Chapter 8

Huisken's Theorem

In this chapter we see that it is possible to completely describe the global in time behavior of the mean curvature flow in the special setting of compact, convex hypersurfaces. The result, due to G. Huisken [**291**], states that the hypersurface remains convex and contracts to a single point at the final time in such a way that, after rescaling by the square root of the remaining time, the resulting family of hypersurfaces converges smoothly to a round sphere.

Theorem 8.1 (Huisken [**291**]). *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$, $n \geq 2$, be a maximal solution to mean curvature flow defined on the maximal time interval $[0,T)$. If $X_0 \doteqdot X(\cdot, 0)$ is a convex embedding, then $X_t \doteqdot X(\cdot,t)$ is a convex embedding for all $t > 0$, X_t converges uniformly to a constant $p \in \mathbb{R}^{n+1}$ as $t \to T$, and the rescaled embeddings $\widetilde{X}_t : M^n \to \mathbb{R}^{n+1}$ defined by*

$$\widetilde{X}_t(x) \doteqdot \frac{X_t(x) - p}{\sqrt{2n(T-t)}} \tag{8.1}$$

converge uniformly in the smooth topology to a smooth embedding whose image coincides with the unit sphere S^n.

Figure 8.1. Gerhard Huisken. Author: Gerd Fischer. Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach.

Eventually, we will present *four* different proofs of this result, each of which illustrates important techniques that arise later in other settings. Huisken's argument, presented in the following sections, makes use of strong pointwise pinching and gradient estimates for the second fundamental form, yielding direct, quantitative control on the convergence to a round sphere. The remaining proofs invoke indirect "compactness" arguments to varying degrees: A second argument, which we present at the end of this chapter, relies on a bound for the ratio of circumradius to inradius; the convergence to a round sphere then follows from a straightforward application of the Arzelà–Ascoli theorem and the strong maximum principle. An argument of Hamilton, which we present in Section 11.4, combines very weak control on the solution (namely, preservation of the initial curvature pinching) with a classification of possible "blow-ups" to reach the conclusion but requires a more sophisticated compactness theory (which we develop in Section 11.1). The fourth proof, presented in Section 12.3, makes use of the more recently discovered "noncollapsing" phenomena for mean curvature flow, which provide very strong control over the evolution, and the convergence then follows again via Arzelà–Ascoli. This proof, unlike the previous three proofs, also applies to the curve shortening flow, giving a unified proof of the theorems of Gage–Hamilton and Huisken.

8.1. Pinching is preserved

A uniformly convex hypersurface admits a positive lower bound for its mean curvature. An elementary application of the maximum principle shows that this bound is preserved along the mean curvature flow.

Lemma 8.2 (Nonnegative lower bound for H preserved under mean curvature flow)**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact solution of mean curvature flow. If $\min_{M\times\{0\}} H \geq 0$, then*

$$H(x,t) \geq \min_{M\times\{0\}} H \quad \textit{for all} \quad (x,t) \in M \times [0,T)\,. \tag{8.2}$$

Proof. Apply the maximum principle to the evolution equation (6.18) for the mean curvature. □

A more careful analysis yields refined estimates for the maximum and minimum of H.

Lemma 8.3. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact solution of mean curvature flow satisfying*

$$\limsup_{t\nearrow T} \max_{M\times\{t\}} H^2 = \infty\,.$$

Then

$$\min_{M\times\{t\}} H \le \frac{\sqrt{n}}{\sqrt{2(T-t)}}\,. \tag{8.3}$$

If, in addition, there is some $C>0$ *such that* $|\mathrm{II}|^2 \le CH^2$*, then*

$$\max_{M\times\{t\}} H \ge \frac{1}{\sqrt{2C(T-t)}}\,. \tag{8.4}$$

Proof. Recalling the evolution equation (6.18) for the mean curvature, we estimate

$$\partial_t H \ge \Delta H + \frac{1}{n}H^3 \quad \text{and} \quad \partial_t H \le \Delta H + C\,H^3,$$

with the second inequality holding provided $|\mathrm{II}|^2 \le CH^2$.

Set $\overline{H}(t) \doteqdot \max_{M\times\{t\}} H$ and $\underline{H}(t) \doteqdot \min_{M\times\{t\}} H$. Let $\{t_i\}_{i\in\mathbb{N}}$ be a sequence of times converging to T and satisfying $\lim_{i\to\infty} \overline{H}(t_i) = \infty$. For each $i\in\mathbb{N}$, let $\overline{\varphi}_i$ and $\underline{\varphi}_i$ be the solutions of the ODEs

$$\begin{cases} \overline{\varphi}_i' = C\overline{\varphi}_i^3 & \text{in } [0,t_i]\,,\\ \overline{\varphi}_i(t_i) = \overline{H}(t_i) \end{cases}$$

and

$$\begin{cases} \underline{\varphi}_i' = \frac{1}{n}\underline{\varphi}_i^3 & \text{in } [0,t_i]\,,\\ \underline{\varphi}_i(t_i) = \underline{H}(t_i)\,, \end{cases}$$

respectively. That is,

$$\overline{\varphi}_i(t) = \frac{1}{\sqrt{\overline{H}^{-2}(t_i) + 2C(t_i - t)}} \qquad \text{and} \qquad \underline{\varphi}_i(t) = \frac{1}{\sqrt{\underline{H}^{-2}(t_i) + \frac{2}{n}(t_i - t)}}\,.$$

By the contrapositive of the ODE comparison principle,

$$\overline{H} \ge \overline{\varphi}_i \quad \text{and} \quad \underline{H} \le \underline{\varphi}_i \quad \text{in} \quad [0,t_i]\,.$$

Taking $i\to\infty$ yields the claims. □

For a convex solution to mean curvature flow, a purely algebraic calculation implies that $|\mathrm{II}|^2 \le H^2$ and hence

$$H_{\min}(t) \lesssim \frac{1}{\sqrt{T-t}} \lesssim H_{\max}(t)\,.$$

The scaling behavior of the mean curvature flow suggests the stronger conclusion

$$H(\cdot\,,t) \sim \frac{1}{\sqrt{T-t}}\,.$$

We shall soon see that this does indeed hold for convex solutions of the mean curvature flow (although it is not true in general).

Any compact, locally uniformly convex hypersurface is α-**pinched** for some $\alpha > 0$; that is,

$$\frac{\kappa_1}{\kappa_n} \geq \alpha .$$

In particular, $\mathrm{II} \geq \frac{\alpha}{n} Hg$ and $\mathrm{II} \leq \frac{1}{n\alpha} Hg$. A straightforward adaptation of the classical *scalar* maximum principle shows that these *tensor* inequalities are also preserved.

Lemma 8.4 (Pinching is preserved [**291**, Theorem 4.3])**.** *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact, strictly mean convex solution to mean curvature flow.*

(1) *If $\mathrm{II} \geq \alpha Hg$ for some $\alpha \in \mathbb{R}$ at $t = 0$, then the same inequality holds for all $t \in (0, T)$.*
(2) *If $\mathrm{II} \leq CHg$ for some $C \in \mathbb{R}$ at $t = 0$, then the same inequality holds for all $t \in (0, T)$.*

Proof. By (6.15), (6.17), and (6.18), the tensor $S \doteqdot \mathrm{II} - \alpha Hg$ satisfies

$$(\nabla_t - \Delta) S = |\mathrm{II}|^2 S .$$

So the first claim follows from the tensor maximum principle (Theorem 6.14). The second claim is proved similarly. □

8.2. Pinching improves: The roundness estimate

Given an immersed hypersurface $X : M^n \to \mathbb{R}^{n+1}$, recall from Chapter 5 that a point $p \in \mathcal{M}$ is called **umbilic** if the second fundamental form at p is proportional to the metric at p; that is,

$$\mathrm{II}_p = \alpha_p g_p$$

for some $\alpha_p \in \mathbb{R}$. By tracing this, we see that $\alpha_p = \frac{1}{n} H(p)$, so $p \in \mathcal{M}$ is umbilic if and only if the **trace-free second fundamental form**

$$\mathring{\mathrm{II}} \doteqdot \mathrm{II} - \tfrac{1}{n} Hg \tag{8.5}$$

vanishes at p. Another way of saying this is that the norm $|\mathring{\mathrm{II}}|$ of the trace-free second fundamental form is strictly positive at a point p unless p is umbilic. Since the only compact, totally umbilic, connected hypersurfaces of $\mathbb{R}^{n+1}$ are the round spheres (see Exercise 5.5), the norm of the trace-free second fundamental form gives a crude pointwise measure of "**roundness**" for a hypersurface. In fact, since roundness should not depend on scale, we are more interested in the scale-invariant ratio $|\mathring{\mathrm{II}}|^2/H^2$.

It turns out that this measure of roundness is preserved by mean curvature flow.

Proposition 8.5 (Roundness is preserved). *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact, mean convex solution of mean curvature flow. Then*

$$\max_{\mathcal{M}_t} \frac{|\mathring{\mathrm{II}}|^2}{H^2} \le \max_{\mathcal{M}_0} \frac{|\mathring{\mathrm{II}}|^2}{H^2} \quad \textit{for all} \quad t \in [0,T)\,. \tag{8.6}$$

Proof. Recalling (6.17) and (6.18), we observe that

$$\begin{aligned}(\nabla_t - \Delta)\frac{\mathrm{II}}{H} &= \frac{(\nabla_t - \Delta)\,\mathrm{II}}{H} - \frac{\mathrm{II}}{H^2}(\partial_t - \Delta)H + 2\nabla_{\frac{\nabla H}{H}}\frac{\mathrm{II}}{H}\\ &= 2\nabla_{\frac{\nabla H}{H}}\frac{\mathrm{II}}{H}\,.\end{aligned}$$

Thus,

$$\begin{aligned}(\partial_t - \Delta)\frac{|\mathring{\mathrm{II}}|^2}{H^2} &= (\partial_t - \Delta)\frac{|\mathrm{II}|^2}{H^2}\\ &= 2\left\langle (\nabla_t - \Delta)\frac{\mathrm{II}}{H}, \frac{\mathrm{II}}{H}\right\rangle - 2\left|\nabla\frac{\mathrm{II}}{H}\right|^2\\ &= 4\left\langle \nabla_{\frac{\nabla H}{H}}\frac{\mathrm{II}}{H}, \frac{\mathrm{II}}{H}\right\rangle - 2\left|\nabla\frac{\mathrm{II}}{H}\right|^2\\ &= 2\left\langle \nabla\frac{|\mathrm{II}|^2}{H^2}, \frac{\nabla H}{H}\right\rangle - 2\left|\nabla\frac{\mathrm{II}}{H}\right|^2\\ &= 2\left\langle \nabla\frac{|\mathring{\mathrm{II}}|^2}{H^2}, \frac{\nabla H}{H}\right\rangle - 2\left|\nabla\frac{\mathrm{II}}{H}\right|^2\,.\end{aligned} \tag{8.7}$$

The claim now follows from the maximum principle. □

We expect that diffusion should work to *improve* the roundness ratio. Of course, we cannot hope for too much since, for obvious reasons, a "thin" torus (with bounded roundness) cannot be smoothly deformed to a surface with perfect roundness — the sphere. For convex hypersurfaces, there is no such obstruction.

Theorem 8.6 (Roundness estimate [**291**]). *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$, $n \ge 2$, be a compact, convex solution of mean curvature flow. For any $\varepsilon > 0$ there exists a constant C_ε — which depends only on ε and the initial embedding — such that*

$$|\mathring{\mathrm{II}}|(x,t) \le \varepsilon H^2(x,t) + C_\varepsilon \quad \textit{for all} \quad (x,t) \in M^n \times [0,T)\,. \tag{8.8}$$

As we shall see in the course of the proof, the constant C_ε can be controlled in terms of ε, n, $\min_{M^n\times\{0\}} \frac{\kappa_1}{H}$, $\mathrm{Area}(\mathcal{M}_0^n)$, $\max_{M^n\times\{0\}} H$, and T (which, by Exercise 8.1, can be controlled in terms of the circumradius or minimum mean curvature of the initial hypersurface).

The roundness estimate implies that solutions become round at a singularity in the sense that, given any sequence of points $(x_k, t_k) \in M \times [0, T)$ satisfying $H(x_k, t_k) \to \infty$ as $k \to \infty$,

$$\frac{|\mathring{\mathrm{II}}|^2}{H^2}(x_k, t_k) \to 0 \quad \text{as} \quad k \to \infty\,.$$

To prove the theorem, we find constants $\sigma > 0$ and $C < \infty$ such that

$$\frac{|\mathring{\mathrm{II}}|^2 - \varepsilon H^2}{H^2} \leq C H^{-\sigma}\,. \tag{8.9}$$

The theorem then follows by estimating the right-hand side using Young's inequality. So, given any $\varepsilon > 0$ and any $\sigma \in (0, 1)$, define

$$f_{\varepsilon,\sigma} \doteqdot \left(|\mathring{\mathrm{II}}|^2 - \varepsilon H^2\right) H^{\sigma-2}. \tag{8.10}$$

Then we wish to show, for any $\varepsilon > 0$, that $f_{\varepsilon,\sigma}$ is bounded uniformly in time for some $\sigma > 0$. First, an evolution equation.

Lemma 8.7. *Given any $\varepsilon > 0$ and $\sigma \in (0, 1)$, the function $f_{\varepsilon,\sigma}$ defined in* (8.10) *evolves according to*

$$\begin{aligned}(\partial_t - \Delta) f_{\varepsilon,\sigma} = {}& \sigma\, |\mathrm{II}|^2\, f_{\varepsilon,\sigma} + 2\,(1-\sigma) \left\langle \nabla f_{\varepsilon,\sigma}, \frac{\nabla H}{H} \right\rangle \\ & - 2 H^\sigma \left| \nabla \frac{\mathrm{II}}{H} \right|^2 - \sigma\,(1-\sigma)\, f_{\varepsilon,\sigma} \frac{|\nabla H|^2}{H^2}.\end{aligned} \tag{8.11}$$

Proof. The evolution equation for $f_{\varepsilon,\sigma}$ follows from the evolution equations (8.7) for $|\mathring{\mathrm{II}}|^2/H^2$ and (6.18) for H and the "product rule"

$$\begin{aligned}(\partial_t - \Delta)(u v^\alpha) = {}& v^\alpha (\partial_t - \Delta) u + \alpha u v^{\alpha-1} (\partial_t - \Delta) v \\ & - 2\alpha \left\langle \nabla(u v^\alpha), \frac{\nabla v}{v} \right\rangle + \alpha(\alpha+1) u v^\alpha \frac{|\nabla v|^2}{v^2}\,.\end{aligned} \tag{8.12}$$

□

Unfortunately, Theorem 8.6 does not follow from a direct application of the maximum principle — the reaction term is of the wrong sign wherever $f_{\varepsilon,\sigma} > 0$. We need to make better use of the diffusion term using integral estimates.

