutions $P_\bullet$ and $Q_\bullet$ may be combined to form a complex of bimodules for this more general twisted tensor product algebra, and there are conditions under which the bimodules in such a complex are projective, but there are no general guarantees [**196**]. Indeed, there exist algebras of Hochschild dimension 0 for which a twisted tensor product algebra has infinite Hochschild dimension. See, e.g., [**148**, Proposition 3.3.6].

The following definition is due to Van den Bergh [**216**].

Definition 4.1.4. The algebra A is *smooth* if its Hochschild dimension is finite and it has a finite projective resolution as an A^e-module consisting of finitely generated projective modules.

Some authors use the term *homologically smooth* to distinguish this from other notions of smoothness. Note that if A and B are smooth, then so is $A \otimes B$ (and more generally $A \otimes^t B$): this follows from Theorem 4.1.3 and its proof, since the (twisted) tensor product of finitely generated modules is also finitely generated. If A is a finitely generated commutative algebra over k, this notion of smoothness is equivalent to more standard definitions as mentioned already in Section 3.3 (see [**216**]).

Example 4.1.5. A polynomial ring $A = k[x_1, \ldots, x_m]$ is smooth by our work in Example 3.1.3 (the resolution used there consists of finitely generated free modules). A skew polynomial ring $k_{\mathbf{q}}[x_1, \ldots, x_m]$ is smooth by our work in Example 3.2.5. If G is a finite group acting on $A = k[x_1, \ldots, x_m]$ by degree-preserving automorphisms and char(k) does not divide the order of G, then the skew group algebra $k[x_1, \ldots, x_m] \rtimes G$ of Section 3.5 is smooth: the group G acts on the Koszul resolution of $k[x_1, \ldots, x_m]$, as a subcomplex of the bar resolution via the map ϕ of (3.4.12). (The action of G on the bar resolution of A is diagonal, that is, $g \cdot (a_0 \otimes \cdots \otimes a_{n+1}) = {}^g a_0 \otimes \cdots \otimes {}^g a_{n+1}$ for all $g \in G$, $a_0, \ldots, a_{n+1} \in A$.) Our work in Section 3.5 shows that the Koszul resolution may be induced to a projective resolution of $k[x_1, \ldots, x_m] \rtimes G$ as a $(k[x_1, \ldots, x_m] \rtimes G)^e$-module, and this resolution consists of finitely generated modules. More specifically, under the action of G, the Koszul resolution may be viewed as a resolution of $A = k[x_1, \ldots, x_m]$ as a $\mathcal{D}$-module, where $\mathcal{D}$ is defined by equation (3.5.3). Induce from $\mathcal{D}$ to $(A \rtimes G)^e$. The map (3.5.4) is an isomorphism between $A \rtimes G$ and the $\mathcal{D}$-module A induced to an $(A \rtimes G)^e$-module.

Example 4.1.6. In contrast, the quantum complete intersections of Example 3.2.6, $A = k\langle x_1, x_2\rangle(x_1x_2 - qx_2x_1,\ x_1^2,\ x_2^2)$, are not smooth. We explained there that A has infinite global dimension, forcing its Hochschild dimension to be infinite as well.

We will look closely at some algebras with Hochschild dimension 0 or 1.

Definition 4.1.7. An algebra A is *separable* if $\dim(A) = 0$. It is *quasi-free* if $\dim(A) \leq 1$.

This notion of quasi-free algebras is due to Cuntz and Quillen [**55**]. Quasi-free algebras have also been called *Cuntz-Quillen smooth* or *formally smooth*.

By definition, A is separable if and only if it is projective as an A^e-module. Another equivalent condition to separability is that any derivation from A to an A-bimodule is inner. Indeed, if A is separable, then $\mathrm{HH}^1(A, M) = 0$ for all A-bimodules M. By our work in Section 1.2, the vanishing of $\mathrm{HH}^1(A, M)$ is equivalent to the statement that any derivation from A to M is inner. Conversely, suppose that A is not projective as an A^e-module. Let K_1 be the first syzygy module of A in a given projective resolution $P_\bullet$ of A as an A^e-module. We claim that $\mathrm{HH}^1(A, K_1) \neq 0$. To see this, note that by the definitions, $\mathrm{HH}^1(A, K_1) \cong \mathrm{Hom}_{A^e}(K_1, K_1)/\mathrm{Im}(i_1^*)$, where i_1 is the embedding of K_1 into P_0 as in diagram (A.2.4). This quotient is nonzero, as if not, then the identity map from K_1 to K_1 is in the image of i_1^*, which implies that the short exact sequence $0 \to K_1 \to P_0 \to A \to 0$ splits and A is projective as an A^e-module, a contradiction.

We will look at some equivalent conditions to quasi-freeness in the next section. For now, we consider some examples and implications.

Example 4.1.8. $A = k[x]$ has Hochschild dimension 1 by our work in Example 1.1.18, and so is quasi-free. However, $k[x, y]$ has Hochschild dimension 2 by our work in Example 3.1.3, and so is not quasi-free. Thus the tensor product of two quasi-free algebras is not always quasi-free, and similarly for twisted tensor products. However, the free product of two quasi-free algebras is always quasi-free [**55**, Proposition 5.3]. It follows that the tensor algebra $T(V)$ of a finite-dimensional vector space V is quasi-free.

A quasi-free algebra is hereditary: if A is quasi-free, then there is a projective resolution $0 \to P_1 \to P_0 \to A \to 0$ of A as an A^e-module. For any A-module M, we may apply $- \otimes_A M$ to this sequence to obtain a projective resolution of M as an A-module, as explained in Section 2.5. Therefore the projective dimension $\mathrm{pdim}_A(M)$ of M is at most 1, and so $\mathrm{gldim}_l\, A \leq 1$. By definition then, A is hereditary.

We define the *Jacobson radical* (or simply *radical*) of A, denoted $\mathrm{rad}(A)$, to be the intersection of all the maximal left ideals of A. This can be shown to be the same as the intersection of all the maximal right ideals, and thus is a two-sided ideal. If A is finite dimensional as a vector space over k, we say that A is *semisimple* if $\mathrm{rad}(A) = 0$. Equivalently, every A-module is a direct sum of simple A-modules. The Wedderburn-Artin Theorem states that every semisimple algebra over k is a direct sum of matrix algebras over division rings. See, e.g., [**121**, Theorem IX.3.3].

Any semisimple algebra is separable, since if A is semisimple, then so is A^e, and so all A^e-modules are projective. For example, if G is a finite group whose order is not divisible by the characteristic of k, then the group algebra kG is semisimple by Maschke's Theorem, and so kG is separable.

Exercise 4.1.9. Find the Hochschild dimensions of the following algebras:

(a) A skew polynomial ring $k_{\mathbf{q}}[x_1, \ldots, x_m]$.

(b) A skew group algebra $k_{\mathbf{q}}[x_1, \ldots, x_m] \rtimes G$ in case the characteristic of k does not divide the order of G.

Exercise 4.1.10. Find the Hochschild dimension of $k[x]/(x^n)$. (See Example 2.5.10.)

Exercise 4.1.11. Find the Hochschild dimension of each of the algebras $k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1,\ x_1^{n_1},\ x_2^{n_2})$ discussed in Example 3.2.2. (See Exercise 3.2.12 and Examples 3.2.6 and 4.1.6.)

4.2. Noncommutative differential forms

We study the quasi-free algebras of Definition 4.1.7 in more detail in this section. To this end, we introduce a noncommutative version of Kähler differentials. This material and notation are from Cuntz and Quillen [**55**] and Ginzburg [**88**].

Let A be a k-algebra and for each $n \geq 0$, let

(4.2.1) $$\Omega^n_{nc}A = A \otimes (\overline{A})^{\otimes n},$$

where $\overline{A} = A/k$ is the vector space quotient (the field k is identified with $k \cdot 1$ as a vector subspace of A). We write elements of $\overline{A}$ via notation from A, viewing $\overline{A}$ noncanonically as a vector space direct summand of A, when this will not cause confusion. We will identify the vector space $\Omega^1_{nc}A$ with the kernel of the multiplication map $\pi : A \otimes A \to A$, at the same time giving it the structure of an A-bimodule, as follows. This will indicate a connection with the Heller operator of similar notation Ω (see Section A.2) and also a comparison to Kähler differentials [**223**] of similar notation Ω^1_{com} in the special case that A is commutative.

Consider $\Omega^1_{nc}A$ to be an A-bimodule under the following actions:

(4.2.2) $$c(a \otimes b) = ca \otimes b \quad \text{and} \quad (a \otimes b)c = a \otimes bc - ab \otimes c$$

for all $a, b, c \in A$. Let $j : A \otimes \overline{A} \to A \otimes A$ be given by $j(a \otimes b) = ab \otimes 1 - a \otimes b$ for $a, b \in A$. Then the sequence

(4.2.3) $$0 \to \Omega^1_{nc}A \xrightarrow{j} A \otimes A \xrightarrow{\pi} A \to 0$$

is an exact sequence of A-bimodules. To see this, note that j maps $\Omega^1_{nc}A$ isomorphically onto $\mathrm{Ker}(\pi)$. Due to exactness of the bar resolution $B(A)$ of the A^e-module A given in (1.1.4), $\mathrm{Ker}(\pi) = \mathrm{Im}(d_1)$, but this is precisely the image of j. Further, j is injective. Assume that $j(\sum_i a_i \otimes b_i) = 0$ for some elements a_i, b_i. Then, since $b_i \in \overline{A}$ and

$$j(\sum_i a_i \otimes b_i) = (\sum_i a_i b_i) \otimes 1 - \sum_i a_i \otimes b_i,$$

we have $\sum_i a_i b_i = 0$ (the tensor product is over the field k). It follows that $\sum_i a_i \otimes b_i = j(\sum_i a_i \otimes b_i) = 0$. Thus the sequence (4.2.3) is exact as claimed, and $\Omega^1_{nc}A$ is a first syzygy module of A as an A-bimodule.

As one important property of the A-bimodule $\Omega^1_{nc}A$, we claim that for all A-bimodules M,

$$\text{(4.2.4)} \qquad \operatorname{Der}(A, M) \cong \operatorname{Hom}_{A^e}(\Omega^1_{nc}A, M),$$

where $\operatorname{Der}(A, M)$ is the space of k-derivations from A to M, defined in Section 1.2. This isomorphism follows immediately from our work in Section 1.2 interpreting Hochschild cohomology in degree 1, as this is precisely the space of Hochschild 1-cocycles, that is, the 1-cochains on the bar resolution that factor through the first syzygy module. The isomorphism (4.2.4) can be interpreted as saying that $\Omega^1_{nc}A$ represents the functor $\operatorname{Der}(A, -)$ on the category of A-bimodules.

In comparison, for commutative algebras A, we consider A-modules rather than A-bimodules. The Kähler differentials, defined next, represent the functor $\operatorname{Der}(A, -)$ on the category of A-modules [**223**].

Definition 4.2.5. Let A be a commutative algebra. The A-module of *Kähler differentials* $\Omega^1_{com}A$ is the A-module with one generator da for each $a \in A$ and $dc = 0$ for all $c \in k$. Relations are

$$d(a + b) = da + db \quad \text{and} \quad d(ab) = adb + bda$$

for all $a, b \in A$.

We claim that $\Omega^1_{com}A \cong \operatorname{Ker}\pi/(\operatorname{Ker}\pi)^2 \cong (\Omega^1_{nc}A)/(\operatorname{Ker}\pi)^2$ and that $\Omega^1_{com}A \cong \operatorname{HH}_1(A)$. See Exercises 4.2.14 and 4.2.15.

