Preface

The present book arises from notes of a master's course that the first author delivered from the academic course 2011/12 until 2017/18 for the Master's Degree in Mathematics at Universidad Complutense de Madrid. The prerequisites are basic courses in linear algebra, elementary topology and algebraic topology, differential geometry, complex analysis, and partial differential equations.

This is a somewhat non-standard course on differential topology, that is, its main focus is the geometry and topology of manifolds, trying to touch on many of its ramifications. Being of an introductory nature, it cannot aspire to include all material on the interplay of geometry and topology. On the other hand, we consider it important in a text of this type to give complete proofs of some of the important landmarks in the development of the area. For this reason, we have decided to introduce the different aspects of the theory of manifolds in arbitrary dimension, but then at every chapter we move to give important results and full proofs for the case of dimension 2, that is, surfaces. The title of the book arises from this consideration. The content of the book is distributed into six chapters, following the philosophy of going from the soft to the hard. In Chapter 1, we start with the topological side of the course, introducing manifolds as natural objects to study (natural from the geometrical and from the physical points of view) and setting up the core problem of the classification of manifolds. Smooth manifolds are introduced, where a differentiable structure on a manifold is the way to make sense of the concept of differentiation, which is needed to pose (and solve) physical problems on a manifold. The chapter contains complete proofs of two important results in the case of surfaces: the existence of triangulations and the classification of compact surfaces.

Chapter 2 introduces the theory of algebraic topology. This is the area of mathematics that constructs algebraic invariants to study the topology of spaces. For manifolds, topology is the way to go from the local to the global. By definition, a manifold is a space that locally looks like a Euclidean space (which is the model of the space in which we live). So locally there is no information on a manifold, and the global information is encoded in the algebraic topology invariants: homotopy and homology groups. We introduce de Rham cohomology by its paramount importance, where the differentiable structure of a smooth manifold is used to get implications in the algebraic topology of the manifold, giving a link between geometry and topology.

We delve in Chapter 3 into Riemannian geometry. Both from the geometric and from the physical points of view, it is natural that a smooth manifold can have more structure. For example, this could be a way to measure the lengths, angles, or whatever other concept of geometric significance can be computed locally (at a point). These give rise to different types of geometric structures. We focus on Riemannian metrics, as these were historically the first ones to be studied, and the most thoroughly considered in differential geometry, but by no means the only interesting ones. A metric allows us to define curvature, which helps in understanding how much a manifold is intrinsically bended. We introduce the theory in arbitrary dimensions and focus on the case of surfaces. The chapter includes the proof of the very important Gauss-Bonnet theorem for surfaces, which links the curvature of a surface to its global topology. We include a discussion of orbifolds (which are like manifolds but with singular points) and their Riemannian structures, where the curvature appears concentrated. This point of view is of interest in itself.

To deepen on the relation that takes us from the local to the global, it is necessary to focus on manifolds which look the same from one point to another (homogeneous, isotropic). From the physical point of view, according to the Einstein conception of the universe, isotropic spaces arise as cosmological models in which the physical mechanisms that rule the natural phenomena are the the same at every point of the space, and with no preferred directions. From the geometric point of view, these are the spaces where we can move figures from one point to another. It is natural that they are at the heart of the origin of geometry, first with the Euclidean geometry and later with the non-Euclidean geometries. Chapter 4 focuses on the specific study of the three possibilities for isotropic surfaces, i.e., constant curvature surfaces. We will analyse the case of positive curvature which gives rise to spherical and projective geometry, the case of zero curvature which is Euclidean geometry, and the case of negative curvature which produces hyperbolic geometry. Our main focus is on surfaces, so we give a classification of compact surfaces with these types of geometries. The classification is thoroughly given for the case of tori (zero curvature), and an introduction is given to Teichmüller theory for the important topic of hyperbolic metrics on compact surfaces.