First, we need an estimate for the good curvature gradient term (i.e., the *second* to last term in (8.11) — it turns out that the final term, due to its dependence on σ, is not as useful as it appears; we will simply discard it).

Lemma 8.8. *Let $\mathcal{M}^n \hookrightarrow \mathbb{R}^{n+1}$, $n \geq 2$, be a convex hypersurface satisfying $\mathrm{II} \geq \alpha H g$ for some $\alpha > 0$. Then*

$$\left| \nabla \frac{\mathrm{II}}{H} \right|^2 \geq \gamma \frac{|\nabla \mathrm{II}|^2}{H^2}\,, \tag{8.13}$$

where $\gamma > 0$ depends only on α and n.

Proof. Observe that

$$\left|\nabla\frac{\mathrm{II}}{H}\right|^2 = H^{-4}|H\nabla\,\mathrm{II} - \nabla H\otimes\mathrm{II}\,|^2\,.$$

Thus, it suffices to prove the purely algebraic inequality

$$|\operatorname{tr}(B)\,T - \operatorname{tr}(T)\otimes B|^2 \geq \gamma(\alpha,n)\,\operatorname{tr}(B)^2\,|T|^2 \tag{8.14}$$

for all symmetric, positive definite $B\in\mathbb{R}^n\otimes\mathbb{R}^n$ satisfying $B\geq\alpha\operatorname{tr}(B)\mathrm{I}$ and all totally symmetric $T\in\mathbb{R}^n\otimes\mathbb{R}^n\otimes\mathbb{R}^n$, where I is the standard inner product on $\mathbb{R}^n$ and $\operatorname{tr}(T)$ denotes any choice of the three equivalent traces of T. Moreover, as the estimate is invariant under scaling of either B or T, it suffices to prove (8.14) when $|B|=|T|=1$ (the inequality is trivial if $T=0$). So suppose that there is no $\gamma>0$ for which (8.14) holds on the subset

$$K \doteqdot \{\,(B,T) : B>0,\ B\geq\alpha\operatorname{tr}(B)\,\mathrm{I},\ |B|=|T|=1\,\}.$$

Then, as K is compact and $\operatorname{tr}(B)^2\leq n$, there must be some pair $(B,T)\in K$ satisfying

$$\operatorname{tr}(B)\,T = \operatorname{tr}(T)\otimes B\,. \tag{8.15}$$

Note that $\operatorname{tr}(T)$ cannot be the zero vector as, by assumption, T and $\operatorname{tr}(B)$ are nonzero. Thus, tracing over the first and second components, we find that $\operatorname{tr}(T)$ is an eigenvector of B with eigenvalue $\operatorname{tr}(B)$. But this is impossible as B is positive definite. □

In particular, since the inequality $\mathrm{II}\geq\alpha Hg$ is preserved under mean curvature flow, we can estimate

$$H^\sigma\left|\nabla\frac{\mathrm{II}}{H}\right|^2 \geq \gamma\frac{|\nabla\mathrm{II}|^2}{H^2}H^\sigma \geq \gamma f_{\varepsilon,\sigma}\frac{|\nabla\mathrm{II}|^2}{H^2}\,,$$

where the constant γ depends only on n and the initial curvature pinching $\alpha\doteqdot\min_{M\times\{0\}}\frac{\kappa_1}{H}$. We can now estimate, wherever $f_{\varepsilon,\sigma}>0$,

$$(\partial_t-\Delta)f_{\varepsilon,\sigma}\leq\sigma|\mathrm{II}|^2f_{\varepsilon,\sigma}+2\,(1-\sigma)\left\langle\nabla f_{\varepsilon,\sigma},\frac{\nabla H}{H}\right\rangle-2\gamma f_{\varepsilon,\sigma}\frac{|\nabla\mathrm{II}|^2}{H^2}\,. \tag{8.16}$$

Next, given $p\geq10$, say, set

$$v=v_{\varepsilon,\sigma,p}\doteqdot(f_{\varepsilon,\sigma})_+^{\frac{p}{2}}, \tag{8.17}$$

where $(f_{\varepsilon,\sigma})_+\doteqdot\max\{f_{\varepsilon,\sigma},0\}$ denotes the nonnegative part of $f_{\varepsilon,\sigma}$. Then v is C^2, so that we may legally calculate

$$2v\nabla v=p(f_{\varepsilon,\sigma})_+^{p-1}\nabla f_{\varepsilon,\sigma}$$

and

$$(\partial_t - \Delta)v^2 = pf_{\varepsilon,\sigma}^{p-1}(\partial_t - \Delta)f_{\varepsilon,\sigma} - p(p-1)f_{\varepsilon,\sigma}^{p-2}|\nabla f_{\varepsilon,\sigma}|^2$$
$$\leq pv^2\left(\sigma|\mathrm{II}|^2 + \frac{4\,(1-\sigma)}{p}\left\langle\frac{\nabla v}{v}, \frac{\nabla H}{H}\right\rangle - 2\gamma\frac{|\nabla\mathrm{II}|^2}{H^2}\right) - \frac{4(p-1)}{p}|\nabla v|^2.$$

Estimating the inner product term, using Young's inequality, by

$$4\,(1-\sigma)\,v^2\left\langle\frac{\nabla v}{v}, \frac{\nabla H}{H}\right\rangle \leq 4v^2\frac{|\nabla v|}{v}\frac{|\nabla H|}{H} \leq v^2\left(\frac{|\nabla v|^2}{v^2} + 4\frac{|\nabla H|^2}{H^2}\right)$$

we arrive at

$$(\partial_t - \Delta)v^2 \leq \sigma p|\mathrm{II}|^2 v^2 - 2|\nabla v|^2 - 2\,(\gamma p - 2n)\,v^2\frac{|\nabla\mathrm{II}|^2}{H^2}\,.$$

If we further require that $p > \frac{4n}{\gamma}$, then

$$(\partial_t - \Delta)v^2 \leq \sigma p|\mathrm{II}|^2 v^2 - \gamma p v^2\frac{|\nabla\mathrm{II}|^2}{H^2} - 2|\nabla v|^2\,. \tag{8.18}$$

Using Simons's identity, we can control the first term on the right in L^1 by the two good (i.e., negative) terms.

Proposition 8.9. *For each $n \geq 2$ there exists a constant P with the following property: Let $\mathcal{M}^n \hookrightarrow \mathbb{R}^{n+1}$ be a smooth hypersurface and, given $\alpha > 0$, let $u \in W^{2,1}(\mathcal{M})$ be a function with compact support contained in the set $\{x \in \mathcal{M} : \mathrm{II} \geq \alpha H g\}$. Then*

$$\alpha^4\int_{\mathcal{M}} u^2\,|\mathring{\mathrm{II}}|^2\,d\mu \leq P\int_{\mathcal{M}} u^2\left(\frac{|\nabla\mathrm{II}|^2}{H^2} + \frac{|\nabla\mathrm{II}|}{H}\frac{|\nabla u|}{u}\right)d\mu\,. \tag{8.19}$$

Proof. By Simons's identity (5.50),

$$\nabla_{(i}\nabla_{j)}\mathrm{II}_{kl} - \nabla_{(k}\nabla_{l)}\mathrm{II}_{ij} = \mathrm{C}_{ijkl}\,,$$

where

$$\mathrm{C} \doteqdot \mathrm{II}\otimes\mathrm{II}^2 - \mathrm{II}^2\otimes\mathrm{II}\,.$$

Note that

$$|\mathrm{C}|^2 = \sum_{i,j=1}^n \kappa_i^2\kappa_j^2(\kappa_i - \kappa_j)^2\,.$$

In particular, at any point where $\mathrm{II} \geq \alpha H g$,

$$|\mathrm{C}|^2 \geq \alpha^4 H^4\sum_{i,j=1}^n(\kappa_i - \kappa_j)^2 = 2n\alpha^4 H^4|\mathring{\mathrm{II}}|^2\,. \tag{8.20}$$

Thus, integrating by parts,

$$
\begin{aligned}
\alpha^4 \int_{\mathcal{M}} u^2 |\mathring{\mathrm{II}}|^2 \, d\mu &\le \frac{1}{2n} \int_{\mathcal{M}} |\mathrm{C}|^2 H^{-4} u^2 \, d\mu \\
&= \frac{1}{2n} \int_{\mathcal{M}} \frac{u^2}{H^4} \mathrm{C}^{ijkl} \left(\nabla_i \nabla_j \mathrm{II}_{kl} - \nabla_k \nabla_l \mathrm{II}_{ij} \right) \, d\mu \\
&= \frac{1}{n} \int_{\mathcal{M}} \frac{u^2}{H^4} \mathrm{C}^{ijkl} \nabla_i \nabla_j \mathrm{II}_{kl} \, d\mu \\
&= \frac{1}{n} \int_{\mathcal{M}} \frac{u^2}{H^4} \left(4\mathrm{C}^{ijkl} \frac{\nabla_i H}{H} - 2\mathrm{C}^{ijkl} \frac{\nabla_i u}{u} - \nabla_i \mathrm{C}^{ijkl} \right) \nabla_j \mathrm{II}_{kl} \, d\mu .
\end{aligned}
$$

Using the notation of Section 6.8, we observe that

$$
\mathrm{C} = \mathrm{II} * \mathrm{II} * \mathrm{II} \quad \text{and} \quad \nabla \mathrm{C} = \mathrm{II} * \mathrm{II} * \nabla \mathrm{II}
$$

and hence

$$
\begin{aligned}
\alpha^4 \int_{\mathcal{M}} u^2 \, |\mathring{\mathrm{II}}|^2 \, d\mu &\le \frac{1}{n} \int_{\mathcal{M}} \frac{u^2}{H^4} \left(\mathrm{II} * \frac{\nabla \mathrm{II}}{H} - \mathrm{II} * \frac{\nabla u}{u} - \nabla \mathrm{II} \right) * \mathrm{II} * \mathrm{II} * \nabla \mathrm{II} \, d\mu \\
&\le P \int_{\mathcal{M}} u^2 \left(\frac{|\nabla \mathrm{II}|^2}{H^2} + \frac{|\nabla \mathrm{II}|}{H} \frac{|\nabla u|}{u} \right) d\mu \, ,
\end{aligned}
$$

where we estimated $|\mathrm{II}| \le H$. □

Since $|\mathring{\mathrm{II}}|^2 \ge \varepsilon H^2 \ge \varepsilon |\mathrm{II}|^2$ wherever $f_{\varepsilon,\sigma} > 0$, Proposition 8.9 yields

$$
\begin{aligned}
\int_{\mathcal{M}} |\mathrm{II}|^2 \, v^2 \, d\mu &\le \frac{P}{\varepsilon \alpha^4} \int_{\mathcal{M}} v^2 \left(\frac{|\nabla \mathrm{II}|^2}{H^2} + \frac{|\nabla \mathrm{II}|}{H} \frac{|\nabla v|}{v} \right) d\mu \\
&\le \frac{P}{\varepsilon \alpha^4} \int_{\mathcal{M}} v^2 \left((1+r) \frac{|\nabla \mathrm{II}|^2}{H^2} + r^{-1} \frac{|\nabla v|^2}{v^2} \right) d\mu
\end{aligned}
$$

for any $r > 0$. Choosing $r = p^{\frac{1}{2}}$, we find

$$
\begin{aligned}
\frac{d}{dt} \int_{\mathcal{M}} v^2 d\mu &= \int_{\mathcal{M}} \partial_t v^2 \, d\mu - \int_{\mathcal{M}} v^2 H^2 \, d\mu \\
&\le \sigma p \int_{\mathcal{M}} |\mathrm{II}|^2 \, v^2 \, d\mu - \gamma p \int_{\mathcal{M}} v^2 \frac{|\nabla \mathrm{II}|^2}{H^2} \, d\mu - 2 \int_{\mathcal{M}} |\nabla v|^2 \, d\mu \\
&\le \left(\frac{\sigma p P}{\varepsilon \alpha^4} (1 + p^{\frac{1}{2}}) - \gamma p \right) \int_{\mathcal{M}} v^2 \frac{|\nabla \mathrm{II}|^2}{H^2} d\mu + \left(\frac{\sigma p^{\frac{1}{2}} P}{\varepsilon \alpha^4} - 2 \right) \int_{\mathcal{M}} |\nabla v|^2 d\mu .
\end{aligned}
$$

Choosing p sufficiently large and σ of the order $p^{-\frac{1}{2}}$, we can ensure that the right-hand side is nonpositive.

Lemma 8.10. *There exists $\ell > 0$ depending only on n, α, and ε such that*

$$\frac{d}{dt}\int_{\mathcal{M}} (f_{\varepsilon,\sigma})_+^p \, d\mu \leq 0 \tag{8.21}$$

for all $p \geq \ell^{-1}$ and all $\sigma \leq \ell p^{-\frac{1}{2}}$.

Integrating yields an L^2 estimate for the function $v \doteqdot f_{\varepsilon,\sigma}^{\frac{p}{2}}$ (at least when p is large and σ is of the order $p^{-\frac{1}{2}}$). To pass from L^2 to L^∞, we make use of an iteration argument originally due to Stampacchia [**330**, Chapter II, Appendix B]. It is based on the following ingenious iteration lemma.

Lemma 8.11 (Stampacchia's lemma)**.** *Let $\varphi : [k_0, \infty) \to \mathbb{R}$ be a nonnegative, nonincreasing function satisfying*

$$\varphi(h) \leq \frac{C}{(h-k)^\alpha}\varphi(k)^\beta \quad \text{for all} \quad h > k > k_0 \tag{8.22}$$

and for some constants $C > 0$, $\alpha > 0$, and $\beta > 1$. Then

$$\varphi(k_0 + d) = 0\,, \tag{8.23}$$

where $d^\alpha = C\varphi(k_0)^{\beta-1}2^{\frac{\alpha\beta}{\beta-1}}$.