Returning to the general case of a not necessarily commutative algebra A, recall the definition (4.2.1) of $\Omega^n_{nc}A$.

Definition 4.2.6. Let A be an algebra. The space of *noncommutative differential forms* on A is

$$\Omega_{nc}A = \bigoplus_{n \geq 0} \Omega^n_{nc}A.$$

We will see that $\Omega_{nc}A$ is a differential graded algebra with

$$\begin{aligned} d(a_0 \otimes \cdots \otimes a_n) &= 1 \otimes a_0 \otimes \cdots \otimes a_n, \\ (a_0 \otimes \cdots \otimes a_n)(a_{n+1} \otimes \cdots \otimes a_r) &= \sum_{i=0}^{n} (-1)^{n-i} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_r \end{aligned}$$

for all $a_0, \ldots, a_r \in A$, a theorem of Cuntz and Quillen [**55**, Proposition 1.1]. Moreover, $\Omega_{nc}A$ is universal with respect to differential graded algebras whose degree 0 term is the target of an algebra homomorphism from A.

Theorem 4.2.7. *The space $\Omega_{nc}A$ of noncommutative differential forms on an algebra A is a differential graded algebra with differential and multiplication given above, unique such that*

$$a_0(da_1)\cdots(da_n) = a_0 \otimes \cdots \otimes a_n$$

for all $a_0, \ldots, a_n \in A$. Moreover, for any differential graded algebra Γ and algebra homomorphism $u : A \to \Gamma^0$, there is a unique differential graded algebra homomorphism $u_* : \Omega_{nc}A \to \Gamma$ that extends u.

Proof. It may be checked directly that $\Omega_{nc}A$ is indeed a differential graded algebra. (See Exercise 4.2.16 or [**55**, Proposition 1.1].)

For the second statement, let Γ be a differential graded algebra and $u : A \to \Gamma^0$ an algebra homomorphism. Define $u_* : \Omega_{nc}A \to \Gamma$ by

$$u_*(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = u(a_0)du(a_1)\cdots du(a_n).$$

It may be checked that u_* is a homomorphism of differential graded algebras, and it is uniquely determined. □

Definition 4.2.8. A *square-zero extension* of A is an algebra R such that $A \cong R/I$ for some ideal I of R with $I^2 = 0$. Two square-zero extensions R, R' of A are *equivalent* if they are isomorphic as algebras via an isomorphism that induces the identity map on A.

The ideal I in the definition is necessarily an A-bimodule since $I^2 = 0$: given an element $a = r + I \in A$ for some $r \in R$, and given $x \in I$, define $ax = rx$ and $xa = xr$. Conversely, every A-bimodule M determines a square-zero extension: let $R = A \oplus M$ and define $(a_1, m_1)\cdot(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2)$ for all $a_1, a_2 \in A$ and $m_1, m_2 \in M$. This is called the *trivial extension.* More generally, let $f : A \otimes A \to M$ be a Hochschild 2-cocycle, that is,

$$af(b \otimes c) + f(a \otimes bc) = f(ab \otimes c) + f(a \otimes b)c$$

for all $a, b, c \in A$ as in (1.2.2). Then $R = A \oplus M$ is a ring with multiplication

$$(a_1, m_1)(a_2, m_2) = (a_1a_2, a_1m_2 + m_1a_2 + f(a_1 \otimes a_2)) \tag{4.2.9}$$

for all $a_1, a_2 \in A$ and $m_1, m_2 \in M$. (Associativity is equivalent to the above Hochschild 2-cocycle condition.) This connection between Hochschild 2-cocycles and extensions was refined by Hochschild [**114**] to the following theorem.

Theorem 4.2.10. *Let M be an A-bimodule. Then $\mathrm{HH}^2(A, M)$ is in one-to-one correspondence with equivalence classes of square-zero extensions of A by M.*

Proof. We have already seen that a representative element of $\mathrm{HH}^2(A, M)$, at the chain level, determines a square-zero extension of A by M. A calculation shows that cohomologous cocycles correspond to equivalent square-zero extensions. Conversely, given a square-zero extension R of A by $I = M$, choose an A-bimodule isomorphism $A \oplus I \to R$ that sends each element of I to itself in R and when composed with the quotient map $R \to A$ is the identity on A. This is possible due to the A-bimodule structure of I described earlier. Let $a, b \in A$ and $x, y \in I$, and identify (a, x) and (b, y) with their images in R. Then

$$\begin{aligned}(a,x)(b,y) &= (a,0)(b,0) + (a,0)(0,y) + (0,x)(b,0) + (0,x)(0,y)\\ &= (a,0)(b,0) + (1,0)a(0,y) + (0,x)b(1,0)\\ &= (a,0)(b,0) + (0,ay) + (0,xb),\end{aligned}$$

since $I^2 = 0$. Necessarily $(a,0)(b,0) = (ab, f(a \otimes b))$ for some function $f : A \otimes A \to I$ that is a Hochschild 2-cocycle. A calculation shows that a different choice of map $A \oplus I \to R$ yields a cohomologous cocycle. □

Square-zero extensions, noncommutative differential forms, and quasi-free algebras are all related as stated in the following theorem. By a *lifting* of a square-zero extension R of A, we mean an A-bimodule structure on R extending that on I and an A-bimodule homomorphism $A \to R$ that is a section of the quotient map $R \to A$.

Theorem 4.2.11. *The following are equivalent for an algebra A:*

(i) *A is quasi-free.*

(ii) *$\Omega^1_{nc}A$ is a projective A^e-module.*

(iii) $\mathrm{HH}^2(A, M) = 0$ *for all A-bimodules M.*

(iv) *For any square-zero extension R of A, there is a lifting $A \to R$.*

Proof. We have identified $\Omega^1_{nc}A$ with the first syzygy of A as an A^e-module. If A is quasi-free, there exists a projective resolution $P_\bullet$ of A as an A^e-module of length 1, that is, $P_i = 0$ for all $i \geq 2$. Thus the first syzygy module is P_1, a projective module. By Schanuel's Lemma (Lemma A.2.5), any other first syzygy module is projective as well, so in particular $\Omega^1_{nc}A$ is projective. Thus (i) implies (ii). Conversely, if $\Omega^1_{nc}A$ is a projective A^e-module, then the Hochschild dimension of A is at most 1, so A is quasi-free, that is, (ii) implies (i).

By dimension shifting (Theorem A.3.3),

$$\mathrm{HH}^2(A, M) \cong \mathrm{Ext}^1_{A^e}(\Omega^1_{nc}A, M).$$

So (ii) implies (iii). Conversely, assume that $\mathrm{Ext}^1_{A^e}(\Omega^1_{nc}A, M) = 0$ for all A-bimodules M, so that every A^e-extension of $\Omega^1_{nc}A$ splits. Map a projective

A^e-module onto $\Omega^1_{nc}A$, and consider the extension of $\Omega^1_{nc}A$ by the kernel of this map. It splits, forcing $\Omega^1_{nc}A$ itself to be projective. So (iii) implies (ii).

Finally, if $\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M, then every square-zero extension splits by Theorem 4.2.10. So if R is a square-zero extension of A, there is a lifting $A \to R$. That is, (iii) implies (iv). Conversely, let R be a square-zero extension of A by an ideal I. Assume there is a lifting $A \to R$. The lifting is a splitting of the sequence of A-bimodules, $0 \to I \to R \to A \to 0$. So if any square-zero extension lifts, then in particular the square-zero extension $A \oplus M$ lifts for any A-bimodule M, and so the sequence $0 \to M \to A \oplus M \to A \to 0$ splits as a sequence of A-bimodules. By Theorem 4.2.10, $\mathrm{HH}^2(A, M) = 0$ for all A-bimodules M. Thus (iv) implies (iii). □

Schelter [**190**] proposed a condition similar to (iv) in the theorem as a definition of smoothness for noncommutative algebras.

In case A is commutative, consider the related condition that for every *commutative* square-zero extension R of A, there is a lifting $A \to R$. This is equivalent to smoothness for commutative algebras [**223**, Section 9.3.1] but is a weaker condition than being quasi-free. In fact, a commutative algebra is smooth in the sense of Definition 4.1.4 if and only if it is smooth in this classical sense [**216**].

Exercise 4.2.12. Verify that equations (4.2.2) do indeed give $\Omega^1_{nc}A$ the structure of an A-bimodule, that is, (a) the action is well-defined, and (b) it is an A-bimodule action.

Exercise 4.2.13. Verify that the map j in the sequence (4.2.3) is a well-defined A-bimodule map.

Exercise 4.2.14. Show that if A is a commutative algebra, then the A-module $\Omega^1_{com}A$ of Definition 4.2.5 may equivalently be defined as

$$A \otimes A/(ab \otimes c - a \otimes bc + ac \otimes b \mid a, b, c \in A)$$

by considering the left A-module map from $A \otimes A$ to $\Omega^1_{com}A$ that sends $1 \otimes b$ to db for all $b \in A$. This can also be identified with Hochschild homology $\mathrm{HH}_1(A)$.

Exercise 4.2.15. Verify the isomorphism $\Omega^1_{com}A \cong \operatorname{Ker}\pi/(\operatorname{Ker}\pi)^2$ for a commutative algebra A by using Exercise 4.2.14 and the map $A \otimes A \to I$ that sends $a \otimes b$ to $ab \otimes 1 - a \otimes b$ for all $a, b \in A$.

Exercise 4.2.16. Verify the claims from the proof of Theorem 4.2.7:

(a) $\Omega_{nc}A$ is a differential graded algebra. In particular, check that $d^2 = 0$, the multiplication is associative, and the differential is a graded derivation.

(b) The map u_* is a homomorphism of differential graded algebras, and is unique.

Exercise 4.2.17. Verify that formula (4.2.9) defines an associative multiplication on $A \oplus M$.

4.3. Van den Bergh duality and Calabi-Yau algebras

For some smooth algebras, both commutative and noncommutative, there exists a duality between Hochschild homology and cohomology, an analog of Poincaré duality in geometry. We state this duality in Theorem 4.3.2. A special case is Corollary 4.3.8 for Calabi-Yau algebras, which are defined in this section.

In general, if P is a left A^e-module, then $\mathrm{Hom}_{A^e}(P, A^e)$ is a right A^e-module by setting $(f \cdot (a \otimes b))(p) = bf(p)a$ for all $a, b \in A$, $p \in P$, and $f \in \mathrm{Hom}_{A^e}(P, A^e)$. This gives the structure of a right A^e-module to the Hochschild cohomology space of A with coefficients in A^e, that is, to $\mathrm{HH}^i(A, A^e)$ for each i. This action appears in the statement of Theorem 4.3.2 below.

Definition 4.3.1. An A-bimodule U is *invertible* if there is an A-bimodule V such that $U \otimes_A V \cong A$ and $V \otimes_A U \cong A$ as A-bimodules.

The invertible A-bimodules correspond one-to-one with autoequivalences of the category A-mod, that is, equivalences between the category of A-modules and itself, given by tensor products with the invertible A-bimodules. The identity autoequivalence is given by the invertible A-bimodule A.

An invertible A-bimodule gives a duality between Hochschild homology $\mathrm{HH}_*(A)$ and cohomology $\mathrm{HH}^*(A)$ under some conditions, as stated in the following theorem of Van den Bergh [**216**, Theorem 1].