Chapter 5 analyses a very important enrichment that a manifold may admit, namely that of a complex manifold, in which the tangent space has a natural complex structure. This upgrade has deep implications from the point of view of differential geometry that echoes in the global topology of the manifold. For instance, this complex setting induces a new enhanced algebraic structure on de Rham cohomology, relating it with another cohomology intrinsically tied to the complex structure called Dolbeault cohomology. But this enhancement also has important consequences from from the point of view of algebra, building a bridge between complex manifolds and (smooth) projective varieties, that is the zero locus of complex polynomials. These varieties are the main objects of study in algebraic geometry. Following the general philosophy of the book, we particularize to the case of surfaces with complex structures, that is complex curves. These structures are equivalent to conformal structures (Riemannian metrics up to a variable dilation factor) thanks to the uniformization theorem. Strikingly, the classification of complex curves parallels that of surfaces with constant curvature. A very important result that we fully prove in the chapter is the degree-genus formula. It says that the topology of a complex curve (its genus) is given by the degree of the planar model (a polynomial in the complex plane which defines the complex curve). The chapter ends with the classification of elliptic curves, which parallels that of tori given in Chapter 5.

The final chapter, Chapter 6, moves into the links of geometry and the theory of partial differential equations, consisting of the study of differential equations on smooth manifolds. The more relevant equations have their origin in geometrical questions or are of physical significance. This area is commonly known as global analysis, since its main features are the implications of the global nature of geometrical spaces on the properties of differential equations. The local aspect, the study of differential equations on open sets of the Euclidean space, are studied in mathematical analysis. We review the theory of harmonic forms and its interplay with de Rham cohomology, which is an elliptic problem on a compact manifold. The chapter gives a proof of the existence of metrics of constant curvature on a conformal structure of a compact surface. Our proof in the positive curvature case hinges on a nice trick using orbifolds. In particular, we give a complete proof of the uniformization theorem with analytical techniques. We end up with a brief introduction to the Ricci flow, a current topic which has produced very strong results in geometrization.

This manuscript can be used to deliver a course at postgraduate level. As such, this book may serve as a reference for a first course that explores the interface between differential topology and algebraic topology. With this objective, the course should be focused on the material of Chapter 1, with special attention to the classification of compact triangulated surfaces, Chapter 2, especially singular homology and its interplay with de Rham cohomology and Chapter 3, reaching the celebrated Gauss-Bonet theorem as an interplay between topology and geometry.

However, the main aim of this book is to serve as basic reference for a postgraduate course, at the level of a master's course or an advanced PhD course depending on the background of the targeted audience. In this way, in a half-year course, most of the material of Chapters 1, 2, 3, 4, and 5 may be covered (perhaps omitting the review of known topics). With a view towards a course presenting a more analytic perspective, the material of Chapter 2 can be reduced (especially higher homotopy groups, Seifert-van Kampen theorem, and simplicial homology) and the final part of Chapter 5 (degree-genus formula and elliptic curves) may be replaced with the contents of Chapter 6.

Following this aspiration as a textbook, each chapter has been complemented with an extensive collection of problems, ranging from easier to very difficult ones. Some of them complete results appearing inthe text, and others give hints to profound ramifications that have not been treated. Each chapter ends with a list of topics for further study and with a number of references. The topics have been chosen so that they can be proposed to students, who will write small dissertations, as the first author has successfully done during the years that he has delivered the master's course. The references are divided into basic reading (i.e., texts where the content of the chapter is treated in full), bibliography for the topics for further study, and references which go beyond of the content of the chapter or that have been mentioned in the text.

Most of the topics of the book can be found in other texts with more specific aims. Some of the material has been written in a review form (such as homology theory, Riemannian geometry, or differential equations on manifolds). We hope that this book serves as motivation for learning all these aspects by reading deeper treatises which cover the different theories at large. Our aim is to focus in the interconnections between all these aspects. Modern geometry (from the mid-twentieth century) has seen the most important advances produced on the interaction with algebra, physics, or analysis.

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