Proof. Consider the sequence of numbers k_r given by

$$k_r = k_0 + d - \frac{d}{2^r}, \quad r = 0, 1, 2, \ldots .$$

By assumption,

$$\varphi(k_{r+1}) \leq C\frac{2^{(r+1)\alpha}}{d^\alpha}\varphi(k_r)^\beta \quad \text{for all} \quad r = 0, 1, \ldots . \tag{8.24}$$

We will prove by induction that

$$\varphi(k_r) \leq \varphi(k_0)2^{-r\mu} \tag{8.25}$$

for all $r \in \mathbb{N}$, where $\mu \doteqdot \frac{\alpha}{\beta-1} > 0$. Clearly (8.25) holds trivially for $r = 0$. Supposing (8.25) holds up to some integer r, we find by (8.24) and the definition of d that

$$\varphi(k_{r+1}) \leq C\frac{2^{(r+1)\alpha}}{d^\alpha}\varphi(k_0)^\beta 2^{-r\mu\beta} = \varphi(k_0)2^{-(r+1)\mu}\,.$$

The claim (8.25) follows. Now, by the monotonicity assumption,

$$0 \leq \varphi(k_0 + d) \leq \varphi(k_r) \qquad \text{for all} \quad r = 0, 1, \ldots .$$

But, by (8.25), $\varphi(k_r) \to 0$ as $r \to \infty$. $\square$

Readers unfamiliar with Stampacchia iteration may wish to attempt Exercise 8.4 before proceeding.

Given any $k \geq k_0 \doteqdot \sup_{\sigma\in(0,1)} \sup_{M\times\{0\}} f_{\varepsilon,\sigma}$, set

$$v_k^2(x,t) \doteqdot \left(f_{\varepsilon,\sigma}(x,t) - k\right)_+^p \quad \text{and} \quad U_k(t) \doteqdot \{x \in M : v_k(x,t) > 0\}\,.$$

We want to apply Stampacchia's lemma to the function

$$k \mapsto \|U_k\| \doteqdot \int_0^T \int_{U_k(t)} d\mu(\cdot,t)\,dt\,.$$

Note that

$$(h-k)^p\,\|U_h\| \leq \int_0^T \int_{U_k} v_k^2\,d\mu\,dt\,. \tag{8.26}$$

We need to estimate the right-hand side in terms of $\|U_k\|$.

First observe that

$$\frac{d}{dt}\int v_k^2\,d\mu + \int |\nabla v_k|^2\,d\mu + \int v_k^2 H^2\,d\mu \leq \sigma p \int_{U_k} f_{\varepsilon,\sigma}^p H^2\,d\mu \tag{8.27}$$

for all $p \geq \min\{4, \frac{2n}{\gamma}\}$ (this is proved by integrating an estimate similar to (8.18)). To exploit the good gradient term, we make use of the Sobolev inequality of Michael and Simon [**400**] (see also [**305**]). For functions on submanifolds, the Sobolev inequality includes an additional term involving the mean curvature.

Theorem 8.12 (Sobolev inequality)**.** *Let M^n be a smoothly immersed submanifold of $\mathbb{R}^{n+k}$ of dimension $n \geq 2$ and codimension $k \geq 1$ and let p be any number satisfying $1 < p < n$. Then every function $u \in W_0^{1,p}(M)$ satisfies*

$$\left(\int_M |u|^{p^*}\,d\mu\right)^{\frac{1}{p^*}} \leq S\left[\int_M \left(|\nabla u|^p + |\vec{H}|^p|u|^p\right)\,d\mu\right]^{\frac{1}{p}}, \tag{8.28}$$

where $\frac{1}{p^*} \doteqdot \frac{1}{p} - \frac{1}{n}$ and S is a constant which depends only on n and p.

The Sobolev inequality cannot be applied in the critical case $n = p$, which we shall require when $n = 2$. In that case, we instead make use of the following Poincaré inequality, which is obtained from the $p = 1$ case of the Sobolev inequality using Hölder's inequality (albeit with a constant which now depends also on the measure of the support of u).

Corollary 8.13 (Poincaré inequality). *Let M^n be a smoothly immersed submanifold of $\mathbb{R}^{n+k}$ of dimension $n \geq 2$ and codimension $k \geq 1$ and let q be any number satisfying $q \geq 1$. Then every function $u \in W_0^{1,n}(M)$ satisfies*

$$(8.29)\quad \left(\int_M |u|^{1^*q}\,d\mu\right)^{\frac{1}{1^*q}} \leq S|\mathrm{spt}\,u|^{\frac{1}{1^*q}}\left[\int_M \left(|\nabla u|^n + |\vec{H}|^n|u|^n\right)d\mu\right]^{\frac{1}{n}},$$

where $\frac{1}{1^*} \doteqdot 1-\frac{1}{n}$ and S is a constant which depends only on n and q.

Applying the Sobolev inequality (with $p=2$) when $n \geq 3$ and the Poincaré inequality when $n=2$ then yields[1]

$$\frac{d}{dt}\int v_k^2\,d\mu + \frac{1}{S}\left(\int v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}} \leq \sigma p\int_{U_k} f_{\varepsilon,\sigma}^p H^2\,d\mu\,,$$

where, for $n=2$, we define $2^* \doteqdot 1^*q$ for some $q \geq 1$ and S is a constant which depends only on n when $n \geq 3$ and depends only on $|\mathcal{M}_0|$ and q (which we may take to be fixed) when $n=2$. Integrating this in time and noting that $U_k(0)=\emptyset$ whenever $k > k_0$, we find

$$(8.30)\quad \sup_{[0,T)}\left(\int_{U_k} v_k^2\,d\mu\right) + \int_0^T\left(\int v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}} dt \leq S\sigma p\int_0^T\int_{U_k} H^2 f_{\varepsilon,\sigma}^p\,d\mu\,dt\,.$$

We need to convert the left-hand side into the space-time L^2-norm of v_k. We achieve this using the interpolation inequality for L^p spaces. Recall that this allows us to estimate, for any $\vartheta \in (0,1)$ and any $r, q > 1$,

$$\|f\|_{q_0} \leq \|f\|_q^{\vartheta}\,\|f\|_r^{1-\vartheta}\,,$$

where $\frac{1}{q_0} \doteqdot \frac{\vartheta}{q} + \frac{1-\vartheta}{r}$. Setting $r=1$, $q = 2^*/2 > 1$, and $\vartheta = 1/(2-\frac{1}{q}) \in (0,1)$ we find $q_0 = \frac{1}{\vartheta}$ and hence

$$\int_{U_k} v_k^{2q_0}\,d\mu \leq \left(\int_{U_k} v_k^2\,d\mu\right)^{q_0-1}\left(\int_{U_k} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}}.$$

Young's inequality now yields

$$\begin{aligned}\left(\int_0^T\int_{U_k} v_k^{2q_0}\,d\mu\,dt\right)^{\frac{1}{q_0}} &\leq \left(\sup_{[0,T)}\int_{U_k} v_k^2\,d\mu\right)^{\frac{q_0-1}{q_0}}\left(\int_0^T\left(\int_{U_k} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}} dt\right)^{\frac{1}{q_0}}\\ &\leq \sup_{[0,T)}\int_{U_k} v_k^2\,d\mu + \int_0^T\left(\int_{U_k} v_k^{2^*}\,d\mu\right)^{\frac{2}{2^*}} dt\,.\end{aligned}$$

[1] We remark that the argument of [**291**] differs slightly from the one presented here, in that it uses the "$p=1$" Sobolev inequality and the Hölder inequality. The simplification (at least when $n \geq 3$) using the "$p=2$" inequality was suggested to us by Klaus Ecker.

Returning to (8.30), the Hölder inequality then yields the desired estimate

$$\begin{aligned}(8.31)\qquad \int_0^T\!\!\int_{U_k} v_k^2\,d\mu\,dt &\le \|U_k\|^{1-\frac{1}{q_0}}\left(\int_0^T\!\!\int_{U_k} v_k^{2q_0}\,d\mu\,dt\right)^{\frac{1}{q_0}}\\ &\le S\sigma p\|U_k\|^{1-\frac{1}{q_0}}\int_0^T\!\!\int_{U_k} H^2 f_{\varepsilon,\sigma}^p\,d\mu\,dt\,.\end{aligned}$$

The right-hand side can be bounded in terms of an appropriate power of $\|U_k\|$ using Hölder's inequality. Indeed, for any $r\ge 1$,

$$\begin{aligned}(8.32)\qquad \int_0^T\!\!\int_{U_k} H^2 f_{\varepsilon,\sigma}^p\,d\mu\,dt &\le \|U_k\|^{1-\frac{1}{r}}\left(\int_0^T\!\!\int_{U_k} H^{2r} f_{\varepsilon,\sigma}^{pr}\,d\mu\,dt\right)^{\frac{1}{r}}\\ &= \|U_k\|^{1-\frac{1}{r}}\left(\int_0^T\!\!\int_{U_k} f_{\varepsilon,\sigma'}^{pr}\,d\mu\,dt\right)^{\frac{1}{r}},\end{aligned}$$

where $\sigma' \doteqdot \sigma+\frac{2}{p}$. Let ℓ be as in Lemma 8.10. If $\sigma\le\frac{\ell}{2}(pr)^{-\frac{1}{2}}$ and $p\ge\max\{\ell^{-1},\frac{16r}{\ell^2}\}$, then $pr\ge\ell^{-1}$ and $\sigma'\doteqdot\sigma+\frac{2}{p}\le\ell(pr)^{-\frac{1}{2}}$. Thus we may apply Lemma 8.10 to estimate the right-hand side of (8.32) to obtain

$$(8.33)\qquad \int_0^T\!\!\int_{U_k} H^2 f_{\varepsilon,\sigma}^p\,d\mu\,dt \le C\|U_k\|^{1-\frac{1}{r}}\,,$$

where $C\doteqdot k_0^p\,(T\mu_0(M))^{\frac{1}{r}}$.

Putting together estimates (8.26), (8.31), and (8.33), we arrive at

$$\|U_h\|\le\frac{SC\sigma p}{(h-k)^p}\|U_k\|^\gamma$$

for all $h>k\ge k_1$, where $\gamma\doteqdot 2-\frac{1}{q_0}-\frac{1}{r}$. Now fix any $r>\frac{q_0}{q_0-1}$ (so that $\gamma>1$) and any $p>\max\{\ell^{-1},\frac{16r}{\ell^2}\}$ and choose $\sigma<\frac{\ell}{2}(pr)^{-\frac{1}{2}}$ sufficiently small that $\sigma p\le 1$. Then we may apply Stampacchia's lemma to conclude that $\|U_k\|=0$ for all $k\ge k_1+d$, where $d^p=SC2^{\frac{\gamma p}{\gamma-1}+1}\|U_{k_1}\|^{\gamma-1}$. That is,

$$f_{\varepsilon,\sigma}\le k_1+d\le K\doteqdot k_1+SC2^{\frac{\gamma p}{\gamma-1}+1}(|\mathcal{M}_0|T)^{\gamma-1}\,.$$

Young's inequality then yields

$$|\mathring{\mathrm{II}}|^2\le\varepsilon H^2+KH^{-\sigma}\le 2\varepsilon H^2+C_\varepsilon\,.$$

This completes the proof of the roundness estimate.

8.3. A gradient estimate for the curvature

Next, we use the roundness estimate to derive an estimate for the gradient of the second fundamental form. Roughly speaking, such an estimate implies that the hypersurface looks umbilic in a neighborhood of a singularity and not only at the singular point.

Theorem 8.14 (Gradient estimate [**291**,**303**]). *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a solution of mean curvature flow for $n \geq 2$ with compact, convex initial datum. For any $\varepsilon > 0$ there exists a constant C_ε — which depends only on ε and the initial embedding — such that*

$$|\nabla \mathrm{II}(x,t)| \leq \varepsilon H^2(x,t) + C_\varepsilon \quad \textit{for all} \quad (x,t) \in M \times [0,T)\,. \tag{8.34}$$

Proof. Recalling Lemma 6.22, we may estimate

$$\begin{aligned}(\partial_t - \Delta)|\nabla \mathrm{II}|^2 &= -2|\nabla^2 \mathrm{II}|^2 + \sum_{i+j+k=1} \nabla^i \mathrm{II} * \nabla^j \mathrm{II} * \nabla^k \mathrm{II} * \nabla \mathrm{II} \\ &\leq c(n)|\mathrm{II}|^2|\nabla \mathrm{II}|^2 - 2|\nabla^2 \mathrm{II}|^2\,.\end{aligned}$$

We will control the bad term using the good term in the evolution equation for $|\mathrm{II}|^2$. The following estimate, which exploits the Codazzi equation to improve the constant in the rough estimate $|\nabla H|^2 \leq n|\nabla \mathrm{II}|^2$, is crucial.