Theorem 4.3.2. *Let A be a smooth algebra. Assume that there is a positive integer d for which $\mathrm{HH}^i(A, A^e) = 0$ for all $i \neq d$ and that $U = \mathrm{HH}^d(A, A^e)$ is an invertible A-bimodule. There is an isomorphism of vector spaces*

$$\mathrm{HH}^n(A, M) \cong \mathrm{HH}_{d-n}(A, U \otimes_A M)$$

for all A-bimodules M and $n \in \{0, \dots, d\}$, and $\mathrm{HH}^n(A, M) = 0$ for $n > d$.

As a consequence of the statement, the integer d in the theorem is equal to the Hochschild dimension $\dim(A)$ of A: by hypothesis, $\mathrm{HH}^d(A, A^e) \neq 0$, implying $d \leq \dim(A)$. By equation (A.2.10), there exists an A^e-module M for which $\mathrm{HH}^{\dim(A)}(A, M) \neq 0$, and so by the last statement of the theorem, $d \geq \dim(A)$.

Definition 4.3.3. If the hypotheses of the theorem are satisfied, we call U the *dualizing bimodule* of A and we say that A has *Van den Bergh duality.*

Proof of Theorem 4.3.2. The proof is a special case of a proof in [**136**], and we choose the same indexing for ease of comparison. Since A is smooth, there is an A^e-projective resolution $(P_\bullet, \delta_\bullet)$ of A such that each P_i is finitely generated and $P_i = 0$ for all $i > n = \dim(A)$. (Note that $d \leq n$ since $U = \mathrm{HH}^d(A, A^e) \neq 0$ by hypothesis, and as explained above, it will follow from the proof and equation (A.2.10) that $d = n$.) Let $Q_\bullet \xrightarrow{\varepsilon} M$ be an A^e-projective resolution of M. Since U is invertible, the functor $U \otimes_A -$ is a category equivalence, and so $U \otimes_A Q_\bullet$ is an A^e-projective resolution of $U \otimes_A M$. Let

$$C_{p,q} = \mathrm{Hom}_{A^e}(P_{-p}, Q_q)$$

for all $p \leq 0$, $q \geq 0$. We claim that since P_{-p} is finitely generated and projective, for each p, q, there is an isomorphism of vector spaces,

$$\mathrm{Hom}_{A^e}(P_{-p}, A^e) \otimes_{A^e} Q_q \xrightarrow{\sim} C_{p,q} \tag{4.3.4}$$

given by $f \otimes y \mapsto (x \mapsto (-1)^{pq} f(x) y)$ for all $f \in \mathrm{Hom}_{A^e}(P_{-p}, A^e)$ and $y \in Q_q$. Indeed, in case $P_{-p} = A^e$, this is clearly an isomorphism, as it is if P_{-p} is a free module of finite rank. The claim then follows for any finitely generated projective A^e-module, and thus for each P_{-p}. Note that the tensor product here is taken over A^e instead of over A.

The rest of the proof uses a comparison of two spectral sequences for a bicomplex (Section A.7). Consider the columns in the following diagram, where $n = \dim(A)$:

$$\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & \mathrm{Hom}_{A^e}(P_n, Q_2) & \longleftarrow & \cdots & \longleftarrow & \mathrm{Hom}_{A^e}(P_0, Q_2) & \longleftarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & \mathrm{Hom}_{A^e}(P_n, Q_1) & \longleftarrow & \cdots & \longleftarrow & \mathrm{Hom}_{A^e}(P_0, Q_1) & \longleftarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longleftarrow & \mathrm{Hom}_{A^e}(P_n, Q_0) & \longleftarrow & \cdots & \longleftarrow & \mathrm{Hom}_{A^e}(P_0, Q_0) & \longleftarrow & 0 \\
 & & \downarrow & & & & \downarrow & & \\
 & & 0 & & & & 0 & &
\end{array}$$

We will apply the two spectral sequences described in Section A.7 to the double complex $C_{\bullet,\bullet}$.

Let E'' denote the first spectral sequence described in Section A.7 for a double complex, given specifically by equation (A.7.5). That is, E'' begins with vertical differentials only, and we write $E''_1 \cong \mathrm{H}''(C)$ and

$E''_2 \cong \mathrm{H}'(\mathrm{H}''(C))$. Since each P_i is projective and $Q_\bullet \xrightarrow{\varepsilon} M$ is exact, the ith column of the above diagram is exact when augmented with $\mathrm{Hom}_{A^e}(P_i, M)$. Thus we find that E''_1 consists of $\mathrm{Hom}_{A^e}(P_\bullet, M)$ in the bottom row, with 0 in all other positions. It follows that E''_2 consists of $\mathrm{HH}^*(A, M)$ in the bottom row, with zero differentials. So the spectral sequence collapses and this is the cohomology of $C_{\bullet,\bullet}$.

For comparison, let E' denote the second spectral sequence described in Section A.7 for a double complex, given on page 233. That is, E' begins with horizontal differentials only, and we write $E'_1 \cong \mathrm{H}'(C)$ and $E'_2 \cong \mathrm{H}''(\mathrm{H}'(C))$. By hypothesis, E'_1 consists of $U \otimes_{A^e} Q_\bullet$ as the $-d$th column and 0 in all other positions. It follows that E'_2 is $\mathrm{Tor}^{A^e}_{d-*}(U, M)$, with zero differentials. We claim that $U \otimes_{A^e} Q_j \cong A \otimes_{A^e} (U \otimes_A Q_j)$ as A^e-modules for all j. To see this, first note that it is true for a free A^e-module since $U \xrightarrow{\sim} A \otimes_{A^e} (U \otimes_A A^e)$ via the map $u \mapsto 1 \otimes (u \otimes (1 \otimes 1))$ which has inverse $a \otimes (u \otimes (b \otimes c)) \mapsto$ $caub$. Each Q_j is projective, so is a direct summand of a free module, and this isomorphism preserves such a direct sum. Therefore $\mathrm{Tor}^{A^e}_{d-*}(U, M) \cong$ $\mathrm{HH}_{d-\bullet}(A, U \otimes_A M)$, and this is the cohomology of $C_{\bullet,\bullet}$, completing the proof. □

We next show that polynomial rings have Van den Bergh duality.

Example 4.3.5. Let $A = k[x]$. Consider the Koszul resolution (1.1.19) of A as an A^e-module. Apply $\mathrm{Hom}_{k[x]^e}(-, k[x]^e)$ to obtain

$$0 \longleftarrow \mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \longleftarrow \mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \longleftarrow 0.$$

Under the isomorphism $\mathrm{Hom}_{k[x]^e}(k[x]^e, k[x]^e) \cong \mathrm{Hom}_k(k, k[x]^e) \cong k[x]^e$, this sequence is equivalent to

$$0 \longleftarrow k[x]^e \longleftarrow k[x]^e \longleftarrow 0,$$

the nonzero map being given by multiplication by $x \otimes 1 - 1 \otimes x$. So $\mathrm{HH}^0(k[x], k[x]^e) = 0$ and $\mathrm{HH}^1(k[x], k[x]^e) \cong k[x]$, an invertible $k[x]$-bimodule. The hypotheses of Theorem 4.3.2 are satisfied and so

$$\mathrm{HH}^n(k[x], M) \cong \mathrm{HH}_{1-n}(k[x], M)$$

for all $k[x]$-bimodules M and $n = 0, 1$. A similar argument applies to a polynomial ring in more indeterminates: in Example 4.1.5 we explained that $k[x_1, \ldots, x_m]$ is smooth. We find that

$$\mathrm{HH}^m(k[x_1, \ldots, x_m], k[x_1, \ldots, x_m]^e) \cong k[x_1, \ldots, x_m],$$

while $\mathrm{HH}^i(k[x_1, \ldots, x_m], k[x_1, \ldots, x_m]^e) = 0$ for $i \neq m$, and so by Theorem 4.3.2,

$$\mathrm{HH}^n(k[x_1, \ldots, x_m], M) \cong \mathrm{HH}_{m-n}(k[x_1, \ldots, x_m], M)$$

for all $k[x_1, \ldots, x_m]$-bimodules M and $n = 0, \ldots, m$.

Definition 4.3.6. A smooth algebra A is *Calabi-Yau* if it has Van den Bergh duality with dualizing bimodule $U \cong A$.

Example 4.3.7. As a consequence of our work in Example 4.3.5, polynomial rings $k[x_1, \ldots, x_m]$ are Calabi-Yau.

Calabi-Yau algebras were first defined by Ginzburg [**89**] as an analog, in the noncommutative setting, of rings of functions on Calabi-Yau varieties. There is also a notion of a *twisted Calabi-Yau algebra*: in the definition of Calabi-Yau algebra, allow more generally an isomorphism $U \cong A_\sigma$, where σ is an algebra automorphism of A and $A_\sigma = A$ with A^e-module structure twisted by σ on the right, that is, the right action of A on A_σ is given by $a \cdot b = a\sigma(b)$ for all $a, b \in A$. See, e.g., [**94**].

When A is a Calabi-Yau algebra, we may replace U by A in Theorem 4.3.2 and apply the isomorphism $A \otimes_A M \cong M$ to obtain the following corollary.

Corollary 4.3.8. *If A is a Calabi-Yau algebra of Hochschild dimension d, then*

$$\mathrm{HH}^n(A, M) \cong \mathrm{HH}_{d-n}(A, M)$$

for all A-bimodules M and integers n.

Examples of noncommutative Calabi-Yau algebras include some Sklyanin algebras [**163**] and some deformed preprojective algebras [**7**]. Skew group algebras can be Calabi-Yau, and we give details for some of these examples in the next section.

Exercise 4.3.9. Prove that invertible bimodules correspond with autoequivalences of the category A-mod.

Exercise 4.3.10. Let $A = k[x_1, \ldots, x_m]$. Verify the claimed structure of $\mathrm{HH}^i(A, A^e)$ stated in Example 4.3.5.

Exercise 4.3.11. Let $A = k_q[x_1, x_2]$, as defined in Example 3.2.1. Find $\mathrm{HH}^i(A, A^e)$ for each i. Is A Calabi-Yau?

Exercise 4.3.12. Let $A = k\langle x_1, x_2\rangle/(x_1x_2 - qx_2x_1,\ x_1^{n_1},\ x_2^{n_2})$, as in Example 3.2.6. Find $\mathrm{HH}^i(A, A^e)$ for each i. Is A Calabi-Yau? Cf. Exercise 3.2.12.

4.4. Skew group algebras

Let G be a finite group, let k be a field of characteristic not dividing $|G|$, and let V be a kG-module of finite dimension d as a vector space. In this section we show that the skew group algebra $A = S(V) \rtimes G$ has Van den Bergh duality and determine conditions under which it is Calabi-Yau. In Example 3.5.7, we found expressions for the Hochschild cohomology spaces

Recall the standard sign convention (2.3.1) that we make heavy use of in this chapter.

6.1. Coderivations

In this section, we present Stasheff's realization [**207**] of the Gerstenhaber bracket on Hochschild cohomology of A as a graded commutator of coderivations on the tensor coalgebra of A. See also Quillen [**180, 181**] for related constructions and Schlessinger and Stasheff [**191**] for the case of Harrison cohomology. We start by defining the tensor coalgebra.

Let $T = T(A) = \bigoplus_{n\geq 0} A^{\otimes n}$, where we set $A^{\otimes 0} = k$, considered as a complex with differential d_T given by

$$d_T(a_1 \otimes \cdots \otimes a_n) = \sum_{i=1}^{n-1} (-1)^i a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n$$

for all $a_1, \dots, a_n \in A$. Then $T(A)$ is a graded vector space with $T_n(A) = A^{\otimes n}$. Here we will use the notation $\mathrm{Hom}_k(T(A), T(A))$ for the graded vector space whose mth component is

$$\mathrm{Hom}_k(T(A), T(A))_m = \{f \mid f(A^{\otimes n}) \subseteq A^{\otimes (n+m)} \text{ for all } n\}.$$

Then $\mathrm{Hom}_k(T(A), T(A))$ is a graded algebra under composition of functions, and it is a bicomplex whose total complex has differential ∂ given by

$$\partial(f) = d_T f - (-1)^{|f|} f d_T \tag{6.1.1}$$

for all homogeneous functions f (see Section A.5). By abuse of notation, as is customary, we use the same notation and terminology for the bicomplex and its total complex, and it will be clear from context which is meant. Note that $|d_T| = -1$. A calculation shows that

$$\partial(fg) = \partial(f)g + (-1)^{|f|} f \partial(g),$$

where fg denotes composition of these functions. That is, ∂ is a graded derivation on $\mathrm{Hom}_k(T(A), T(A))$. Thus $\mathrm{Hom}_k(T(A), T(A))$ is a differential graded algebra.

The complex $\mathrm{Hom}_k(T(A), T(A))$ has another binary operation given by the graded commutator:

$$[f, g] = fg - (-1)^{|f||g|} gf \tag{6.1.2}$$

for all homogeneous $f, g \in \mathrm{Hom}_k(T(A), T(A))$. By virtue of being a graded commutator, it enjoys a graded Jacobi identity just as in Lemma 1.4.3(ii). Calculations show that

$$\partial([f, g]) = [\partial(f), g] + (-1)^{|f|} [f, \partial(g)]$$

and

$$\partial(f) = [d_T, f]. \tag{6.1.3}$$

Define a k-linear map $\Delta_T : T(A) \to T(A) \otimes T(A)$ by

$$\begin{aligned}\Delta_T(a_1 \otimes \cdots \otimes a_n) &= 1 \otimes (a_1 \otimes \cdots \otimes a_n) + (a_1 \otimes \cdots \otimes a_n) \otimes 1 \\ &\quad + \sum_{i=1}^{n-1} (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} \otimes \cdots \otimes a_n)\end{aligned}$$

for all $a_1, \ldots, a_n \in A$. Under this map, $T(A)$ is a differential graded coalgebra, that is, Δ_T is a chain map, $(\Delta_T \otimes 1)\Delta_T = (1 \otimes \Delta_T)\Delta_T$, and $(\varepsilon \otimes 1)\Delta_T = 1 = (1 \otimes \varepsilon)\Delta_T$, where $\varepsilon : T(A) \to k$ projects onto $T(A)_0 = k$. Some authors write $T^c(A)$ for this differential graded coalgebra in order to distinguish it from the algebra on the same underlying vector space. Let $\Delta_T^{(2)} : T(A) \to T(A) \otimes T(A) \otimes T(A)$ be defined by

$$\Delta_T^{(2)} = (\Delta_T \otimes 1_T)\Delta_T = (1_T \otimes \Delta_T)\Delta_T.$$

Definition 6.1.4. A *graded coderivation* on $T(A)$ is a graded k-linear map $f : T(A) \to T(A)$, of some degree j, for which

$$\Delta_T f = (f \otimes 1_T + 1_T \otimes f)\Delta_T$$

as functions from $T(A)$ to $T(A) \otimes T(A)$. Denote by $\mathrm{Coder}(T(A))$ the vector space spanned by the graded coderivations on $T(A)$.

A calculation shows that the space $\mathrm{Coder}(T(A))$ is closed under the graded commutator bracket (6.1.2) by its definition (recalling the sign convention (2.3.1)). Also note that the differential d_T is itself a coderivation since Δ_T is a chain map. $\mathrm{Coder}(T(A))$ is thus a subcomplex of the Hom complex $(\mathrm{Hom}_k(T(A), T(A)), \partial)$ since $\partial = [d_T, -]$ as noted in (6.1.3).

The following connection with Hochschild cohomology goes back to work of Stasheff [**207**] (see also Quillen [**180, 181**]). Let $B = B(A)$ be the bar resolution (1.1.4) of A as an A^e-module, so that $B_n = A^{\otimes(n+2)}$ for all $n \geq 0$. We take the differential d^* on $\mathrm{Hom}_k(T(A), A) \cong \mathrm{Hom}_{A^e}(B, A)$ to be that induced by the differential d on the bar resolution of A given by equation (1.1.2), which in turn is related to the differential d_T on $T(A)$:

$$\begin{aligned}&d^*(f)(a_1 \otimes \cdots \otimes a_m) \\ &= a_1 f(a_2 \otimes \cdots \otimes a_m) + f d_T(a_1 \otimes \cdots \otimes a_m) + (-1)^m f(a_1 \otimes \cdots \otimes a_{m-1}) a_m\end{aligned}$$

for all $a_1, \ldots, a_m \in A$ and $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$. Note that the degree of such a function f is taken to be $m-1$ in our context here (not m as in other contexts). A calculation shows that f may be extended uniquely to a coderivation $D_f : T(A) \to T(A)[1-m]$ as follows:

$$D_f = (1_T \otimes f \otimes 1_T)\Delta_T^{(2)}, \tag{6.1.5}$$

where if $l < m$, we interpret D_f to be 0 on $A^{\otimes l}$. On elements then, applying the sign convention (2.3.1), we have

(6.1.6)
$$D_f(a_1 \otimes \cdots \otimes a_l) = \sum_{i=1}^{l-m+1} (-1)^u a_1 \otimes \cdots \otimes a_{i-1} \otimes f(a_i \otimes \cdots \otimes a_{i+m-1}) \otimes a_{i+m} \otimes \cdots \otimes a_l$$

for all $a_1, \dots, a_l \in A$, where $u = (m-1)(i-1)$. (Existence and uniqueness of D_f is due to the corresponding truncated complex being cofree in a certain category of coalgebras; see, for example, [**153**, Section II.3.7].)

Next we show that the complex of coderivations is isomorphic to the bar complex $\mathrm{Hom}_k(T(A), A) \cong \mathrm{Hom}_{A^e}(B, A)$ from which Hochschild cohomology is obtained.

Theorem 6.1.7. *The complex* $(\mathrm{Coder}(T(A)), \partial)$ *is a subcomplex of the complex* $(\mathrm{Hom}_k(T(A), T(A)), \partial)$ *that is isomorphic, as a differential graded vector space, to* $(\mathrm{Hom}_k(T(A), A), d^*)$.

Proof. We have already seen that the space $\mathrm{Coder}(T(A))$ is a subcomplex of $(\mathrm{Hom}_k(T(A), T(A)), \partial)$, since the differential d_T is a coderivation, $\partial = [d_T, -]$, and $\mathrm{Coder}(T(A))$ is closed under bracket. Given an element f of $\mathrm{Hom}_k(T(A), A)$, it extends uniquely to a coderivation D_f from $T(A)$ to $T(A)$ given by (6.1.5). On the other hand, given a coderivation from $T(A)$ to $T(A)$, its composition with projection onto $T_1(A) = A$ is an element of $\mathrm{Hom}_k(T(A), A)$. A calculation shows that the differential ∂ on $\mathrm{Coder}(T(A))$ corresponds to d^* on $\mathrm{Hom}_k(T(A), A)$. $\square$

As a consequence of the theorem, Hochschild cohomology $\mathrm{HH}^*(A)$ is the cohomology of the complex $(\mathrm{Coder}(T(A)), \partial)$. We may realize the Gerstenhaber bracket in a natural way on $\mathrm{Coder}(T(A))$ as follows. Recall the degree shift by 1 here in making comparisons to earlier sections.

Theorem 6.1.8. *The bracket* (6.1.2) *induces the Gerstenhaber bracket of Definition* 1.4.1 *on Hochschild cohomology* $\mathrm{HH}^*(A)$ *under the isomorphism of complexes given in Theorem* 6.1.7.

Proof. The isomorphism of Theorem 6.1.7 sends cochains f, g on the bar resolution $B = B(A)$ to their corresponding coderivations D_f, D_g on $T(A)$ given by formula (6.1.6). We claim that the formula (6.1.2) applied to D_f, D_g coincides with Definition 1.4.1 of Gerstenhaber bracket. To see this, note that projecting values of $[D_f, D_g] = D_f D_g - (-1)^{|D_f||D_g|} D_g D_f$ onto $T_1(A) = A$ yields the formula

(6.1.9)
$$f D_g - (-1)^{|D_f||D_g|} g D_f$$

for their bracket as an element of $\mathrm{Hom}_k(T(A), A)$. If $f \in \mathrm{Hom}_k(A^{\otimes m}, A)$ and $g \in \mathrm{Hom}_k(A^{\otimes n}, A)$, then $|D_f| = m-1$ and $|D_g| = n-1$. Applying the formula (6.1.6) for D_f and D_g in terms of f and g, and comparing to the formula for the Gerstenhaber bracket in Definition 1.4.1, we see that they are the same. □

Exercise 6.1.10. Verify that Δ_T is a chain map.

Exercise 6.1.11. Verify that $\mathrm{Coder}(T(A))$ is closed under the graded commutator bracket (6.1.2).

Exercise 6.1.12. Verify that the differential d_T is a coderivation.

Exercise 6.1.13. Verify that the differential ∂ is a graded derivation with respect to the graded commutator (6.1.2).

Exercise 6.1.14. Verify that (6.1.5) (equivalently, (6.1.6)) defines a coderivation on $T(A)$.

6.2. Derivation operators

In this section, we present Suárez-Álvarez' methods [**212**] for computing Gerstenhaber brackets with elements of homological degree 1 on an arbitrary resolution. These methods may be used for example to find the Lie structure on degree 1 Hochschild cohomology $\mathrm{HH}^1(A)$ and the structure of its module $\mathrm{HH}^*(A)$. Suárez-Álvarez worked in a broader context of derivation operators and actions on Ext. Here we consider only that part of his theory that is directly relevant to the Gerstenhaber bracket on $\mathrm{HH}^*(A)$, and refer to [**212**] for more general results.

Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A with differential d. Let $f : A \to A$ be a derivation, so that it represents an element of $\mathrm{HH}^1(A)$, as explained in Section 1.2. Let $f^e : A^e \to A^e$ be defined by

$$f^e = f \otimes 1 + 1 \otimes f, \tag{6.2.1}$$

and note that f^e is a derivation on A^e. Functions satisfying equation (6.2.3) below are termed *derivation operators* (or more specifically f^e*-operators*). More generally, the notion of a δ-operator, for any derivation δ on an algebra, is defined in [**212**].

The following lemma is related to work of Gopalakrishnan and Sridharan [**95**].

Lemma 6.2.2. *Let $f : A \to A$ be a derivation. There is a k-linear chain map $\tilde{f}_\bullet : P_\bullet \to P_\bullet$ lifting f with the property that for each n,*

$$\tilde{f}_n((a \otimes b) \cdot x) = f(a)xb + a\tilde{f}_n(x)b + axf(b) \tag{6.2.3}$$

for all $a, b \in A$ and $x \in P_n$. Moreover, $\tilde{f}_\bullet$ is unique up to A^e-module chain homotopy.