Lemma 8.15. *On any smooth hypersurface $X : M^n \to \mathbb{R}^{n+1}$,*

$$|\nabla \mathrm{II}|^2 \geq \tfrac{3}{n+2}|\nabla H|^2. \tag{8.35}$$

Proof. We decompose

$$\nabla \mathrm{II} = E + F$$

into its trace part

$$E_{ijk} \doteqdot \tfrac{1}{n+2}\left(\nabla_i H g_{jk} + \nabla_j H g_{ki} + \nabla_k H g_{ij}\right)$$

and its trace-free part $F_{ijk} \doteqdot \nabla_i \mathrm{II}_{jk} - E_{ijk}$. As E and F are orthogonal (a fact we invite the reader to check) we conclude

$$|\nabla \mathrm{II}|^2 \geq |E|^2 = \tfrac{3}{n+2}|\nabla H|^2\,. \qquad \square$$

By the roundness estimate, given any $\varepsilon > 0$ there is a constant $C_\varepsilon > 0$ such that $|\mathring{\mathrm{II}}|^2 \leq \varepsilon H^2 + C_\varepsilon$. Thus, fixing any $\varepsilon > 0$,

$$G_\varepsilon \doteqdot 2C_\varepsilon + \varepsilon H^2 - |\mathring{\mathrm{II}}|^2 \geq C_\varepsilon > 0\,.$$

Similarly, there is a constant $C_0 > 0$ such that $|\mathring{\mathrm{II}}|^2 \leq \frac{(n-1)}{n(n+2)}H^2 + C_0$, which, setting $\delta \doteqdot \frac{3}{n+2}$, ensures that

$$G_0 \doteqdot 2C_0 + \delta H^2 - |\mathrm{II}|^2 = 2C_0 + \tfrac{2(n-1)}{n(n+2)}H^2 - |\mathring{\mathrm{II}}|^2 \geq C_0 + \tfrac{(n-1)}{n(n+2)}H^2 > 0\,.$$

We compute

$$(\partial_t - \Delta)G_\varepsilon = 2|\mathrm{II}|^2(G_\varepsilon - 2C_\varepsilon) + 2|\nabla \mathrm{II}|^2 - 2\left(\tfrac{1}{n} + \varepsilon\right)|\nabla H|^2 .$$

Since $G_\varepsilon \geq C_\varepsilon$, we can estimate $G_\varepsilon - 2C_\varepsilon \geq -G_\varepsilon$ and, assuming $\varepsilon \leq \varepsilon_0 \doteqdot \frac{n-1}{n(n+2)}$, we can estimate

$$|\nabla \mathrm{II}|^2 - \left(\tfrac{1}{n} + \varepsilon\right)|\nabla H|^2 \geq \frac{\kappa}{2}|\nabla \mathrm{II}|^2 ,$$

where $\kappa \doteqdot \frac{2(n-1)}{3n}$. Thus,

$$(\partial_t - \Delta)G_\varepsilon \geq\; -2|\mathrm{II}|^2 G_\varepsilon + \kappa |\nabla \mathrm{II}|^2 .$$

Similarly,

$$(\partial_t - \Delta)G_0 \geq\; -2|\mathrm{II}|^2 G_0 .$$

We seek a bound[2] for the ratio $\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}$. Note that, at a local spatial maximum of $\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}$,

$$0 = \nabla_k \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0} = 2\frac{\langle \nabla_k \nabla \mathrm{II}, \nabla \mathrm{II}\rangle}{G_\varepsilon G_0} - \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}\left(\frac{\nabla_k G_\varepsilon}{G_\varepsilon} + \frac{\nabla_k G_0}{G_0}\right) .$$

In particular,

$$4\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}\left\langle \frac{\nabla G_\varepsilon}{G_\varepsilon}, \frac{\nabla G_0}{G_0}\right\rangle \leq \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}\left|\frac{\nabla G_\varepsilon}{G_\varepsilon} + \frac{\nabla G_0}{G_0}\right|^2 \leq 4\frac{|\nabla^2 \mathrm{II}|^2}{G_\varepsilon G_0} .$$

Suppose that $\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}$ reaches a new interior minimum at (x_0, t_0). Then

$$\begin{aligned}
0 &\leq (\partial_t - \Delta)\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0} \\
&= \frac{(\partial_t - \Delta)|\nabla \mathrm{II}|^2}{G_\varepsilon G_0} - \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}\left(\frac{(\partial_t - \Delta)G_\varepsilon}{G_\varepsilon} + \frac{(\partial_t - \Delta)G_0}{G_0}\right) \\
&\quad + \frac{2}{G_\varepsilon G_0}\left\langle \nabla \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}, \nabla(G_\varepsilon G_0)\right\rangle + 2\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}\left\langle \frac{\nabla G_\varepsilon}{G_\varepsilon}, \frac{\nabla G_0}{G_0}\right\rangle \\
&\leq \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}\left((c+4)|\mathrm{II}|^2 - \kappa \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon}\right)
\end{aligned}$$

at (x_0, t_0). It follows that

$$\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0} \leq \frac{(c+4)}{\kappa}\frac{|\mathrm{II}|^2}{G_0} \leq \frac{(c+4)}{\kappa}\frac{n(n+2)}{n-1}\frac{|\mathrm{II}|^2}{H^2} \leq \frac{(c+4)}{\kappa}\frac{n(n+2)}{n-1}$$

[2]Eventually, we will want to take ε arbitrarily small, but we will need to keep δ positive and independent of ε, which explains why we do not simply consider $\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon^2}$.

at any local parabolic maximum of $\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}$. On the other hand, if no such point exists, then

$$\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0} \leq C_0 \doteqdot \max_{M\times\{0\}} \frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0}\,.$$

We conclude that

$$\frac{|\nabla \mathrm{II}|^2}{G_\varepsilon G_0} \leq C \doteqdot \max\left\{C_0, \frac{(c+4)}{\kappa}\frac{n(n+2)}{n-1}\right\}.$$

The claim now follows from Young's inequality. □

We will need the following consequence of the gradient estimate.

Corollary 8.16. *Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$, where $n \geq 2$, be a maximal solution of mean curvature flow evolving from a compact, convex initial embedding $X_0 : M^n \to \mathbb{R}^{n+1}$. Then*

$$\lim_{t\to T} \frac{H_{\max}}{H_{\min}} = 1 \quad \text{and} \quad \lim_{t\to T} \operatorname{diam}_t(M) = 0\,, \tag{8.36}$$

where $H_{\max} \doteqdot \max_{M\times\{t\}} H$, $H_{\min} \doteqdot \min_{M\times\{t\}} H$, and $\operatorname{diam}_t(M) \doteqdot \max_{x,y\in M} d_{g(t)}(x,y)$ is the intrinsic diameter of $(M^n, g(t))$.

Proof. By the gradient estimate (Theorem 8.14), for every $\eta > 0$ there is a constant $C_\eta < \infty$ such that

$$|\nabla H| \leq \frac{1}{2}\eta^2 H^2 + C_\eta\,.$$

Since $H_{\max}(t) \to \infty$ as $t \to T$, there is, for every $\eta > 0$, some point $(x_\eta, t_\eta) \in M \times [0,T)$ such that

$$H_\eta^2 \doteqdot H^2(x_\eta, t_\eta) = H^2_{\max}(t_\eta) \geq 8C_\eta/\eta^2$$

and hence

$$|\nabla H|(x, t_\eta) \leq \eta^2 H(x_\eta, t_\eta)^2$$

for all $x \in M$. Now let γ be a unit speed geodesic (with respect to the metric at time t_η) through $\gamma(0) = x_\eta$. Then, for each $s \leq L \doteqdot \eta^{-1}H_\eta^{-1}$, the mean value theorem provides some $s_0 \in (0,s)$ such that

$$H(\gamma(s), t_\eta) = H_\eta + s\nabla_{\gamma'(s_0)} H(\gamma(s_0), t_\eta) \geq H_\eta(1-\eta)\,. \tag{8.37}$$

Next, recall from the trace of the Gauß equation that

$$\mathrm{Rc} = H\mathrm{II} - \mathrm{II}^2\,.$$

Thus, applying the estimate $\mathrm{II} \geq \frac{\alpha}{n} g$,

$$\mathrm{Rc}(\gamma', \gamma') \geq (n-1)\frac{\alpha^2}{n^2}H^2 \geq (n-1)\frac{\alpha^2}{n^2}H_\eta^2(1-\eta)\,.$$

If $\eta < \frac{1}{2}$, then

$$\mathrm{Rc}(\gamma', \gamma') \geq (n-1)K^2 g\,,$$

where $K \doteqdot \frac{\alpha}{2n} H_\eta$. Choosing further $\eta \leq \frac{\alpha}{2n\pi}$, we obtain $L \geq \pi K^{-1}$. Myers's theorem then implies that every point of M_{t_η} is reached by a geodesic of length at most L and we conclude from (8.37) that

$$H_{\min}(t_\eta) \geq (1-\eta) H_{\max}(t_\eta)\,.$$

Finally, since $H_{\min}$ is nondecreasing,

$$H^2_{\max}(t) \geq (1-\eta)^2 H^2_{\max}(t_\eta) \geq \frac{1}{4} H^2_\eta \quad \text{for all} \quad t \geq t_\eta\,.$$

We leave it to the reader to check that the above arguments then hold for all $t \geq t_\eta$. We conclude that, given any $\eta \leq \min\{\frac{\alpha}{2n\pi}, \frac{1}{2}\}$, there is some time $t_\eta \in [0, T)$ such that

$$\mathrm{diam}_t(M) \leq \frac{1}{\eta H_{\max}(t)} \quad \text{and} \quad H_{\min}(t) \geq (1-\eta) H_{\max}(t)$$

for all $t > t_\eta$. The corollary follows. □

8.4. Huisken's theorem

We now have all the ingredients in place needed to prove Huisken's theorem (Theorem 8.1).

8.4.1. Convergence to a point. By the avoidance principle, the precompact regions $\Omega_t \subset \mathbb{R}^{n+1}$ bounded by the time slices $\mathcal{M}_t$ satisfy $\Omega_{t_2} \subset \Omega_{t_1}$ for all $0 \leq t_1 \leq t_2 < T$. Since $\mathrm{diam}(\mathcal{M}_t) \to 0$ as $t \to T$ (Corollary 8.16), we conclude that $\bigcap_{t\in[0,T)} \Omega_t$ is a single point, p say.

This proves the first part of Theorem 8.1. To complete the proof, we need to obtain appropriate decay estimates for the rescaled solution and its derivatives.

8.4.2. Estimates in C^0 after rescaling. We first show that the images of the rescaled embeddings converge in the Hausdorff topology to the unit sphere.

By Lemma 8.3 and Corollary 8.16, the mean curvature satisfies

$$\sqrt{2n(T-t)}\,\tfrac{1}{n} H(x,t) \to 1 \ \text{ as } \ t \to T \tag{8.38}$$

uniformly in $x \in M$. Integration in time then yields

$$\limsup_{t\to T} \frac{\sup_{x\in M} \|X(x,t) - p\|}{\sqrt{2n(T-t)}} \leq \limsup_{t\to T} \frac{\sup_{x\in M} \int_t^T H(x,s)\,ds}{\sqrt{2n(T-t)}} = 1\,. \tag{8.39}$$

To obtain the reverse inequality, we first combine the roundness estimate (Theorem 8.6) with the uniform curvature blow-up (Corollary 8.16) to deduce

$$\frac{|\mathring{\mathrm{II}}|^2}{H^2}(x,t) \to 0 \ \text{ as } \ t \to T$$

uniformly in $x \in M$. It follows that

$$\sqrt{2n(T-t)}\,\frac{\mathrm{II}_{(x,t)}(v,v)}{g_{(x,t)}(v,v)} \to 1 \ \text{ as } \ t \to T \tag{8.40}$$

uniformly in $x \in M$ and $v \in T_xM \setminus \{0\}$. In particular,

$$\liminf_{t\to T} \frac{\rho_-(t)}{\sqrt{2n(T-t)}} \geq \liminf_{t\to T} \frac{1}{\sqrt{2n(T-t)}\,\max\limits_{\mathcal{M}\times\{t\}} \kappa_n} = 1\,, \tag{8.41}$$

where

$$\rho_-(t) \doteqdot \sup\{r : B_r(q) \subset \Omega_t \text{ for some } q \in \mathbb{R}^{n+1}\}$$

is the **inradius** of $\mathcal{M}_t$. It follows that

$$\frac{\|p_t - p\|}{\sqrt{2n(T-t)}} \to 0\,,$$

where p_t is the center of the largest ball enclosed by $\mathcal{M}_t$. Indeed, whenever $p_t \neq p$, we can choose $x_t \in \mathcal{M}$ such that p_t lies on the segment joining $X(x_t,t)$ and p, so that

$$\|p - p_t\| = \|X(x_t,t) - p\| - \|X(x_t,t) - p_t\| \leq \|X(x_t,t) - p\| - \rho_-(t) \tag{8.42}$$

and the claim follows from (8.39) and (8.41). Estimating

$$\frac{\|X(x,t) - p\|}{\sqrt{2n(T-t)}} \geq \frac{\|X(x,t) - p_t\|}{\sqrt{2n(T-t)}} - \frac{\|p_t - p\|}{\sqrt{2n(T-t)}} \to 1 \ \text{ as } \ t \to T$$

uniformly in $x \in \mathcal{M}$, we may now conclude that

$$\frac{\|X(x,t) - p\|}{\sqrt{2n(T-t)}} \to 1 \ \text{ as } \ t \to T$$

uniformly in $x \in M$. This proves Hausdorff convergence of the image hypersurfaces to the unit sphere. To obtain C^0 convergence of the parametrizations, it will suffice to show that the derivative of the rescaled family of embeddings converges.

8.4.3. Estimates in C^1 after rescaling. Recalling the evolution equation (6.15) for the induced metric g we find, for any $x \in M$ and any nonzero $v \in \mathbb{R}^n$, that the rescaled metric satisfies

$$\partial_t \log \frac{g_{(x,t)}(v,v)}{2n(T-t)} = \frac{2n}{2n(T-t)} - 2H(x,t)\frac{\mathrm{II}_{(x,t)}(v,v)}{g_{(x,t)}(v,v)}\,.$$

Thus, to prove that the rescaled metric has a nondegenerate limit, it suffices to show that

$$-\infty < \int_0^T \left(\frac{1}{2n(T-t)} - \tfrac{1}{n}H(x,t)\frac{\mathrm{II}_{(x,t)}(v,v)}{g_{(x,t)}(v,v)}\right) dt < \infty$$

for all $x \in \mathcal{M}$ and $v \in T_x\mathcal{M} \setminus \{0\}$.

We first consider the lower bound. Recalling that

$$\frac{1}{n}H_{\min} \leq \frac{1}{\sqrt{2n(T-t)}}\,,$$

we may estimate

$$\tfrac{1}{n}H\frac{\mathrm{II}(v,v)}{g(v,v)} - \frac{1}{2n(T-t)} \leq \tfrac{1}{n}H|\mathring{\mathrm{II}}| + \frac{1}{n^2}\left(H_{\max}^2 - H_{\min}^2\right)\,. \tag{8.43}$$

In order to estimate these terms effectively, we will need to obtain a better decay rate for the terms on the right than given by Corollary 8.16 and (8.40). The following lemma will allow us to bound both terms by an integrable power of the remaining time.