Proof. We wish to define each $\tilde{f}_i$ so that it satisfies equation (6.2.3), and so that the following diagram commutes:

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\mu_P} & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \tilde{f}_1} & & \downarrow{\scriptstyle \tilde{f}_0} & & \downarrow{\scriptstyle f} & & \\ \cdots & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\mu_P} & A & \longrightarrow & 0 \end{array}$$

If P_0 is free as an A^e-module, choose a free basis $\{x_j \mid j \in J\}$, where J is some indexing set. Since μ_P is surjective, for each $j \in J$, there exists a $y_j \in P_0$ such that $\mu_P(y_j) = f(\mu_P(x_j))$. Set $\tilde{f}_0(x_j) = y_j$. Extend to P_0 by requiring

$$\tilde{f}_0((a \otimes b) \cdot x_j) = f(a)x_j b + a y_j b + a x_j f(b)$$

for all $a, b \in A$ and $j \in J$. Note the rightmost square in the diagram indeed commutes since f is a derivation and the action of A^e on A is by left and right multiplication. If P_0 is not free, we may realize it as a direct summand of a free module and argue similarly.

Now $\tilde{f}_0 d_1$ has image contained in the image of d_1, since $\mu_P \tilde{f}_0 d_1 = f \mu_P d_1 = 0$. We may apply the same argument as above to define $\tilde{f}_1$, and so on. Thus we have a k-linear chain map $\tilde{f}_\bullet$ satisfying (6.2.3).

If $\tilde{f}_\bullet$ and $\tilde{f}'_\bullet$ are two such k-linear chain maps, then $\tilde{f}_\bullet - \tilde{f}'_\bullet$ is a chain map lifting the zero map from A to A. Since each of $\tilde{f}_\bullet$, $\tilde{f}'_\bullet$ satisfies (6.2.3), their difference $\tilde{f}_\bullet - \tilde{f}'_\bullet$ is A^e-linear, and so it is A^e-chain homotopic to 0. □

A standard example is given by functions on the bar resolution, as we explain next.

Example 6.2.4. Let B be the bar resolution on A, and let $f : A \to A$ be a derivation. For each i, let

$$\tilde{f}_i(a_0 \otimes \cdots \otimes a_{i+1}) = \sum_{j=0}^{i+1} a_0 \otimes \cdots \otimes a_{j-1} \otimes f(a_j) \otimes a_{j+1} \otimes \cdots \otimes a_{i+1}$$

for all $a_0, \ldots, a_{i+1} \in A$ and extend k-linearly. Then $\tilde{f}_\bullet : B \to B$ is a derivation operator, that is, it satisfies equation (6.2.3).

The following theorem is due to Suárez-Álvarez [**212**]. For Hochschild cocycles defined on a resolution P other than the bar resolution B, their Gerstenhaber bracket is defined as a function on P via chain maps between P and B. (See Exercise 6.2.11.) This is the definition of Gerstenhaber bracket used in the theorem.

Theorem 6.2.5. *Let $f : A \to A$ be a derivation. Let P be a projective resolution of A as an A^e-module. Let $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be a cocycle, and let $\tilde{f}_n : P_n \to P_n$ be a map satisfying* (6.2.3). *The Gerstenhaber bracket of f and g is represented by*

$$[f, g] = fg - g\tilde{f}_n \qquad (6.2.6)$$

as a cocycle on P_n.

Proof. First suppose that P is the bar resolution $B = B(A)$. Let $\tilde{f}_n$ be chosen as in Example 6.2.4. Then formula (6.2.6) agrees with the historical formula of Definition 1.4.1 for the Gerstenhaber bracket, since f is a 1-cocycle. Also, since $gd = 0$ and $\tilde{f}_\bullet$ is unique up to chain homotopy as stated in Lemma 6.2.2, the element of Hochschild cohomology given by formula (6.2.6) does not depend on the choice of $\tilde{f}_\bullet$.

If P is not the bar resolution, let $\theta : B \to P$ and $\iota : P \to B$ be comparison maps, that is, chain maps lifting the identity map on A. Identify the derivation f with a cocycle on B as described in Section 1.2. The Gerstenhaber bracket of f and g is by definition $[f, g\theta]\iota$, where $[f, g\theta]$ denotes the Gerstenhaber bracket defined as usual on B (see Exercise 6.2.11). Let $\tilde{f}'_\bullet :$ $B \to B$ be a k-linear chain map satisfying (6.2.3) on B. A calculation shows that for each i, the function $\theta\tilde{f}'_i - \tilde{f}_i\theta$ is in fact an A^e-module homomorphism. Since $\theta\tilde{f}'_\bullet - \tilde{f}_\bullet\theta$ lifts the zero map from A to A, it must be A^e-chain homotopic to 0. By our arguments in the first paragraph above, $[f, g\theta] = fg\theta - g\theta\tilde{f}'_n$ represents the Gerstenhaber bracket of f and $g\theta$ at the chain level on B. Using the notation $\sim$ to indicate that cocycles are cohomologous, on P we have

$$[f, g\theta]\iota \;\sim\; fg\theta\iota - g\theta\tilde{f}'_n\iota \;\sim\; fg\theta\iota - g\tilde{f}_n\theta\iota \;\sim\; fg - g\tilde{f}_n,$$

since $\theta\iota$ is chain homotopic to the identity map and $gd = 0$. □

The above proof of Theorem 6.2.5 via Lemma 6.2.2 is constructive, giving rise to a method for computing Gerstenhaber brackets with 1-cocycles. We illustrate this derivation operator method next with a small example. Other examples are in the literature; e.g., [**147**, **159**]. Used in combination with the relation given in Lemma 1.4.7 between cup product and Gerstenhaber bracket, the derivation operator method sometimes suffices to compute the full Gerstenhaber algebra structure on Hochschild cohomology.

Example 6.2.7. Let $A = k[x, y]$. We will find a general formula for the Gerstenhaber bracket of a 1-cocycle with a 2-cocycle on the Koszul resolution P given by (3.1.4), using formula (6.2.6). Other brackets may be found similarly. Let $f = x^i y^j \frac{\partial}{\partial x}$, a derivation on A. Let $g = qx^* \wedge y^*$ for some

$q \in A$. We first find functions $\tilde{f}_0, \tilde{f}_1, \tilde{f}_2$ as in Lemma 6.2.2:

$$\begin{aligned}
\tilde{f}_0(a \otimes b) &= f(a) \otimes b + a \otimes f(b), \\
\tilde{f}_1(a \otimes x \otimes b) &= f(a) \otimes x \otimes b + \sum_{l=1}^{j} ax^i y^{j-l} \otimes y \otimes y^{l-1} b \\
&\quad + \sum_{l=1}^{i} ax^{i-l} \otimes x \otimes x^{l-1} y^j b + a \otimes x \otimes f(b), \\
\tilde{f}_1(a \otimes y \otimes b) &= f(a) \otimes y \otimes b + a \otimes y \otimes f(b), \\
\tilde{f}_2(a \otimes x \wedge y \otimes b) &= f(a) \otimes x \wedge y \otimes b + \sum_{l=1}^{i} ax^{i-l} \otimes x \wedge y \otimes x^{l-1} y^j b \\
&\quad + a \otimes x \wedge y \otimes f(b)
\end{aligned}$$

for all $a, b \in A$. By Theorem 6.2.5, setting $p = x^i y^j$,

$$\begin{aligned}
[f, g](x \wedge y) &= (fg - g\tilde{f}_2)(x \wedge y) \\
&= f(q) - g\left(\sum_{l=1}^{i} x^{i-l} \otimes x \wedge y \otimes x^{l-1} y^j\right) \\
&= p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p).
\end{aligned}$$

So $[f, g] = (p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p))x^* \wedge y^*$.

Exercise 6.2.8. Verify that f^e, defined by (6.2.1), is a derivation on A^e.

Exercise 6.2.9. Verify that $\tilde{f}_\bullet$ as defined in Example 6.2.4 is indeed a k-linear chain map on the bar resolution $B = B(A)$ and that it satisfies (6.2.3).

Exercise 6.2.10. Verify the claim in the first sentence of the proof of Theorem 6.2.5, that is, formula (6.2.6) agrees with the historical definition of Gerstenhaber bracket as defined on the bar resolution.

Exercise 6.2.11. Let P be a projective resolution of A as an A^e-module, and let B be the bar resolution of A. Let $\theta : B \to P$ and $\iota : P \to B$ be comparison maps. Define a bilinear operation $[\ ,\ ]$ on $\mathrm{Hom}_{A^e}(P_\bullet, A)$ as follows. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$. Let $[f, g] = [f\theta, g\theta]\iota$, where the bracket on the right side is the Gerstenhaber bracket of Definition 1.4.1. Show that this induces a well-defined operation $[\ ,\ ]$ on Hochschild cohomology that agrees with the Gerstenhaber bracket under the isomorphism induced by the chain maps θ, ι.

Exercise 6.2.12. Let $A = k[x, y]$, notation as in Example 6.2.7. Find $[x^i y^j \frac{\partial}{\partial x}, qx^*]$ and $[x^i y^j \frac{\partial}{\partial x}, qy^*]$. What more must be computed to obtain all possible Gerstenhaber brackets among elements of $\mathrm{HH}^*(A)$?

6.3. Homotopy liftings

In this section we present Volkov's approach to brackets on Hochschild cohomology expressed directly on an arbitrary resolution, and explain how results of Sections 6.1 and 6.2 fit with this approach. More details and applications may be found in [**220**].

Let $P \xrightarrow{\mu_P} A$ be a projective resolution of A as an A^e-module with differential d. We work with the Hom complex $\mathrm{Hom}_{A^e}(P, P)$ in which the differential $\mathbf{d}$ is given by

$$\mathbf{d}(f) = df - (-1)^m fd$$

for all A^e-maps $f : P \to P[-m]$. (See Section A.1 for the degree shift notation $P[-m]$ and Section A.5 for the Hom complex.) The Hom complex is quasi-isomorphic to $\mathrm{Hom}_{A^e}(P, A)$ via the augmentation μ_P. We use the notation $\sim$ in this section to indicate that two functions are cohomologous in $\mathrm{Hom}_{A^e}(P, P)$. Equivalently, they are cohomologous in $\mathrm{Hom}_{A^e}(P, A)$ after application of μ_P, since μ_P induces a quasi-isomorphism between the complexes $\mathrm{Hom}_{A^e}(P, P)$ and $\mathrm{Hom}_{A^e}(P, A)$.

Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles, that is, $fd = 0$ and $gd = 0$. Inspired by the expression (6.1.9) for the Gerstenhaber bracket obtained in Stasheff's coderivation theory, we aim to express the Gerstenhaber bracket $[f, g]$ analogously as a function on P similar to a graded commutator with respect to function composition. We will define functions $\psi_f : P \to P[1-m]$ and $\psi_g : P \to P[1-n]$ for which the Gerstenhaber bracket is represented at the chain level by

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f. \tag{6.3.1}$$

We caution that we have chosen different sign conventions and notation from that of Volkov [**220**].

We will impose a condition on the functions ψ_f, ψ_g analogous to a property of the circle product stated in Lemma 1.4.5(i). For an m-cocycle f and an n-cocycle g on the bar resolution, this is:

$$(-1)^m (g \circ f)d = (-1)^{mn} f \smile g - g \smile f. \tag{6.3.2}$$

We wish to define an analog of the circle product, as a function on an arbitrary resolution, that has such a relationship to the cup product. With this in mind, let $\Delta_P : P \to P \otimes_A P$ be a diagonal map, that is, an A^e-module chain map lifting the canonical isomorphism $A \to A \otimes_A A$ of A^e-modules. The cup product $f \smile g$ may be represented by the function $(f \otimes g)\Delta_P$ as in formula (2.3.2), where we have suppressed the subscript A on the tensor product symbol between functions since the subscript is clear from the domain $P \otimes_A P$. Accordingly, we require ψ_f to satisfy the following equation analogous to (6.3.2) for all cocycles g in $\mathrm{Hom}_{A^e}(P_n, A)$:

$$(-1)^m g\psi_f d = ((-1)^{mn} f \otimes g - g \otimes f)\Delta_P. \tag{6.3.3}$$

We impose these two conditions (6.3.1) and (6.3.3) on functions ψ_f, ψ_g, and derive from these some further conditions, leading to Definition 6.3.6 below of homotopy lifting. We will show in Theorem 6.3.11 that the conditions are sufficient to define the Gerstenhaber bracket as (6.3.1), justifying this approach.

We consider the second imposed condition (6.3.3) first. Fixing f, since $gd = 0$ and $|\psi_f| = m - 1$, the condition is (recalling the Koszul sign convention (2.3.1)):

$$g\mathbf{d}(\psi_f) = (-1)^m g\psi_f d = ((-1)^{mn} f \otimes g - g \otimes f)\Delta_P = g(f \otimes 1_P - 1_P \otimes f)\Delta_P$$

for all $n \geq 0$ and all n-cocycles g. This will hold if

$$\mathbf{d}(\psi_f) = (f \otimes 1_P - 1_P \otimes f)\Delta_P. \tag{6.3.4}$$

We consider the first imposed condition (6.3.1) in the case that g is the 0-cocycle μ_P, rewriting it as follows. Let B denote the bar resolution on A, and let $\theta : B \to P$ and $\iota : P \to B$ be comparison maps. Then $f\theta$ is a cocycle on B, and so the Gerstenhaber bracket $[f\theta, \mu_B]$ becomes 0 in cohomology by Lemma 1.4.5(ii) since μ_B is simply the multiplication map π on A. Using the historical definition of Gerstenhaber bracket and the comparison maps ι, θ to translate to cocycles on P, the Gerstenhaber bracket of f and μ_P is

$$[f, \mu_P] = [f\theta, \mu_P\theta]\iota = [f\theta, \mu_B]\iota \sim 0.$$

So, if $[f, \mu_P]$ may be expressed as in equation (6.3.1), then setting $\psi = \psi_{\mu_P}$, we have

$$f\psi + (-1)^m \mu_P \psi_f \sim 0. \tag{6.3.5}$$

Note that by its definition, the function $f\psi + (-1)^m \mu_P \psi_f$ takes P_{m-1} to A and condition (6.3.5) is simply requiring ψ_f to take values in P_0 consistent with values of $f\psi$.

In fact these two conditions (6.3.4) and (6.3.5) are sufficient to define the bracket via formula (6.3.1), as we will see in Theorem 6.3.11. Next we will give a name to functions ψ_f having these properties, as in [**220**].

Definition 6.3.6. Let P be a projective resolution of A as an A^e-module, let $\Delta_P : P \to P \otimes_A P$ be a diagonal map, and let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ be a cocycle. An A^e-module homomorphism $\psi_f : P \to P[1 - m]$ is a *homotopy* *lifting of f with respect to Δ_P* if

$$\begin{aligned} \mathbf{d}(\psi_f) &= (f\otimes 1_P - 1_P\otimes f)\Delta_P \quad \text{and} \\ \mu_P\psi_f &\sim (-1)^{m-1} f\psi \end{aligned}$$

for some $\psi : P \to P[1]$ for which $\mathbf{d}(\psi) = (\mu_P\otimes 1_P - 1_P\otimes\mu_P)\Delta_P$.

We will often use the term *homotopy lifting of f* without explicit reference to Δ_P if it is clear from the context which map Δ_P is intended, or in situations where the choice of Δ_P does not matter. We caution again that our homotopy lifting differs from that of Volkov [**220**] by signs.

It may be checked directly that if ψ_f, ψ_g are homotopy liftings for cocycles f, g with respect to Δ_P, then $[f, g]$ as defined in (6.3.1) is a cocycle. We check that if either f or g is a coboundary, then so is $[f, g]$ as defined in (6.3.1): if $f = hd$ for some cochain h, set

$$\psi_f = (-1)^m (h\otimes 1_P - 1_P\otimes h)\Delta_P. \tag{6.3.7}$$

A calculation shows that ψ_f is a homotopy lifting for f. With this choice, $f\psi_g \sim (-1)^{(m-1)(n-1)} g\psi_f$, and so $[f, g]$ is a coboundary.

Example 6.3.8. Let $P = B$, the bar resolution of A, and let Δ_B be the standard diagonal map on B given by formula (2.3.3) (note this corresponds to the map Δ_T of Section 6.1 by writing $B = A\otimes T(A)\otimes A$ and extending Δ_T to an A^e-module homomorphism). Then $(\mu_B\otimes 1_B - 1_B\otimes\mu_B)\Delta_B = 0$, and we may let $\psi = 0$ in Definition 6.3.6. Let $f \in \mathrm{Hom}_{A^e}(B_m, A)$ be a cocycle. We may assume without loss of generality that $f(a_0\otimes\cdots\otimes a_{m+1})$ is 0 whenever at least one of $a_1, \ldots, a_m$ is in the field k, since f is cohomologous to such a function (as may be seen by mapping to the reduced bar resolution). Let

$$\begin{aligned} &\psi_f(a_0\otimes\cdots\otimes a_{l+1}) \\ &= \sum_{i=1}^{l-m+1} (-1)^u a_0\otimes\cdots\otimes a_{i-1}\otimes f(a_i\otimes\cdots\otimes a_{i+m-1})\otimes a_{i+m}\otimes\cdots\otimes a_{l+1}, \end{aligned}$$

where $u = (m-1)(i-1)$, for all $l \geq m$ and $a_0, \ldots, a_{l+1} \in A$, and take ψ_f to be the zero map on B_l for $l \leq m-1$. Then ψ_f is a homotopy lifting of f with respect to Δ_B. A calculation shows that with this choice of ψ_f and a similar choice of ψ_g, the bracket $[f, g]$ as given by formula (6.3.1) is precisely the Gerstenhaber bracket as defined on the bar resolution in Definition 1.4.1.

We may view ψ_f defined by the formula in the example as a coderivation on B, or restrict to $T(A) \cong k\otimes T(A)\otimes k \hookrightarrow A\otimes T(A)\otimes A = B$ to obtain a coderivation $\psi_f|_{T(A)}$ on $T(A)$ as in Definition 6.1.4; see also formula (6.1.6). If f is a 1-cocycle, then $\psi_f|_{T(A)}$, viewed another way, may be extended to a derivation operator in the sense of Lemma 6.2.2 (see Example 6.2.4). Thus homotopy liftings encompass these two views—coderivations on the tensor coalgebra and derivation operators on the bar resolution—that were introduced in Sections 6.1 and 6.2.

We next state a needed existence and uniqueness result for homotopy liftings.

Lemma 6.3.9. *Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ be a cocycle, and let $\Delta_P : P \to P \otimes_A P$ be a diagonal map. There is a homotopy lifting $\psi_f : P \to P[1-m]$ of f with respect to Δ_P. Moreover, it is unique up to chain homotopy.*

Proof. First we show existence of a homotopy lifting ψ of μ_P with respect to Δ_P. Consider the function $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$ in the Hom complex $\mathrm{Hom}_{A^e}(P, P)$. Apply the quasi-isomorphism μ_P to $\mathrm{Hom}_{A^e}(P, A)$. Note that $\mu_P(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = \mu_P \otimes \mu_P - \mu_P \otimes \mu_P = 0$, and so under the quasi-isomorphism from $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$ to $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$, the map $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ becomes 0. Further, a calculation shows that

$$\mathbf{d}(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = 0,$$

and therefore $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ is a boundary in $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$. Precomposing with the chain map Δ_P, we see that $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P = \mathbf{d}(\psi)$ for some $\psi : P \to P[1]$, as claimed.

Next we show existence of functions ψ_f satisfying the conditions (6.3.4) and (6.3.5). Now $(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a chain map from P to $P[-m]$ since $fd = 0$. Applying μ_P, since $|\mu_P| = 0$, we have

$$\begin{aligned} \mu_P(f \otimes 1_P - 1_P \otimes f)\Delta_P &= f(1_P \otimes \mu_P - \mu_P \otimes 1_P)\Delta_P \\ &= -f\mathbf{d}(\psi) \;\; = \;\; -f\psi d, \end{aligned}$$

that is, applying the quasi-isomorphism from $\mathrm{Hom}_{A^e}(P, P)$ to $\mathrm{Hom}_{A^e}(P, A)$ given by μ_P, we find that $\mu_P(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a coboundary. Consequently, in $\mathrm{Hom}_{A^e}(P, P)$, the function $(f \otimes 1_P - 1_P \otimes f)\Delta_P$ is a coboundary, so that

$$(f \otimes 1_P - 1_P \otimes f)\Delta_P = \mathbf{d}(\psi_f)$$

for some ψ_f, that is, condition (6.3.4) holds. We will show that some of the functions ψ_f satisfying (6.3.4) also satisfy condition (6.3.5). Combining the above two equations, we now have

$$\mu_P \mathbf{d}(\psi_f) = -f\psi d,$$

and since $\mathbf{d}(\psi_f) = d\psi_f + (-1)^m \psi_f d$ and $\mu_P d = 0$, this is equivalent to

$$((-1)^m \mu_P \psi_f + f\psi)d = 0.$$

However, we want $(-1)^m \mu_P \psi_f + f\psi \sim 0$. Set $g = (-1)^m \mu_P \psi_f + f\psi$, viewed as a map from P_{m-1} to A. We have seen that g is a cocycle, and thus it corresponds to a chain map $g_\bullet$ from P to $P[1-m]$. Define $\psi'_f = \psi_f - (-1)^m g_\bullet$.

Since $g_\bullet$ is a chain map, $\mathbf{d}(\psi'_f) = \mathbf{d}(\psi_f)$, and so ψ'_f also satisfies (6.3.4). Additionally we now have

$$(-1)^m \mu_p \psi'_f + f\psi = (-1)^m \mu_P \psi_f + f\psi - g = 0,$$

by definition of g, and so ψ'_f also satisfies (6.3.5). Replacing ψ_f by ψ'_f, we have shown that there exists a homotopy lifting of f with respect to Δ_P.

Finally, we show uniqueness up to chain homotopy. Let ψ_f and ψ'_f be two homotopy liftings of f with respect to Δ_P. Then $\mathbf{d}(\psi_f - \psi'_f) = 0$ and $\mu_P(\psi_f - \psi'_f) \sim 0$. Again, μ_P gives rise to the quasi-isomorphism from $\mathrm{Hom}_{A^e}(P,P)$ to $\mathrm{Hom}_{A^e}(P,A)$ and this implies $\psi_f - \psi'_f \sim 0$, as claimed. Note that this argument does not depend on choice of homotopy ψ for $(\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P$, again as any two will be homotopic. □

The following theorem and proof are [**220**, Theorem 4].

Theorem 6.3.10. *Let P be a projective resolution of A as an A^e-module with diagonal map $\Delta_P : P \to P \otimes_A P$. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles. The element of Hochschild cohomology* $\mathrm{HH}^*(A)$ *represented by $[f,g]$ as defined by formula* (6.3.1) *is independent of choice of resolution P, of diagonal map Δ_P, and of homotopy liftings ψ_f and ψ_g.*

Proof. We will prove independence of choices in the reverse order from what is stated. Independence of choice of ψ_f and ψ_g is immediate from the uniqueness of ψ_f and ψ_g up to chain homotopy stated in Lemma 6.3.9, since $fd = 0$ and $gd = 0$.