Lemma 8.17. *There are constants $N < \infty$, $C < \infty$, and $\delta > 0$ such that*

$$N\frac{|\mathring{\mathrm{II}}|^2}{H^2} + \frac{|\nabla \mathrm{II}|^2}{H^4} \leq C(T-t)^{\delta}\,. \tag{8.44}$$

Proof. Recalling Lemma 6.22, we may estimate

$$(\partial_t - \Delta)|\nabla \mathrm{II}|^2 \leq (4+c)|\mathrm{II}|^2|\nabla \mathrm{II}|^2 - 2|\nabla^2 \mathrm{II}|^2\,,$$

where c is a constant which depends only on n. Thus,

$$\begin{aligned}(\partial_t - \Delta)\frac{|\nabla \mathrm{II}|^2}{H^4} &\leq c\frac{|\nabla \mathrm{II}|^2}{H^2} - 2\frac{|\nabla^2 \mathrm{II}|^2}{H^4} + 8\left\langle \nabla\frac{|\nabla \mathrm{II}|^2}{H^4}, \frac{\nabla H}{H}\right\rangle \\ &\leq c\frac{|\nabla \mathrm{II}|^2}{H^2} - 2\frac{|\nabla^2 \mathrm{II}|^2}{H^4} + 16n\frac{|\nabla^2 \mathrm{II}||\nabla \mathrm{II}|^2}{H^5} \\ &\leq c\frac{|\nabla \mathrm{II}|^2}{H^2} - \frac{|\nabla^2 \mathrm{II}|^2}{H^4} + 64n^2\frac{|\nabla \mathrm{II}|^4}{H^6}\,.\end{aligned}$$

Since $H \sim (T-t)^{-\frac{1}{2}} \to \infty$, the gradient estimate (Theorem 8.14) implies that

$$\frac{|\nabla \mathrm{II}|^2}{H^2} \leq H^2$$

and hence

$$(\partial_t - \Delta)\frac{|\nabla \mathrm{II}|^2}{H^4} \leq -\frac{|\nabla^2 \mathrm{II}|^2}{H^4} + (c + 64n^2)\frac{|\nabla \mathrm{II}|^2}{H^2}$$

for t sufficiently close to T. By decomposing $\nabla^2\,\mathrm{II}$ into certain (orthogonal) symmetric and skew-symmetric parts and recalling (8.20), we can estimate

$$|\nabla^2 \mathrm{II}|^2 \geq \gamma H^4 |\mathring{\mathrm{II}}|^2\,, \tag{8.45}$$

where $\gamma > 0$ depends only on n and the initial pinching constant α. Thus,

$$(\partial_t - \Delta)\frac{|\nabla \mathrm{II}|^2}{H^4} \leq -\gamma|\mathring{\mathrm{II}}|^2 + c'\frac{|\nabla \mathrm{II}|^2}{H^2}$$

for t sufficiently close to T, where $c' \doteqdot c + 64n^2$.

On the other hand, the roundness estimate (Theorem 8.6) and Lemma 8.8 yield

$$\begin{aligned}(\partial_t - \Delta)\frac{|\mathring{\mathrm{II}}|^2}{H^2} &\leq 4\left\langle \nabla\frac{|\mathring{\mathrm{II}}|^2}{H^2}, \frac{\nabla H}{H}\right\rangle - 2\gamma'\frac{|\nabla \mathrm{II}|^2}{H^2} \\ &\leq 16n\frac{|\nabla \mathrm{II}|^2}{H^2}\frac{|\mathring{\mathrm{II}}|}{H} - 2\gamma'\frac{|\nabla \mathrm{II}|^2}{H^2} \leq -\gamma'\frac{|\nabla \mathrm{II}|^2}{H^2}\end{aligned}$$

for t sufficiently close to T, where $\gamma' > 0$ is a constant which depends only on n and the initial pinching constant α. Setting $N \doteqdot \frac{c'+\delta}{\gamma'}$, where δ is the positive solution of

$$(c' + \delta)\delta = \gamma\gamma'\,,$$

we find

$$\begin{aligned}(\partial_t - \Delta)\left(\frac{|\nabla \mathrm{II}|^2}{H^4} + N\frac{|\mathring{\mathrm{II}}|^2}{H^2}\right) &\leq -\gamma|\mathring{\mathrm{II}}|^2 - \delta\frac{|\nabla \mathrm{II}|^2}{H^2} \\ &= -\delta H^2\left(N\frac{|\mathring{\mathrm{II}}|^2}{H^2} + \frac{|\nabla \mathrm{II}|^2}{H^4}\right) \\ &\leq -\frac{\delta n}{4(T-t)}\left(N\frac{|\mathring{\mathrm{II}}|^2}{H^2} + \frac{|\nabla \mathrm{II}|^2}{H^4}\right)\end{aligned}$$

for t sufficiently close to T. The comparison principle then yields

$$\log\left(\frac{|\nabla \mathrm{II}|^2}{H^4} + N\frac{|\mathring{\mathrm{II}}|^2}{H^2}\right)\Bigg|_{t_0}^{t} \leq \frac{\delta n}{4}\log(T-t)\Big|_{t_0}^{t}$$

for all $t > t_0$ for t_0 sufficiently close to T. The claim follows. $\square$

Corollary 8.18. *There are constants $C < \infty$ and $\delta > 0$ such that*

$$H_{\max} - H_{\min} \leq C(T-t)^{\delta} H_{\max}\,. \tag{8.46}$$

Proof. This is proved by integrating the gradient bound of the previous lemma along geodesics, as in Corollary 8.16. See Exercise 8.8. □

We conclude that

$$\int_0^T \partial_t \left(\frac{g_{(x,t)}(v,v)}{2n(T-t)} \right) dt > -\infty \,.$$

To obtain the upper bound, observe that

$$H\frac{\mathrm{II}(v,v)}{g(v,v)} - \frac{1}{2(T-t)} \geq -H|\mathring{\mathrm{II}}| - \frac{1}{n}\left(H_{\max}^2 - H_{\min}^2\right) + \frac{1}{n}H_{\max}^2 - \frac{1}{2(T-t)} \,.$$

Lemma 8.19. *There are constants $C < \infty$ and $\delta > 0$ such that*

$$\frac{1}{2n(T-t)} - \frac{1}{n^2}H_{\max}^2 \leq C(T-t)^{\delta-1} \,. \tag{8.47}$$

Proof. By Lemma 8.17, we may estimate

$$(\partial_t - \Delta)H = |\mathrm{II}|^2 H = \left(\frac{|\mathring{\mathrm{II}}|^2}{H^2} + \frac{1}{n} \right) H^3 \leq \left(C(T-t)^\delta + \frac{1}{n} \right) H^3 \,.$$

The ODE comparison principle then yields

$$H_{\max}^{-2}(t) \leq 2\left(\frac{C}{\delta+1}(T-t)^{\delta+1} + \frac{1}{n}(T-t) \right)$$

and hence

$$\frac{1}{n^2}H_{\max}^2(t) - \frac{1}{2n(T-t)} \geq -\frac{\frac{nC}{\delta+1}(T-t)^\delta}{2n(T-t)\left[\frac{nC}{\delta+1}(T-t)^\delta + 1\right]} \,.$$

The claim follows. □

It follows that

$$\int_0^T \partial_t \left(\frac{g_{(x,t)}(v,v)}{2n(T-t)} \right) dt < \infty \,.$$

We can now conclude that the rescaled metrics converge uniformly as $t \to T$ to a nondegenerate limit metric.

8.4.4. Estimates in C^∞ after rescaling. We have already proved that the rescaled Weingarten tensor converges to the identity map (8.40) and that its derivative decays to zero (Lemma 8.17). The Bernstein estimates (Theorem 6.24) yield bounds for all higher derivatives of the rescaled curvature, so we can obtain decay estimates by interpolation.

Proposition 8.20 (Interpolation inequality for tensor fields)**.** *Let T be a smooth tensor field of rank (p, q) on a compact Riemannian n-manifold (M^n, g) equipped with a metric connection ∇. For every $k \in \mathbb{N}$ there exists a constant $c_k < \infty$, which depends only on n, $p + q$, and k, such that*

$$|\nabla^k T|^2_{C^0} \leq c_k |\nabla^{k-1} T|_{C^0} |\nabla^{k+1} T|_{C^0} . \tag{8.48}$$

Proof. Let T be a tensor field of covariant order p and contravariant order q. It suffices to prove the claim when $k = 1$ since, for other values of k, the claim follows immediately upon replacing T with $\nabla^{k-1}T$.

Consider first the case that $|T|_{C^0} |\nabla^2 T|_{C^0} = 0$. We claim that $|\nabla T|_{C^0} = 0$. This is certainly the case if $|T|_{C^0} = 0$, so suppose that $|\nabla^2 T|_{C^0} = 0$. Consider the function $f : M^n \to \mathbb{R}$ defined by

$$f(x) \doteqdot \frac{1}{2} |T_x|^2 .$$

If $|\nabla T|_{C^0} \neq 0$, then we can find $x \in M^n$ and a unit vector $v \in T_x M^n$ such that $\nabla_v T_x \neq 0$. Let $\gamma : \mathbb{R} \to M^n$ be the geodesic with $(\gamma(0), \gamma'(0)) = (x, v)$. Observe that

$$\frac{d}{ds} |\nabla_{\gamma'} T|^2 = 2g(\nabla_{\gamma'} \nabla_{\gamma'} T, \nabla_{\gamma'} T) = 0 .$$

So $|\nabla_{\gamma'} T|^2 \equiv |\nabla_v T_x|^2 \neq 0$. A similar calculation yields

$$\frac{d^2}{ds^2} (f \circ \gamma) = |\nabla_{\gamma'} T|^2 .$$

It follows that f grows linearly along γ. But this is impossible since M^n is compact (and hence f is bounded).

So we may assume that $|T|_{C^0} |\nabla^2 T|_{C^0} \neq 0$. Fix $x \in M^n$, unit vectors $u, u_1, \ldots, u_p \in T_x M^n$, and unit covectors $\alpha_1, \ldots, \alpha_q \in T_x^* M^n$. Let $\gamma : \mathbb{R} \to M^n$ be the unit speed geodesic with $\gamma'(0) = u$. Extend $u, u_1, \ldots, u_p$ and $\alpha_1, \ldots, \alpha_q$ by parallel translation to form vector and covector fields, respectively, along γ. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(s) = T(u_1, \ldots, u_p, \alpha_1, \ldots, \alpha_q)(\gamma(s)) .$$

Recalling the interpolation inequality (3.44) for real-valued functions, we obtain at the point x that

$$|\nabla_u T(u_1, \ldots, u_p, \alpha_1, \ldots, \alpha_q)| = f'(0) \leq 2 \sup_{s \in \mathbb{R}} |f(s)|^{1/2} \sup_{s \in \mathbb{R}} |f''(s)|^{1/2} .$$

It follows that

$$|\nabla T|_0^2 \leq 4 |T|_0 |\nabla^2 T|_0 ,$$

where $|\cdot|_0$ is the operator norm. The claim follows since the operator norm is equivalent to the norm induced by g, with the constant depending only on n and $p + q$. $\square$

Lemma 8.21. *For every $\gamma > 0$ and each $k \in \mathbb{N}$ there exists a constant $C_k < \infty$ such that*

$$(T-t)^{1+k}|\nabla^k \mathrm{II}|^2 \le C_k (T-t)^{\gamma}\,. \tag{8.49}$$

Proof. By combining the decay estimate of Lemma 8.17 with the tensor interpolation inequalities of Proposition 8.20 as in the proof of Grayson's theorem (see Lemma 3.27), it suffices to bound the rescaled derivatives $(T-t)^{k+1}|\nabla^k \mathrm{II}|^2$ by some constant, $\widetilde{C}_k$ say, for each $k \ge 2$. We can achieve this using the rapid smoothing estimates (Theorem 6.24) exactly as in the proof of Grayson's theorem (see Lemma 3.25). □

To obtain estimates in C^∞ for the rescaled embeddings, we adapt the proof of Theorem 6.20. We recall that the rescaled embeddings $\widetilde{X}(\cdot,t) : \mathcal{M}^n \to \mathbb{R}^{n+1}$ are defined for $t \in [0,T)$ by

$$\widetilde{X}(\cdot,t) \doteqdot \frac{X(\cdot,t)-p}{\sqrt{2n(T-t)}}\,.$$

As in the proof of Theorem 6.20, we work with a fixed (time-independent) metric $\langle\cdot,\cdot\rangle$ and connection D (the initial ones, say) on $T\mathcal{M}^n$.

Lemma 8.22. *There exists $\lambda > 0$ such that*

$$\widetilde{g}_t \ge \lambda \widetilde{g}_0$$

for all $t \in [0,T)$, where $\widetilde{g}_t$ is the metric induced by $\widetilde{X}(\cdot,t)$.

There exist $\delta > 0$ and, for each $m \in \mathbb{N}$, $C_m < \infty$ such that

$$|D^m \widetilde{X}| + (T-t)^{1-\delta}|D_t D^m \widetilde{X}| \le C_m\,. \tag{8.50}$$

Proof. The first claim was proved in Section 8.4.3. So we are free to work with either metric.

Observe that

$$2(T-t)\partial_t \widetilde{X} = \widetilde{X} - \tfrac{1}{n}\widetilde{H}\widetilde{\mathsf{N}}\,.$$

In particular,

$$\begin{aligned} 2(T-t)D_t\widetilde{X}_* &= \widetilde{X}_* - D\widetilde{H}\otimes\widetilde{\mathsf{N}} - \tfrac{1}{n}\widetilde{H}\widetilde{X}_*\widetilde{\mathrm{L}} \\ &= \left(1-\tfrac{1}{n^2}\widetilde{H}^2\right)\widetilde{X}_* - D\widetilde{H}\otimes\widetilde{\mathsf{N}} - \widetilde{H}\widetilde{X}_*\left(\widetilde{\mathrm{L}}-\tfrac{1}{n}\widetilde{H}\mathrm{I}\right)\,. \end{aligned}$$

It follows from Lemma 8.17 and Corollary 8.18 that

$$|D_t\widetilde{X}_*| \le C_1(T-t)^{\delta-1}$$

for some $\delta > 0$ and $C_1 < \infty$. Integrating then yields a bound for $|\widetilde{X}_*|$, which proves the claim when $m = 1$. It is not hard to see that the claim follows in general, for if we differentiate $2(T-t)D_t\widetilde{X}_*$ a further m times, then, assuming that the claim holds up to order m, we see that each factor will contain bounded terms multiplied by either a nontrivial derivative of the curvature, the factor $\left(1 - \frac{1}{n^2}\widetilde{H}^2\right)$, or the factor $\left(\widetilde{\mathrm{L}} - \frac{1}{n}\widetilde{H}\mathrm{I}\right)$, each of which is bounded by a constant multiple of $(T-t)^{\delta-1}$. So we obtain

$$|D_t D^{m+1}\widetilde{X}| \leq C_1(T-t)^{\delta-1}\,.$$

Integration then yields a bound for $|D^{m+1}\widetilde{X}|$. □

Thus,

$$\partial_t |D^m \widetilde{X}|^2 \leq 2C_m^2 (T-t)^{\delta-1}\,.$$

Integrating, we find that $|D^m X|$ is Cauchy in t for each $m \in \mathbb{N}$.