Let Δ_P and Δ'_P be two diagonal maps. Then $\Delta'_P - \Delta_P = \mathbf{d}(u)$ for some $u : P \to (P \otimes_A P)[1]$. Let ψ_f and ψ_g be homotopy liftings of f and g with respect to Δ_P. Let

$$\psi'_f = \psi_f + (-1)^m (f \otimes 1_P - 1_P \otimes f)u,$$

and similarly ψ'_g. A calculation shows that ψ'_f and ψ'_g are homotopy liftings of f and g with respect to Δ'_P, respectively. We find that

$$\begin{aligned}
&f\psi'_g - (-1)^{(m-1)(n-1)} g\psi'_f \\
&= f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f + (-1)^n f(g \otimes 1_P - 1_P \otimes g)u \\
&\qquad - (-1)^{(m-1)(n-1)}(-1)^m g(f \otimes 1_P - 1_P \otimes f)u \\
&= f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f,
\end{aligned}$$

so these two expressions give the same bracket $[f,g]$ via formula (6.3.1). Thus the formula is independent of choice of diagonal map.

Let $Q \xrightarrow{\mu_Q} A$ be another projective resolution of A as an A^e-module, and let $\Delta_Q : Q \to Q \otimes_A Q$ be a diagonal map. Let $\iota : P \to Q$ and $\theta : Q \to P$ be chain maps lifting the identity map on A. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and

$g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles on P. Then $f\theta$ and $g\theta$ are cocycles on Q. Let $\psi_{f\theta}$ be a homotopy lifting for $f\theta$ with respect to Δ_Q. Set $\psi_f = \theta\psi_{f\theta}\iota$. We first check that ψ_f is a homotopy lifting for f with respect to $\Delta_P = (\theta \otimes \theta)\Delta_Q\iota$:

$$\begin{aligned}
\mathbf{d}(\psi_f) &= \theta\mathbf{d}(\psi_{f\theta})\iota \\
&= \theta(f\theta \otimes 1_Q - 1_Q \otimes f\theta)\Delta_Q\iota \\
&= (f \otimes 1_P - 1_P \otimes f)(\theta \otimes \theta)\Delta_Q\iota \\
&= (f \otimes 1_P - 1_P \otimes f)\Delta_P,
\end{aligned}$$

so ψ_f satisfies equation (6.3.4).

Set $\psi_P = \theta\psi_Q\iota$, where ψ_Q satisfies $\mathbf{d}(\psi_Q) = (\mu_Q \otimes 1_Q - 1_Q \otimes \mu_Q)\Delta_Q$ as well as $(-1)^m\mu_Q\psi_{f\theta} + f\theta\psi_Q \sim 0$. Then

$$\begin{aligned}
\mathbf{d}(\psi_P) = \theta\mathbf{d}(\psi_Q)\iota &= \theta(\mu_Q \otimes 1_Q - 1_Q \otimes \mu_Q)\Delta_Q\iota \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)(\theta \otimes \theta)\Delta_Q\iota \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)\Delta_P,
\end{aligned}$$

since $\theta\mu_Q = \mu_P\theta$. It follows, since $(-1)^m\mu_Q\psi_{f\theta} + f\theta\psi_Q \sim 0$, that by the definitions of ψ_f and of ψ_P above,

$$\begin{aligned}
(-1)^m\mu_P\psi_f + f\psi_P &= (-1)^m\mu_Q\psi_{f\theta}\iota + f\theta\psi_Q\iota \\
&= ((-1)^m\mu_Q\psi_{f\theta} + f\theta\psi_Q)\iota \sim 0,
\end{aligned}$$

that is, ψ_f satisfies (6.3.5). Therefore ψ_f is a homotopy lifting of f with respect to Δ_P, and we may similarly define a homotopy lifting of g.

Finally, formula (6.3.1) applied to f, g on P yields

$$\begin{aligned}
[f, g] &= f\theta\psi_{g\theta}\iota - (-1)^{(m-1)(n-1)}g\theta\psi_{f\theta}\iota \\
&= [f\theta, g\theta]\iota,
\end{aligned}$$

so the chain map ι takes $[f\theta, g\theta]$ to $[f, g]$. Thus the bracket does not depend on choice of resolution. □

As a consequence of Theorem 6.3.10, the bracket given by formula (6.3.1) agrees with the Gerstenhaber bracket of Definition 1.4.1 on Hochschild cohomology.

Theorem 6.3.11. *Let P be a projective resolution of A as an A^e-module. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles on P, and let ψ_f and ψ_g be homotopy liftings of f and g, as in Definition* 6.3.6*. The bracket given at the chain level by*

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)}g\psi_f$$

induces the Gerstenhaber bracket on Hochschild cohomology $\mathrm{HH}^*(A)$.

Proof. In Example 6.3.8, we saw that taking P to be the bar resolution recovers the Gerstenhaber bracket of Definition 1.4.1 from formula (6.3.1). By Theorem 6.3.10, it is independent of choices. □

Remark 6.3.12. In practice, often $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$, in which case Definition 6.3.6 of homotopy lifting can be simplified by taking $\psi = 0$. See Example 6.3.8, the next Section 6.4, and [**220**, Remark 1].

Exercise 6.3.13. Let ψ_f, ψ_g be homotopy liftings of f, g as in Definition 6.3.6.

(a) Check directly that $[f, g]$ as defined in (6.3.1) is a cocycle.
(b) Check directly that if either f or g is a coboundary, then so is $[f, g]$ as defined in (6.3.1). (See (6.3.7) for a homotopy lifting of a coboundary and verify first that it is indeed a homotopy lifting.)

Exercise 6.3.14. Verify that in the context of Example 6.3.8, the formula for ψ_f indeed yields the classical Gerstenhaber bracket as defined on the bar resolution in Definition 1.4.1.

6.4. Differential graded coalgebras

Some of the results of the previous sections lead to effective computational techniques for the Lie structure on Hochschild cohomology. In particular, we explain in this section some settings in which the theory of homotopy liftings can be simplified for computational purposes. Formula (6.4.2) below gives homotopy liftings for all cocycles f in terms of a diagonal map Δ_P and an additional function ϕ_P under some conditions that we discuss next.

Let $\phi_P : P \otimes_A P \to P[1]$ be a homotopy for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$. That is,

$$\mathbf{d}(\phi_P) = \mu_P \otimes 1_P - 1_P \otimes \mu_P. \tag{6.4.1}$$

To see that such a homotopy exists, consider the quasi-isomorphism μ_P from $\mathrm{Hom}_{A^e}(P \otimes_A P, P)$ to $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$, which takes $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ to 0. Since μ_P is a chain map, $\mathbf{d}(\mu_P \otimes 1_P - 1_P \otimes \mu_P) = 0$. It follows that in $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$, the map $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ is a cocycle that becomes 0 (which is a coboundary) after applying the quasi-isomorphism μ_P, so it must be a coboundary in $\mathrm{Hom}_{A^e}(P \otimes_A P, A)$.

Let

$$\Delta_P^{(2)} = (\Delta_P \otimes 1_P)\Delta_P.$$

Note that in general this may not be the same as $(1_P \otimes \Delta_P)\Delta_P$. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ with $fd = 0$, and let $\psi_f : P \to P[1 - m]$ be defined by

$$\psi_f = \phi_P(1_P \otimes f \otimes 1_P)\Delta_P^{(2)}. \tag{6.4.2}$$

Here the map $1_P \otimes f \otimes 1_P$ is considered to be a map from $P \otimes_A P \otimes_A P$ to $P \otimes_A P$ (upon applying the canonical isomorphism $P \otimes_A A \otimes_A P \cong P \otimes_A P$). We will see next that under some conditions, ψ_f is a homotopy lifting of f with respect to Δ_P as in Definition 6.3.6. In this case, homotopy liftings ψ_f for all cocycles f may be found in terms of these two maps Δ_P and ϕ_P via formula (6.4.2).

Example 6.4.3. Consider the bar resolution $B = B(A)$ of A. We may identify $B_i \otimes_A B_j$ with $A \otimes A^{\otimes i} \otimes A \otimes A^{\otimes j} \otimes A$ under the isomorphism

$$(A \otimes A^{\otimes i} \otimes A) \otimes_A (A \otimes A^{\otimes j} \otimes A) \xrightarrow{\sim} A \otimes A^{\otimes i} \otimes A \otimes A^{\otimes j} \otimes A$$
$$(a_0 \otimes \cdots \otimes a_{i+1}) \otimes_A (a'_0 \otimes \cdots \otimes a'_{j+1}) \mapsto$$
$$a_0 \otimes (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1} a'_0) \otimes (a'_1 \otimes \cdots \otimes a'_j) \otimes a'_{j+1}$$

for all $a_0, \ldots, a_{i+1}, a'_0, \ldots, a'_{j+1} \in A$. Define $\phi_B : B \otimes_A B \to B[1]$ by

$$\phi_B(a_0 \otimes (a_1 \otimes \cdots \otimes a_i) \otimes (a_{i+1}) \otimes (a_{i+2} \otimes \cdots \otimes a_{i+j+1}) \otimes a_{i+j+2})$$
$$= (-1)^i a_0 \otimes \cdots \otimes a_{i+j+2}$$

for all $a_0, \ldots, a_{i+j+2} \in A$, that is, up to sign, we have just removed parentheses. Then ψ_f defined as in (6.4.2) agrees with that given in Example 6.3.8.

In the rest of this section we prove results under some hypotheses that will be satisfied for example by Koszul algebras as defined in Section 3.4. A result of Buchweitz, Green, Snashall, and Solberg [**39**] for Koszul algebras guarantees that the standard embedding $\iota : P \to B$ of the Koszul resolution P into the bar resolution B of A (see (3.4.4)) is preserved by the diagonal map in the sense that the diagonal map Δ_B of the bar resolution given by formula (2.3.3) takes $\iota(P)$ to $\iota(P) \otimes_A \iota(P)$. Thus we may define a diagonal map Δ_P on P via this embedding. It follows that Δ_P is coassociative, that is, $(\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes \Delta_P)\Delta_P$ and $(\mu_P \otimes 1_P)\Delta_P = 1_P = (1_P \otimes \mu_P)\Delta_P$, and we say that P is a *counital differential graded coalgebra.* In this case, in Definition 6.3.6 of homotopy lifting of an m-cocycle f on P, we may take $\psi = 0$, and so condition (6.3.5) becomes $\mu_P \psi_f \sim 0$. We may in fact assume under these conditions that $\psi_f|_{P_{m-1}} = 0$. This simplifies the work of finding homotopy liftings (and it simplifies many of the proofs of the previous section under these additional hypotheses). In fact formula (6.4.2) always defines a homotopy lifting in the case that Δ_P, μ_P give P a coalgebra structure, as we see next.

Lemma 6.4.4. *Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A that is a counital differential graded coalgebra. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ be a cocycle, and let $\psi_f : P \to P[1-m]$ be defined by formula* (6.4.2)*. Then ψ_f is a homotopy lifting of f with respect to Δ_P.*

Proof. Let $\phi_P : P \otimes_A P \to P[1]$ be a homotopy for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$, that is, equation (6.4.1) holds. Set $\Delta_P^{(2)} = (\Delta_P \otimes 1_P)\Delta_P = (1_P \otimes \Delta_P)\Delta_P$. Since $\Delta_P^{(2)}$ and $1_P \otimes f \otimes 1_P$ are chain maps,

$$\begin{aligned}
\mathbf{d}(\psi_f) &= \mathbf{d}(\phi_P)(1_P \otimes f \otimes 1_P)\Delta_P^{(2)} \\
&= (\mu_P \otimes 1_P - 1_P \otimes \mu_P)(1_P \otimes f \otimes 1_P)\Delta_P^{(2)} \\
&= (\mu_P \otimes f \otimes 1_P - 1_P \otimes f \otimes \mu_P)\Delta_P^{(2)} \\
&= ((f \otimes 1_P)(\mu_P \otimes 1_P \otimes 1_P) - (1_P \otimes f)(1_P \otimes 1_P \otimes \mu_P))\Delta_P^{(2)} \\
&= (f \otimes 1_P - 1_P \otimes f)\Delta_P.
\end{aligned}$$

Note that $\psi_f|_{P_{m-1}} = 0$ by its definition, and as explained above, we may take $\psi = 0$ in Definition 6.3.6. Therefore ψ_f is a homotopy lifting of f. □

The following theorem is a reworked version of [**165**, Theorem 3.2.5], which has stronger hypotheses, and of [**165**, Lemma 3.4.1], which has somewhat different hypotheses.