We can now conclude that the rescaled embeddings $\widetilde{X}(\cdot, t)$ converge in $C^\infty(M^n, \mathbb{R}^{n+1})$ as $t \to T$ to an embedding whose image coincides with the unit sphere: Choose a finite number of compact sets $K_1, \ldots, K_\mu$ whose interiors cover M and charts $\varphi_1 : U_1 \to \mathbb{R}^n, \ldots, \varphi_\mu : U_\mu \to \mathbb{R}^n$ with $K_i \subset U_i$ for each $i = 1, \ldots, \mu$. Set $\widetilde{X}_i \doteqdot \widetilde{X} \circ \varphi_i$. Since the components of the metric and connection with respect to each coordinate chart are uniformly bounded in each of the corresponding compact sets, equations (6.86) and (6.87) and a simple induction argument imply that $D\widetilde{X}_i$ is Cauchy with respect to t in the space $C^k(K_i, \mathbb{R}^n)$ for each $k \in \mathbb{N}$. Since $C^k(K_i, \mathbb{R}^{n+1})$ is complete, it follows that $D\widetilde{X}_i(\cdot, t)$ converges in $C^\infty(U_i, \mathbb{R}^{n+1})$ for each $i = 1, \ldots, \mu$. We can then conclude, as in the proof of Grayson's theorem, that $\widetilde{X}_i(\cdot, t)$ converges in $C^\infty(K_i, \mathbb{R}^{n+1})$ for each $i = 1, \ldots, \mu$: Let p_t be the center of an inscribed sphere at each time t. Then $\frac{X_i(\cdot,t) - p_t}{\sqrt{2n(T-t)}}$ converges in $C^\infty(U_i, \mathbb{R}^{n+1})$ for each $i = 1, \ldots, \mu$. The arguments of Section 8.4.2 show that rescaled distance $\frac{\|p - p_t\|}{\sqrt{2n(T-t)}}$ goes to zero as $t \to T$, so in fact $\widetilde{X}_i(\cdot, t)$ converges. Pasting the pieces together, we conclude that $\widetilde{X}(\cdot, t)$ converges in the compact-open C^∞ topology to a smooth embedding as $t \to T$. Its image must be the unit sphere by (8.42).

8.5. Regularity of the arrival time

The estimates for the second fundamental form and its derivatives, derived in the previous section, imply that the arrival time of a convex solution to mean curvature flow, a priori only Lipschitz regular, is in fact of class C^2 [**296**, Theorem 6.1].

Theorem 8.23. *The arrival time of a convex, compact solution to mean curvature flow is of class C^2.*

Proof. Let $\{\mathcal{M}^n_t\}_{t\in[0,T)}$ be a solution to mean curvature flow with $\mathcal{M}^n_0$ bounding a bounded, convex region Ω. Recall that the arrival time $u : \overline{\Omega} \to \mathbb{R}$ is defined by

$$u(X) = t \text{ if and only if } X \in \mathcal{M}^n_t\,.$$

Since the hypersurfaces move inward and converge to a single point p as $t \to T$, u is well-defined if we set $u(p) \doteqdot T$. The claim is that $u \in C^2(\Omega)$. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a smooth family of parametrizations $X(\cdot, t)$ of $\mathcal{M}^n_t$. Then

$$u(X(x,t)) = t\,. \tag{8.51}$$

Fix a point $q = X(x,t)$ in Ω and local g_t-normal coordinates $\{x^i\}_{i=1}^n$ for M^n about x. Choose the basis $\{e_i\}_{i=1}^{n+1}$ for $\mathbb{R}^{n+1}$ so that $e_{n+1} = \mathsf{N}(x,t)$ and $e_i = \partial_i X(x,t)$ for each $i = 1, \ldots, n$. Differentiating (8.51) yields the identities

$$Du \cdot \partial_i X = 0 \text{ and } -HDu \cdot \mathsf{N} = 1\,. \tag{8.52}$$

Thus,

$$Du = -\frac{\mathsf{N}}{H}\,.$$

Differentiating (8.52) at the point $q = X(x,t)$ then yields

$$D^2u = \begin{pmatrix} -\mathrm{II}/H & \nabla H/H^2 \\ \nabla H/H^2 & -\partial_t H/H^3 \end{pmatrix}. \tag{8.53}$$

By Lemmas 8.17 and 8.21,

$$\frac{\partial_t H}{H^3} = \frac{\Delta H}{H^3} + \frac{|\mathrm{II}|^2}{H^2} \to \frac{1}{n}$$

as $t \to T$. Thus, by Lemma 8.17,

$$D^2 u \to \frac{1}{n}\,\mathrm{I}$$

as $q \to p$. The claim follows. □

In the 1-dimensional setting, u is even C^3 [**334**]. This is not true in higher dimensions, however [**477**].

8.6. Huisken's theorem via width pinching

We shall now present a quite different proof of Huisken's theorem, which appears in [**26**] in the context of a more general class of hypersurface flows. The principal observation is the general fact that pinching of the principal curvatures implies pinching of the widths. Recall that the **width** $w(z)$ of a compact, convex set $\Omega \subset \mathbb{R}^{n+1}$ in a given direction $z \in S^n$ is defined by

$$w(z) \doteqdot \sigma(z) + \sigma(-z)\,, \tag{8.54}$$

where $\sigma : S^n \to \mathbb{R}$ is the **support function** of Ω, defined by

$$\sigma(z) \doteqdot \sup_{X\in\Omega} \langle X, z\rangle \ .$$

Define the **maximum** and **minimum widths** of Ω by

$$w_+ = \max_{z\in S^n} w(z) \ \text{ and } \ w_- = \min_{z\in S^n} w(z)\,, \tag{8.55}$$

respectively. Observe that the maximum width is just the diameter of Ω,

$$w_+ = \max_{X,Y\in\Omega} \|X - Y\|\,.$$

If Ω is open and its boundary $\mathcal{M} \doteqdot \partial\Omega$ is smooth and locally uniformly convex, then

$$\sigma(z) = \left\langle \mathsf{N}^{-1}(z), z\right\rangle \ ,$$

where $\mathsf{N} : \partial\Omega \to S^n$ is the Gauß map of $\mathcal{M}$, and hence

$$\sigma(\mathsf{N}(X)) = \langle X, \mathsf{N}(X)\rangle$$

for any $X \in \partial\Omega$. We will abuse notation by defining $\sigma : \mathcal{M} \to \mathbb{R}$ by

$$\sigma(X) = \langle X, \mathsf{N}(X)\rangle \ .$$

Define the maximal width w_+ of $\mathcal{M}$ and minimal width w_- of $\mathcal{M}$ to be those of Ω. We have the following *width pinching estimate.*

Lemma 8.24 (Andrews [**26**]). *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be a bounded, locally uniformly convex open set with smooth boundary $\mathcal{M} = \partial\Omega$. If the principal curvatures of $\mathcal{M}$ are pointwise pinched, $\kappa_n(x) \leq C\kappa_1(x)$ for all $x \in \mathcal{M}$ for some $C < \infty$, then the widths of $\mathcal{M}$ are pinched by the same constant*:

$$w_+ \leq Cw_-\,. \tag{8.56}$$

Proof. We first prove the claim when $n = 2$. Let e be any unit vector in $\mathbb{R}^3$ and set $N \doteqdot \mathsf{N}^{-1}(e)$ and $S \doteqdot \mathsf{N}^{-1}(-e)$, so that $w(e) = \langle N - S, e\rangle$. Define the curves $\Gamma_h \doteqdot \mathcal{M} \cap \{X \cdot e = h\}$ of constant "height", h. Each curve Γ_h is a convex planar curve and hence its tangent vector $\mathsf{T}(h, \cdot)$ and curvature $\kappa(h, \cdot)$ are defined almost everywhere. We claim that

$$\int_{\mathcal{M}} \mathrm{II}(\mathsf{T}, \mathsf{T})\, d\mu = \int_{w(S)}^{w(N)} \int_{\Gamma_h} \kappa\, ds_h\, dh = 2\pi w(e)\,, \tag{8.57}$$

where s_h is the arc length parameter of Γ_h. The lemma then follows easily:

$$w_+ \leq \frac{1}{2\pi}\int_{\mathcal{M}} \kappa_n d\mu \leq \frac{C}{2\pi}\int_{\mathcal{M}} \kappa_1 d\mu \leq Cw_-\,.$$

To prove (8.57), define $\mathsf{B} \doteqdot \mathsf{T} \times \mathsf{N}$ so that $\{\mathsf{T}, \mathsf{B}\}$ is a consistently oriented, almost everywhere defined basis for $T\mathcal{M}$. Since $\nabla h \propto \mathsf{T}^\perp$, the volume form $d\mu$ for $\mathcal{M}$ can be written as

$$d\mu = \phi\, ds \wedge dh$$

for some L^1 function ϕ, where, abusing notation, ds is the 1-form dual to T. Since ∇h is the projection of e onto $T\mathcal{M}$,

$$1 = d\mu(\mathsf{T},\mathsf{B}) = \phi\, dh(\mathsf{B}) = \phi\, \langle \mathsf{T}\times\mathsf{N}, e\rangle = \phi\,\langle e\times\mathsf{T},\mathsf{N}\rangle\ ,$$

where we used the total antisymmetry of the vector triple product. On the other hand, since the normal vector to Γ_h is equal to $e\times\mathsf{T}$, the Gauß–Weingarten equation implies that

$$\mathrm{II}(\mathsf{T},\mathsf{T}) = \kappa\,\langle e\times\mathsf{T},\mathsf{N}\rangle\,.$$

We conclude that

$$\phi\,\mathrm{II}(\mathsf{T},\mathsf{T}) = \kappa\,.$$

The identity (8.57) follows.

If $n\geq 3$, then the same argument can be applied to the projection of Ω onto any 3-dimensional subspace of $\mathbb{R}^{n+1}$. So we only need to show that, given any 3-dimensional subspace Π of $\mathbb{R}^{n+1}$, the boundary $\Sigma \doteqdot \partial\pi(\Omega)$ of the projection $\pi(\Omega)$ of Ω onto Π satisfies the same pinching estimate as $\mathcal{M}$ almost everywhere. This follows from the fact that the normal is preserved under the projection: Identify S^2 with $\partial\pi(B^{n+1}) = \Pi\cap S^n$. Observe that, for almost every $X\in\mathcal{M}$, $\pi(X)\in\Sigma$ if and only if $\mathsf{N}_{\mathcal{M}}(X) = \mathsf{N}_\Sigma(\pi(X))$ (this can be seen by translating a hyperplane with normal $\mathsf{N}(X)$ from infinity until it touches $\mathcal{M}$). That is, $\mathsf{N}_\Sigma^{-1} = \pi\circ\mathsf{N}_{\mathcal{M}}^{-1}$. Let $\gamma : I\to S^2$ be a curve parametrized by arc length with $\gamma(0) = \mathsf{N}(X)$ and $\frac{d}{ds}\big|_{s=0}\gamma = v\in T_{\mathsf{N}(X)}S^2 = T_X\Sigma$. If N_Σ is smooth at $\pi(X)$ (which is true almost everywhere), then

$$\mathrm{L}_\Sigma^{-1}(v) = \frac{d}{ds}\Big|_{s=0}\mathsf{N}_\Sigma^{-1}\circ\gamma = \frac{d}{ds}\Big|_{s=0}\pi\circ\mathsf{N}_{\mathcal{M}}^{-1}\circ\gamma = d\pi\circ\mathrm{L}_{\mathcal{M}}^{-1}(v)\,.$$

Thus, for any $v\in T_{\mathsf{N}(X)}S^2 = T_X\Sigma$,

$$\left\langle \mathrm{L}_{\mathcal{M}}^{-1}(v), v\right\rangle = \left\langle d\pi\circ\mathrm{L}_\Sigma^{-1}(v), v\right\rangle = \left\langle \mathrm{L}_\Sigma^{-1}(v), v\right\rangle\ . \tag{8.58}$$

The claim follows. □

It will be useful to relate the widths of a convex set Ω to its **circumradius** and **inradius**, which we recall are defined by

$$\rho_+ \doteqdot \inf\left\{r : \Omega\subset B_r(X)\ \text{ for some } X\in\mathbb{R}^{n+1}\right\}\quad\text{and} \tag{8.59a}$$

$$\rho_- \doteqdot \sup\left\{r : B_r(X)\subset\Omega \text{ for some } X\in\mathbb{R}^{n+1}\right\}, \tag{8.59b}$$

respectively. When a smooth hypersurface $\mathcal{M}$ is the boundary of a bounded convex set Ω, we define the circumradius and inradii of $\mathcal{M}$ as those of Ω.

Lemma 8.25. *On any bounded convex body $\Omega\subset\mathbb{R}^{n+1}$,*

$$\rho_+ \leq \tfrac{1}{\sqrt{3}}w_+\ \ \textit{and}\ \ \rho_- \geq \tfrac{1}{n+2}w_-\,. \tag{8.60}$$

Proof. Let S be a sphere of smallest radius which encloses Ω. By translating if necessary, we may arrange that its center is the origin. Choose two points $x, y \in S \cap \Omega$ which maximize the Euclidean distance. The angle between x and y is at least $\frac{2\pi}{3}$ since otherwise we could move S slightly in the direction $x+y$ so that it strictly contains Ω, contradicting the assumption that S has smallest possible radius. Then the distance from x to y is a lower bound for the maximum width w_+ and is at least $\sqrt{3}$ times the radius of S. This proves the first inequality.

Now let S be a sphere of largest radius contained in Ω. By translating if necessary, we may arrange that its center is the origin. We claim that there is a nonempty set of points $P \subset S \cap \partial\Omega$ such that $P \setminus \{z\}$ is linearly independent for any $z \in P$ and such that there is a positive linear combination of the elements of P with value zero. Indeed, if this were not the case, then the convex hull of $S \cap \partial\Omega$ could not contain the origin, and so S could be moved slightly to become properly contained in Ω. Let E be the smallest affine subspace of $\mathbb{R}^{n+1}$ which contains the set P. Note that E has dimension $k-1$, where P has k elements. Let Σ be the simplex $\{y \in E : \langle y, z\rangle \leq \sigma(z) \text{ for all } z \in P\}$, where σ is the support function of Ω. Then Σ contains the projection of Ω onto E since the latter is convex (and hence the intersection of its supporting half-spaces). Hence the minimum width of $\mathcal{M}$ is no greater than the minimum width of Σ, which is the shortest altitude of Σ. This is bounded by the altitude of a regular simplex inscribed by Σ in E, or $k\rho_-$. Since E has dimension at most $n+1$, the second inequality follows. □

Combining Lemmas 8.24 and 8.25, we deduce that a hypersurface with pinched principal curvatures, $\kappa_n(x) \leq C\kappa_1(x)$, satisfies

$$\rho_+ \leq \tfrac{(n+2)}{\sqrt{3}} C\rho_- . \tag{8.61}$$

Now let $\{\Omega_t\}_{t\in[0,T)}$ be a family of precompact convex open sets whose boundaries $\mathcal{M}_t = \partial\Omega_t$ evolve by mean curvature flow and set

$$C \doteqdot \max_{\mathcal{M}_0} \frac{\kappa_n}{\kappa_1} .$$

We suppose that T is the maximal time, so that $\limsup_{t\nearrow T} \max_{\mathcal{M}_t} H = \infty$.

Lemma 8.26. *The inradius $\rho_-(t)$ of Ω_t goes to zero as $t \to T$. Thus, Ω_t converges in the Hausdorff topology to some point $p \in \mathbb{R}^{n+1}$ as $t \to T$.*

Proof. Suppose, to the contrary, that $\rho_0 \doteqdot \lim_{t\to T} \rho_- > 0$ (the limit exists since ρ_- is nonincreasing). After a translation in space, we may arrange that the sphere of radius ρ_0 centered at the origin lies in Ω_t for all $t \in [0, T)$. For each $t \in [0, T)$, we can represent $\mathcal{M}_t$ as a graph over the sphere. That is, we can find a smooth function $r : S^n \times [0,T) \to (\rho_0, \rho_+(0)]$ such that

$$\mathcal{M}_t = \operatorname{graph} r(\cdot, t) = \{r(z,t)\, z : z \in S^n\} . \tag{8.62}$$

For a starshaped hypersurface evolving by the mean curvature flow, we have $\partial_t X = -\frac{H}{\langle z, \mathsf{N}\rangle} z$ (since $(\partial_t X)^\perp = -H\mathsf{N}$). Thus,

$$\partial_t r(z) = -\frac{H(z)}{\langle z, \mathsf{N}(z)\rangle}.$$

Recall from (5.74) and (5.76) that

$$g^{ij} = \frac{1}{r^2}\left(\overline{g}^{ij} - \frac{\overline{\nabla}^i r \overline{\nabla}^j r}{r^2 + \left|\overline{\nabla} r\right|^2}\right) \quad \text{and}$$

$$\mathrm{II}_{ij} = -\frac{r}{\sqrt{r^2 + \left|\overline{\nabla} r\right|^2}}\left(\overline{\nabla}_i \overline{\nabla}_j r - 2\frac{\overline{\nabla}_i r \overline{\nabla}_j r}{r} - r\overline{g}_{ij}\right),$$

respectively. Since $H = g^{ij}\,\mathrm{II}_{ij}$ and $\langle z, \mathsf{N}\rangle = \frac{r}{\sqrt{r^2+|\overline{\nabla} r|^2}}$, we conclude that the radial function r satisfies the equation

$$\partial_t r = \frac{1}{r^2}\left(\overline{g}^{ij} - \frac{\overline{\nabla}^i r \overline{\nabla}^j r}{r^2 + \left|\overline{\nabla} r\right|^2}\right)\left(\overline{\nabla}_i \overline{\nabla}_j r - 2\frac{\overline{\nabla}_i r \overline{\nabla}_j r}{r} - r\overline{g}_{ij}\right). \tag{8.63}$$

Since r is uniformly bounded away from zero, equation (8.63) is uniformly parabolic up to time T, so the regularity theory for uniformly parabolic equations implies that the solution can be extended to a larger time interval (see [**340**, Section 5.5]). Perhaps the easiest way to see this is to observe that the inverse $\mathrm{K} \doteqdot r^{-1}$ satisfies the linear parabolic equation

$$\partial_t \mathrm{K} = a^{ij}\left(\partial_{\theta^i}\partial_{\theta^j}\mathrm{K} - \overline{\Gamma}_{ij}{}^k \partial_{\theta^k}\mathrm{K} + \mathrm{K}\,\overline{g}_{ij}\right)$$

in local coordinates $\{\theta^i\}_{i=1}^n$ for S^n, where

$$a^{ij} \doteqdot \frac{1}{r^2}\left(\overline{g}^{ij} - \frac{\overline{\nabla}^i r \overline{\nabla}^j r}{r^2 + \left|\overline{\nabla} r\right|^2}\right).$$

Since any solution of (8.63) gives rise to a starshaped solution of mean curvature flow via (8.62), this contradicts the maximality of the solution.

The remaining claim follows from (8.61). □

By comparing the solution with the evolution of inscribed and circumscribed spheres, we deduce that

$$\rho_-(t) \le \sqrt{2(T-t)} \le \rho_+(t) . \tag{8.64}$$

We can now deduce, using an argument of Kai-Seng Chou (a.k.a. Kaising Tso) [**506**], that the curvature growth is of the first type.

Lemma 8.27. *There exists* $t_0 \in [0,T)$ *such that*

$$\max_{\mathcal{M}_t} H \leq \frac{C^2}{\sqrt{2n(T-t)}} \quad \text{for all } t \in [t_0, T). \tag{8.65}$$

Proof. Given $t_1 \in [0,T)$, set $\alpha \doteqdot \rho_-(t_1)$. By translating if necessary, we can arrange that the center of a chosen inscribed sphere for Ω_{t_1} is the origin. Then $\sigma(\cdot,t) \geq \rho_-(t) \geq \rho_-(t_1)$ for $t \leq t_1$. Define $q : \mathcal{M} \times [0,t_1] \to \mathbb{R}$ by

$$q \doteqdot \frac{H}{\sigma - \frac{\alpha}{2}}\,.$$

Observe that

$$\begin{aligned}(\partial_t - \Delta) q &= \left\langle \nabla q, \frac{\nabla \sigma}{\sigma - \frac{\alpha}{2}} \right\rangle + \frac{1}{\sigma - \frac{\alpha}{2}} (\partial_t - \Delta) H - \frac{q}{\sigma - \frac{\alpha}{2}} (\partial_t - \Delta)\sigma \\ &= \left\langle \nabla q, \frac{\nabla \sigma}{\sigma - \frac{\alpha}{2}} \right\rangle + |\mathrm{II}|^2 q - \frac{q}{\sigma - \frac{\alpha}{2}} (|\mathrm{II}|^2 \sigma - 2H) \\ &= \left\langle \nabla q, \frac{\nabla \sigma}{\sigma - \frac{\alpha}{2}} \right\rangle - \frac{q}{\sigma - \frac{\alpha}{2}} \left(\frac{\alpha}{2} |\mathrm{II}|^2 - 2H \right).\end{aligned}$$

Estimating $|\mathrm{II}|^2 \geq \frac{1}{n} H^2$ and $\sigma - \frac{\alpha}{2} \geq \frac{\alpha}{2}$, we find

$$(\partial_t - \Delta) q \leq \left\langle \nabla q, \frac{\nabla \sigma}{\sigma - \frac{\alpha}{2}} \right\rangle - q^2 \left(\frac{\alpha^2}{4n} q - 2 \right).$$

Thus, at a point where q achieves a new spatial maximum,

$$0 \leq -q^2 \left(\frac{\alpha^2}{4n} q - 2 \right), \quad \text{which implies} \quad q \leq \frac{8n}{\alpha^2},$$

and hence

$$\max_{\mathcal{M} \times \{t_1\}} q \leq \max \left\{ \max_{\mathcal{M} \times \{0\}} q, \frac{8n}{\alpha^2} \right\}.$$

By Lemma 8.4 we can find some $C < \infty$ such that $\kappa_n \leq C \kappa_1$ and hence, by (8.61), we can estimate $\sigma(\cdot, t_1) \leq \rho_+(t_1) \leq C \rho_-(t_1) = \alpha C$, so that

$$H \leq \left(\sigma - \frac{\alpha}{2} \right) q \leq \frac{C}{2} \max \left\{ \alpha \max_{\mathcal{M} \times \{0\}} q, \frac{8n}{\alpha} \right\}$$

at time t_1. By Lemma 8.26, we can find $t_0 \in [0,T)$ such that $\alpha = \rho_-(t_1) \leq \sqrt{8n / \max_{\mathcal{M} \times \{0\}} q}$ if $t_1 > t_0$. In that case,

$$H(\cdot, t_1) \leq \frac{C}{\alpha}\,.$$

Now, a circumsphere for $\mathcal{M}_{t_1}$ shrinks to a point under mean curvature flow after time $\rho_+(t_1)^2 / 2n$ and hence

$$T \leq t_1 + \frac{\rho_+(t_1)^2}{2n} \leq t_1 + \frac{\alpha^2 C^2}{2n}\,.$$

We conclude that

$$H(\cdot, t_1) \leq \frac{C^2}{\sqrt{2n(T-t_1)}}.$$

The claim follows since $t_1 \in [t_0, T)$ can be chosen arbitrarily. □

We now have enough control to obtain convergence of the solution after rescaling. The strong maximum principle implies that the limit is round.

Corollary 8.28. *Set*

$$A_0(t) \doteqdot \max_{\mathcal{M}\times\{t\}} |\mathring{\mathrm{II}}|^2 \quad \text{and, for each } m \in \mathbb{N}, \quad A_m \doteqdot \max_{\mathcal{M}\times\{t\}} |\nabla^m \mathrm{II}|^2 .$$

Then, for every $m \in \mathbb{N} \cup \{0\}$,

$$\limsup_{t\nearrow T} (T-t)^{-(m+1)} A_m(t) = 0. \tag{8.66}$$

Proof. Suppose, to the contrary, that there is some $\varepsilon > 0$, some $m \in \mathbb{N}\cup\{0\}$, and some sequence of times $t_j \nearrow T$ such that

$$(T-t_j)^{-1} A_m(t_j) \geq \varepsilon \tag{8.67}$$

for all $j \in \mathbb{N}$. Set $\sigma_j \doteqdot \lambda_j^2 t_j$, where $\lambda_j \doteqdot (T-t_j)^{-\frac{1}{2}}$, and consider the sequence $\{\mathcal{M}_t^j\}_{t\in[-\lambda_j^2 t_j, 1), i\in\mathbb{N}}$ of rescaled flows defined by $\mathcal{M}_t^j \doteqdot \partial\Omega_t^j$, where

$$\Omega_t^j \doteqdot \lambda_j \left(\Omega_{\lambda_j^{-2}t+t_j} - p\right) .$$

Observe that $\sigma_j \to \infty$ as $j \to \infty$ and

$$|\mathrm{II}_j|^2(\cdot, t) \leq H_j^2(\cdot, t) = \lambda_j^{-2} H^2\left(\cdot, t_j + \lambda_j^{-2} t\right) \leq \frac{C_0}{1-t},$$

where

$$C_0 \doteqdot \sup_{(x,t)\in\mathcal{M}\times[0,T)} (T-t) H^2(x,t) < \infty .$$

It follows from the arguments in Section 6.8 that, on any compact time interval $J \Subset (-\infty, 0)$, all derivatives of the local parametrizations for the rescaled solutions are bounded uniformly in j and that the induced metrics are uniformly equivalent. Moreover, by Lemma 8.26 and (8.64), the circumradius $\rho_+^j(t)$ of Ω_t^j is bounded by

$$\rho_+^j(t) \leq \frac{(n+2)C}{\sqrt{3}} \sqrt{2(1-t)} .$$

Since each Ω_t^j contains the origin, this implies a C^0 bound on any compact time interval, uniform in j. It follows from the Arzelà–Ascoli theorem and a diagonal subsequence argument that some subsequence of the parametrizations of the rescaled flows converges in the smooth topology to a smooth, compact, convex limit flow, uniformly on compact time intervals. We claim that $\frac{|\mathring{\mathrm{II}}|^2}{H^2}$ is constant on the limit. Recall from (8.7) and Lemma 8.15 that

$$(8.68)\qquad (\partial_t - \Delta)\frac{|\mathring{\mathrm{II}}|^2}{H^2} \le \left\langle \nabla\frac{|\mathring{\mathrm{II}}|^2}{H^2}, \frac{\nabla H}{H}\right\rangle - \gamma\frac{|\nabla \mathrm{II}|^2}{H^2}\,,$$

where $\gamma > 0$ depends only on n and C. We first show that $\max_{\mathcal{M}\times\{t\}} \frac{|\mathring{\mathrm{II}}|^2}{H^2}$ is constant on the limit. Indeed, given any interval $[a, b] \subset (-\infty, 1)$,

$$\max_{\mathcal{M}\times\{b\}} \frac{|\mathring{\mathrm{II}}_j|^2}{H_j^2} - \max_{\mathcal{M}\times\{a\}} \frac{|\mathring{\mathrm{II}}_j|^2}{H_j^2} = \max_{\mathcal{M}} \frac{|\mathring{\mathrm{II}}|^2}{H^2}(\cdot, \lambda_j^{-2}b + t_j) - \max_{\mathcal{M}} \frac{|\mathring{\mathrm{II}}|^2}{H^2}(\cdot, \lambda_j^{-2}a + t_j).$$

By the maximum principle, $\max_{\mathcal{M}_t} \frac{|\mathring{\mathrm{II}}|^2}{H^2}$ is monotone nonincreasing and hence convergent as $t \to T$. Since $\lambda_j \to \infty$, $\lambda_j^{-2}b + t_j$ and $\lambda_j^{-2}a + t_j$ both approach T as $j \to \infty$, and it follows that the right-hand side converges to zero as $j \to \infty$. So $\max_{\mathcal{M}\times\{t\}} \frac{|\mathring{\mathrm{II}}|^2}{H^2}$ is indeed constant on the limit. The strong maximum principle, applied to (8.68), then implies that the ratio $\frac{|\mathring{\mathrm{II}}|^2}{H^2}$ is constant on the limit flow. It then follows from (8.68) that $\nabla \mathrm{II} \equiv 0$ on the limit and we conclude that the limit is a shrinking sphere, contradicting (8.67). □

We can now proceed as in Section 8.4.4 to convert the geometric estimates into estimates for the rescaled embedding and its derivatives and thereby deduce smooth convergence to an embedding of the unit sphere.