Theorem 6.4.5. *Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A that is a counital differential graded coalgebra. Let $f \in \mathrm{Hom}_{A^e}(P_m, A)$ and $g \in \mathrm{Hom}_{A^e}(P_n, A)$ be cocycles. Define ψ_f by formula* (6.4.2)*, and similarly ψ_g. Then*

$$[f, g] = f\psi_g - (-1)^{(m-1)(n-1)} g\psi_f$$

represents the Gerstenhaber bracket of f and g on Hochschild cohomology.

Proof. This follows immediately from Lemma 6.4.4 and Theorem 6.3.11. □

Remark 6.4.6. If the hypotheses of the theorem do not hold, a homotopy ϕ_P for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$ may still be used to define the Gerstenhaber bracket in a similar way, with the addition of some error terms. See [**220**, Corollary 5 and Remark 1] in which the Gerstenhaber bracket is given generally as

(6.4.7)
$$\begin{aligned}
[f, g] = &-f\phi_P(g \otimes 1_P \otimes 1_P - 1_P \otimes g \otimes 1_P + 1_P \otimes 1_P \otimes g)\Delta_P^{(2)} \\
&+ (-1)^{(m-1)(n-1)} g\phi_P(f \otimes 1_P \otimes 1_P - 1_P \otimes f \otimes 1_P + 1_P \otimes 1_P \otimes f)\Delta_P^{(2)}.
\end{aligned}$$

We caution that the function

$$-\phi_P(f \otimes 1_P \otimes 1_P - 1_P \otimes f \otimes 1_P + 1_P \otimes 1_P \otimes f)\Delta_P^{(2)}$$

is not necessarily a homotopy lifting of f; the formula (6.4.7) instead results from a more complicated homotopy lifting as explained in the proof of [**220**, Corollary 5].

In the remainder of this section, we apply Theorem 6.4.5 to an example, a polynomial ring in two indeterminates. The case of n indeterminates is similar, if more notationally unwieldy, and is handled in [**165**, Section 4], showing that formula (6.3.1) indeed yields the familiar Gerstenhaber bracket on Hochschild cohomology of a polynomial ring. In other settings the first computation of Gerstenhaber brackets, or of a related Batalin-Vilkovisky structure, used these techniques (see, e.g., [**101**, **102**, **166**, **220**]).

Example 6.4.8. Let $A = k[x, y]$, and let P be its Koszul resolution (3.1.4), that is, $P = A \otimes \bigwedge^{\bullet} V \otimes A$, where $V = \mathrm{Span}_k\{x, y\}$. Identify $P_i \otimes_A P_j$ with $A \otimes \bigwedge^i V \otimes A \otimes \bigwedge^j V \otimes A$ for each i, j, and identify $\bigwedge^0 V$ with k and $\bigwedge^1 V$ with V. Thus for example, $P_0 \otimes_A P_1 \cong A \otimes k \otimes A \otimes V \otimes A \cong A \otimes A \otimes V \otimes A$, and we use such expressions in our definitions of maps below. We first find a homotopy $\phi_P : P \otimes_A P \to P[1]$ for $\mu_P \otimes 1_P - 1_P \otimes \mu_P$. In degree 2, the map ϕ_P is necessarily 0 since $P_3 = 0$. We define ϕ_P in degrees 0 and 1 on free basis elements:

$$\begin{aligned}
\phi_P(1 \otimes x^i y^j \otimes 1) &= \sum_{l=1}^{j} x^i y^{j-l} \otimes y \otimes y^{l-1} + \sum_{l=1}^{i} x^{i-l} \otimes x \otimes x^{l-1} y^j, \\
\phi_P(1 \otimes x^i y^j \otimes x \otimes 1) &= -\sum_{l=1}^{j} x^i y^{j-l} \otimes x \wedge y \otimes y^{l-1}, \\
\phi_P(1 \otimes x^i y^j \otimes y \otimes 1) &= 0, \\
\phi_P(1 \otimes x \otimes x^i y^j \otimes 1) &= 0, \\
\phi_P(1 \otimes y \otimes x^i y^j \otimes 1) &= \sum_{l=1}^{i} x^{i-l} \otimes x \wedge y \otimes x^{l-1} y^j.
\end{aligned}$$

We use this function ϕ_P, formula (6.4.2) for ψ_f, and the formula of Theorem 6.4.5 to compute some brackets in degree 1. The diagonal map Δ_P is defined by the standard embedding (3.4.12) of P into the bar resolution, followed by the standard diagonal map (2.3.3) on the bar resolution. Consider the cocycles in degree 1 denoted by $x^i y^j x^*$ and $x^i y^j y^*$, where $\{x^*, y^*\}$ is the dual basis to $\{x, y\}$ (e.g., $x^i y^j x^*$ takes x to $x^i y^j$ and y to 0, where x, y are identified with their images in $\bigwedge^1 V$). First we find some values of $\psi_{x^i y^j x^*}$

and $\psi_{x^iy^jy^*}$ via formula (6.4.2):

$$\begin{aligned}
\psi_{x^iy^jx^*}(1\otimes x\otimes 1) &= \phi_P(1\otimes x^iy^j\otimes 1),\\
\psi_{x^iy^jx^*}(1\otimes y\otimes 1) &= 0,\\
\psi_{x^iy^jy^*}(1\otimes x\otimes 1) &= 0,\\
\psi_{x^iy^jy^*}(1\otimes y\otimes 1) &= \phi_P(1\otimes x^iy^j\otimes 1).
\end{aligned}$$

It follows that, for example,

$$\begin{aligned}
&[x^iy^jx^*,\ x^my^nx^*](1\otimes x\otimes 1)\\
&= x^iy^jx^*\psi_{x^my^nx^*}(1\otimes x\otimes 1) - x^my^nx^*\psi_{x^iy^jx^*}(1\otimes x\otimes 1)\\
&= x^iy^jx^*\phi_P(1\otimes x^my^n\otimes 1) - x^my^nx^*\phi_P(1\otimes x^iy^j\otimes 1)\\
&= \sum_{l=1}^{m} x^iy^jx^{m-l}x^{l-1}y^n - \sum_{l=1}^{i} x^my^nx^{i-l}x^{l-1}y^j\\
&= mx^iy^jx^{m-1}y^n - ix^my^nx^{i-1}y^j\\
&= x^iy^j\frac{\partial}{\partial x}(x^my^n) - x^my^n\frac{\partial}{\partial x}(x^iy^j).
\end{aligned}$$

Another calculation shows that the value of this bracket function on $1\otimes y\otimes 1$ is zero. Therefore, for all $p, q\in A$, we have

$$[px^*,\ qx^*] = (p\frac{\partial}{\partial x}(q) - q\frac{\partial}{\partial x}(p))x^*.$$

Similarly we find that

$$\begin{aligned}
[px^*,\ qy^*] &= p\frac{\partial}{\partial x}(q)y^* - q\frac{\partial}{\partial y}(p)x^*,\\
[py^*,\ qy^*] &= (p\frac{\partial}{\partial y}(q) - q\frac{\partial}{\partial y}(p))y^*.
\end{aligned}$$

We may calculate other brackets using the same techniques (cf. Example 6.2.7 and Exercise 6.2.12).

Exercise 6.4.9. Verify that ψ_f as defined in Example 6.4.3 agrees with that given in Example 6.3.8.

Exercise 6.4.10. Let $P \xrightarrow{\mu_P} A$ be a projective resolution of the A^e-module A that is a counital differential graded coalgebra. Show that one choice of chain map $g_\bullet$ corresponding to a cocycle $g \in \mathrm{Hom}_{A^e}(P_n, A)$ is given by $g_m = (1 \otimes g)(\Delta_P)_m$ for all m. (This draws a parallel between the cup product as given by formula (2.2.1) and the Gerstenhaber bracket as given by formula (6.3.1) using (6.4.2).)

Exercise 6.4.11. Verify the formulas for $[px^*, qy^*]$ and $[py^*, qy^*]$ in Example 6.4.8. Find $[px^*, qx^* \wedge y^*]$ and compare with Example 6.2.7.

6.5. Extensions

In this section, we consider Schwede's exact sequence interpretation of the Lie structure on Hochschild cohomology [**194**]. Hermann [**110**] generalized Schwede's results to some exact monoidal categories, and gave a description of the bracket with degree 0 elements in [**108**], completing Schwede's interpretation. We refer to these papers for most of the technical details and proofs, and present just a skimming here.

Let $n \geq 1$, and let $\mathcal{E}xt^n_{A^e}(A, A)$ denote the category whose objects are n-extensions of A by A as an A^e-module, and morphisms are maps of n-extensions. (See Section A.3 for a discussion of maps and equivalence classes of n-extensions.) View $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^e}(A, A)$ as equivalence classes of objects in $\mathcal{E}xt^n_{A^e}(A, A)$. Consider an m-extension and an n-extension of A by A,

$$\mathbf{f}: \qquad 0 \longrightarrow A \xrightarrow{i_M} M_{m-1} \longrightarrow \cdots \longrightarrow M_0 \xrightarrow{\mu_M} A \longrightarrow 0,$$

$$\mathbf{g}: \qquad 0 \longrightarrow A \xrightarrow{i_N} N_{n-1} \longrightarrow \cdots \longrightarrow N_0 \xrightarrow{\mu_N} A \longrightarrow 0.$$

We will not need notation for the unlabeled maps. We assume that all M_j, N_j are projective as left A-modules, and as right A-modules, where needed. See [**194**] for a discussion about such an assumption.

Let P be a projective resolution of A as an A^e-module. Let f and g be an m-cocycle and an n-cocycle on P, corresponding to the generalized extensions $\mathbf{f}$ and $\mathbf{g}$, respectively. So $f \in \mathrm{Hom}_{A^e}(P_m, A)$ may be defined via the following commuting diagram, which exists by the Comparison Theorem (Theorem A.2.7); see also Section A.3. The map 1 from A to A is the identity map. We denote by $\hat{f}_\bullet : P_\bullet \to M_\bullet$ the chain map indicated below, so that $f = \hat{f}_m$.

$$\begin{array}{ccccccccccccc}
\cdots \longrightarrow & P_m & \xrightarrow{d_m} & P_{m-1} & \xrightarrow{d_{m-1}} & \cdots & \xrightarrow{d_1} & P_0 & \xrightarrow{\mu_P} & A & \longrightarrow & 0 \\
& \downarrow f & & \downarrow \hat{f}_{m-1} & & & & \downarrow \hat{f}_0 & & \downarrow 1 & & \\
0 \longrightarrow & A & \xrightarrow{i_M} & M_{m-1} & \longrightarrow & \cdots & \longrightarrow & M_0 & \xrightarrow{\mu_M} & A & \longrightarrow & 0
\end{array}$$