8.7. Notes and commentary

There are yet further approaches to Huisken's convergence result. For example, it is possible to bypass the interpolation arguments of Section 8.4.4 by analyzing the volume-preserving mean curvature flow equation: Since the round spheres are stable solutions of the linearized equation (as radial graphs, say), exponential convergence follows directly from well-known results (see [**374**, Theorem 9.1.2]). This approach is taken, for example, by James McCoy in [**395**].

An inspiration for Huisken's theorem is Hamilton's seminal classification of closed 3-manifolds with positive Ricci curvature using Ricci flow [**259**].

8.7.1. Mean curvature flow in the sphere. Recall that the mean curvature flow is well-defined for hypersurfaces in general Riemannian ambient spaces. A particularly nice situation is when the ambientspace is the round sphere, S^{n+1}. In that case, Huisken was able to prove, using similar techniques to those used to obtain Theorem 8.1, that hypersurfaces satisfying the quadratic pinching condition

$$\begin{cases} |\mathrm{II}|^2 < \frac{1}{n-1}H^2 + 2 & \text{if } n \geq 3, \\ |\mathrm{II}|^2 < \frac{3}{4}H^2 + \frac{4}{3} & \text{if } n = 2 \end{cases}$$

either shrink to a round point in finite time or else converge in the smooth topology to a totally geodesic sphere as $t \to \infty$. In case $n \geq 3$, the pinching condition is sharp in the sense that the tori $S^m(r) \times S^{n-m}(s)$ with $r^2 + s^2 = 1$ lie in S^{n+1} and satisfy

$$|\mathrm{II}|^2 - \frac{1}{n-1}H^2 = 2 + \frac{(n-2)r^2}{(n-1)s^2}\,.$$

Note that the right-hand side can be made arbitrarily close to 2. However, when $n = 2$, the optimal condition should be positive sectional curvature, which is equivalent to

$$|\mathrm{II}|^2 < H^2 + 2\,.$$

The reason for the stronger pinching condition in this case is purely technical, having to do with obtaining a sign on certain terms in the evolution equation for $|\mathrm{II}|^2 - \alpha H^2$.

It turns out that a different (fully nonlinear) flow for surfaces in S^3 does preserve positive Gaußian curvature and contracts such surfaces to round points [**35**].

8.7.2. Mean curvature flow in Riemannian ambient spaces.

Mean curvature flow of hypersurfaces in Riemannian ambient spaces does not preserve positivity of the second fundamental form. However, it does preserve the stronger convexity condition

$$H\mathrm{II} > nKg + \frac{n^2}{H}Lg\,, \tag{8.69}$$

where $-K \leq 0$ is a global lower bound for the sectional curvatures of the ambient space and L is a global bound for the norm of the covariant derivative of the Riemann curvature tensor of the ambient space [**293**]. An argument similar in nature to that of Theorem 8.1 can then be used to prove that hypersurfaces satisfying (8.69) contract to points, becoming asymptotically round in the process, so long as the ambient space has sectional curvatures bounded from above and injectivity radius bounded from below (away from zero) [**293**]. Once again, the form of the pinching condition arises from considerations involving the application the maximum principle.

It turns out that a different (fully nonlinear) curvature flow will preserve the pinching condition

$$\mathrm{II} > \sqrt{K} g \,, \tag{8.70}$$

where again $-K \leq 0$ is a lower bound for the sectional curvatures of the ambient space. While the condition (8.70) is slightly more restrictive than (8.69) in case the ambient space is locally symmetric, it is a significant improvement in the general case since it is unaffected by the derivatives of the ambient curvature. An argument similar in nature to that of Section 8.6 can then be used to prove that hypersurfaces satisfying (8.70) contract to points, becoming asymptotically round in the process, so long as the sectional curvatures of the ambient space are also bounded above and the covariant derivative of its Riemann curvature tensor is bounded [**27**] (note that no lower bound on the injectivity radius is required). In particular, a compact hypersurface satisfying (8.70) is diffeomorphic to a sphere and bounds a disk. Combining this with an argument of Gromov (see [**213**]) then yields the following smooth quarter pinching "twisted" sphere theorem [**27**, Theorem 7-7].

Theorem 8.29. *Let (N^n, g), $n \geq 2$, be a compact, simply connected, smooth Riemannian n-manifold with sectional curvatures in the range $(\frac{1}{4}, 1]$. Then N is diffeomorphic to a "twisted" sphere: A manifold obtained by smoothly gluing two smooth n-disks along their boundaries by an orientation-preserving diffeomorphism of S^{n-1}.*

N. Alikakos and A. Freire showed that certain embedded hypersurfaces of Riemannian manifolds that are close to a small geodesic ball converge, as time tends to infinity, to surfaces of constant mean curvature under the volume-preserving mean curvature flow [**14**].

8.7.3. High-codimension mean curvature flow. Since every immersed Riemannian submanifold possesses a mean curvature vector, the mean curvature flow can also be defined for immersed submanifolds, regardless of codimension. It is rather unmanageable in the general case, however (somewhat to be expected given the Nash [isometric] embedding theorem) — the algebraic structure of the evolution equations become vastly more complicated in high codimension and embeddedness is no longer preserved (imagine a space curve with two "linked" segments. These will move closer together for a short time and can eventually collide under the space-curve shortening flow).

One interesting situation where things become manageable is known as the **Lagrangian mean curvature flow**: Suppose that the ambient Riemannian manifold $(N, \langle \cdot, \cdot \rangle)$ admits a compatible complex structure

$J : TN \to TN$. An immersed submanifold $X : M \to N$ is **Lagrangian** if

$$X^*\omega = 0\,, \tag{8.71}$$

where ω is the induced symplectic form:

$$\omega(u,v) \doteqdot \langle Ju, v\rangle\ .$$

If the ambient manifold has vanishing Ricci curvature (e.g., $(N, \langle\cdot,\cdot\rangle)$ is $\mathbb{C}^n$ with its standard Hermitian product) then it turns out that the Lagrangian condition (8.71) is preserved under mean curvature flow. An interesting difference between codimension-one mean curvature flow and Lagrangian mean curvature flow concerns the curvature blow-up rate at a singularity: Whereas it is widely believed that the codimension-one mean curvature flow generically undergoes only type-I singularities, the opposite is believed for Lagrangian mean curvature flow (see Chen and Li [**142**], Groh, Schwarz, Smoczyk, and Zehmisch [**250**], and Neves [**414**,**416**]). Andreas Savas-Halilaj and Knut Smoczyk [**456**] have recently constructed an explicit example of Lagrangian mean curvature flow (with initial submanifold given by an immersed Whitney sphere) which forms a singularity that converges, after rescaling, to a Grim plane. For a more detailed investigation of Lagrangian mean curvature flow, we direct the reader to the surveys by M.-T. Wang [**519**], A. Neves [**415**], and D. Joyce [**322**].

In the general case, some important early results were obtained by Smoczyk (see [**488**]). For instance, an analogue of Huisken's theorem holds when the evolving submanifold has trivial normal bundle.

More recently, Andrews and Baker [**43**, **81**] showed that the quadratic curvature pinching condition

$$|\mathrm{II}|^2 \le c_n|\vec{H}|^2$$

is preserved under codimension $k \ge 1$ mean curvature flow in $\mathbb{R}^{n+k}$, where $c_n \doteqdot \frac{4}{3n}$ if $2 \le n \le 4$ or $c_n \doteqdot \frac{1}{n-1}$ if $n \ge 4$, and that such submanifolds contract to "round points" in an $(n+1)$-dimensional subspace. The pinching condition was improved by Charlie Baker and Huy The Nguyen when $n = k = 2$ [**82**].

8.7.4. Free boundary mean curvature flow. Axel Stahl obtained short-time existence of solutions to mean curvature flow with free boundary of prescribed contact angle on a fixed "support" hypersurface in Euclidean space. He showed that solutions either exist for all time or develop a curvature singularity [**496**]. He also obtained an analogue of Huisken's theorem in this setting [**495**].

8.8. Exercises

Exercise 8.1. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact, mean convex solution of mean curvature flow. Prove that:

(i) $2nT \le \rho_+(0)$, where $\rho_+(0)$ is the circumradius of $X_0(M^n)$.

(ii) $2nT \le (\frac{1}{n}H_{\min}(0))^{-2}$, where $H_{\min}(0) \doteqdot \min_{M^n\times\{0\}} H$.

Exercise 8.2. Denote by $\mathrm{S}(n)$ the normed linear space of symmetric $n\times n$-matrices equipped with the Hilbert–Schmidt norm, $\|B\| \doteqdot \sqrt{\operatorname{tr}(B^2)}$. Let $\mathrm{R} \doteqdot \{r\mathrm{I} : r > 0\}$ be the positive ray, where I is the identity matrix. Show that

$$d(B,\mathrm{R}) = \sqrt{\|B\|^2 - \tfrac{1}{n}\operatorname{tr}(B)^2}$$

for any $B \in \mathrm{S}(n)$, where d is the induced distance function.

Exercise 8.3. Prove the product rule (8.12) for smooth u and smooth, positive v. Hint: Consider the function $\log(|u|v^\alpha)$ wherever u is nonzero.

Exercise 8.4 (Stampacchia iteration)**.** Given a bounded open set $\Omega \subset \mathbb{R}^n$, where $n \ge 2$, let $v \in H^1_0(\Omega)$ be a weak subsolution of the equation

$$-\operatorname{div}[A(\nabla v) + V] = 0\,,$$

where $A : \Omega \to \mathrm{GL}(n,\mathbb{R})$ is a family of selfadjoint linear transformations satisfying $A \ge \lambda\mathrm{I}$ for some $\lambda > 0$ and $V : \Omega \to \mathbb{R}^n$ is a vector field satisfying $\left(\int_\Omega |V|^p\,dx\right)^{\frac{1}{p}} \le \nu$ for some $p > n$ and $\nu > 0$.

Given $k > 0$, set

$$v_k \doteqdot \max\{v-k, 0\} \quad \text{and} \quad A_k \doteqdot \{x\in\Omega : v_k(x) > 0\}\,.$$

(i) Show that

$$\lambda^2 \int_\Omega |\nabla v_k|^2\,dx \le \int_{A_k} |V|^2\,dx\,.$$

(ii) Assuming $n \ge 3$, use Hölder's inequality and the Sobolev inequality to deduce that

$$\left(\int_\Omega v_k^{2^*}\,dx\right)^{\frac{1}{2^*}} \le \frac{\nu S}{\lambda}|A_k|^{\frac{1}{2}-\frac{1}{p}}\,,$$

where $\frac{1}{2^*} \doteqdot \frac{1}{2} - \frac{1}{n}$ and S is the Sobolev constant.

(iii) Deduce, for any $h > k > 0$, that

$$(h-k)^{2^*}|A_h| \le C^{2^*}|A_k|^{2^*\left(\frac{1}{2}-\frac{1}{p}\right)}\,,$$

where $C \doteqdot \frac{S\nu}{\lambda}$.

(iv) Conclude from Stampacchia's lemma that

$$\|v\|_\infty \le 2^{\frac{n(p-2)}{2(p-n)}}\,C\,|\Omega|^{\frac{1}{n}-\frac{1}{p}}\,.$$

(v) Assuming $n = 2$, derive a bound for $\|v\|_\infty$ using the Poincaré inequality instead of the Sobolev inequality.

Exercise 8.5. Use the Sobolev inequality (Theorem 8.12) and the Hölder inequality to prove the Poincaré-type inequality of Corollary 8.13.

Exercise 8.6. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a convex solution of mean curvature flow satisfying $\kappa_1(\cdot,0) \geq \alpha H(\cdot,0)$ for some $\alpha > 0$.

(1) Verify that the constant C_ε in Theorem 8.6 depends only on n, α, $|\mathcal{M}_0|$, T, $\max_{\mathcal{M}\times\{0\}} H$, and ε.

(2) Set $\rho_+(0) \doteqdot R$ and choose V and Θ so that $|\mathcal{M}_0| \leq VR^n$ and $\max_{\mathcal{M}\times\{0\}} H \leq \Theta R^{-1}$. Show that the constant C_ε can be written as $\mathrm{C}(n,\alpha,V,\Theta,\varepsilon)R^{-2}$.

(3) Show that the same is true if, instead, we set $\min_{\mathcal{M}\times\{0\}} H \doteqdot R^{-1}$.

Exercise 8.7. Fill in the details of the proof of Lemmas 8.15 and 8.17. More precisely, prove that:

(a) $|\nabla \,\mathrm{II}\,|^2 \geq \frac{3}{n+2}|\nabla H|^2$ on any hypersurface of $\mathbb{R}^{n+1}$.

(b) $|\nabla^2 \mathrm{II}|^2 \geq \gamma H^4 |\mathring{\mathrm{II}}|^2$ on a compact, convex hypersurface of $\mathbb{R}^{n+1}$ satisfying $\mathrm{II} \geq \alpha H g$, where γ depends only on n and α.

Exercise 8.8. Prove Corollary 8.18.

Exercise 8.9. Let $\{\Omega_t\}_{t\in[0,T)}$ be a family of bounded convex bodies whose boundaries evolve by mean curvature flow. Use the fact that $\rho_-(t) \to 0$ as $t \to T$ to prove that

$$\rho_-(t) \leq \sqrt{2(T-t)} \leq \rho_+(t)\,.$$

Exercise 8.10. Let $X : M^n \times [0,T) \to \mathbb{R}^{n+1}$ be a compact, uniformly convex solution of mean curvature flow. Use the fact that $\mu_t(M^n) \to 0$ as $t \to T$ to prove that

$$\sqrt{2nT} \geq \alpha n \left(\frac{\mu_0(M^n)}{\mathrm{Area}(S^n)}\right)^{\frac{1}{n}}\,,$$

where $\alpha \doteqdot \min_{M^n\times\{0\}} \frac{\kappa_1}{H}$.