Chapter 1

Topological surfaces

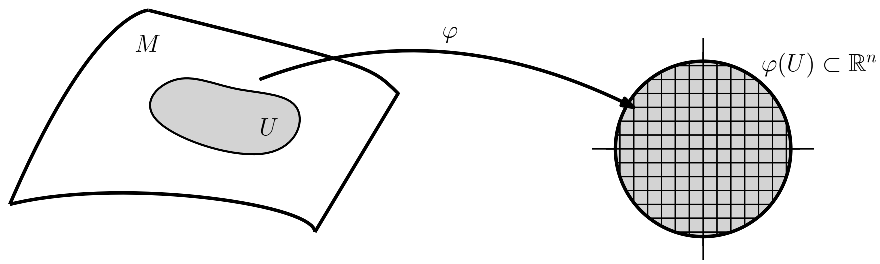
Topology is the area of mathematics which studies topological spaces, i.e., spaces where continuity can be defined. But topological spaces can be very weird, and at some point, one may wonder whether spaces with strange topologies serve any purpose apart from being helpful to understand the theory. Natural spaces are those attached to geometrical problems (that is, about figures that can be drawn and manipulated in our space) or physical problems (that is, particles and systems moving according to some physical laws). In these spaces, we can naturally put coordinates to locate the figures or the particles (at least in a local region of the space), as happens for the surface of the Earth. This leads to the notion of manifold as the most important object to study in topology. In this chapter we introduce manifolds, and we focus on those of dimension 2 (i.e., surfaces), providing a proof of the classification theorem for compact surfaces.

1.1. Topological and smooth manifolds

In order to fix notation within this book, we denote by $\mathbb{R}^n$ the n-dimensional *Euclidean space*, with the usual topology. The *open ball* of radius $r > 0$ centered at $p \in \mathbb{R}^n$ will be denoted $B^n_r(p) = \{x \in \mathbb{R}^n \mid ||x - p|| < r\}$. We will shorten $B^n = B^n_1(0)$, the standard open ball. The *closed ball*, or *disc*, will be denoted $\bar{B}^n_r(p) = \{x \in \mathbb{R}^n \mid ||x - p|| \leq r\}$, and the standard disc is $D^n = \bar{B}^n_1(0)$. The *sphere* will be $S^{n-1}_r(p) = \{x \in \mathbb{R}^n \mid ||x - p|| = r\}$, and the standard sphere will be shortened as $S^{n-1} = S^{n-1}_1(0)$. Finally, when the underlying dimension is clear from the context, we will omit it from the notation.

Definition 1.1. A *topological manifold* M (or just a manifold) is a (non-empty) Hausdorff and second countable topological space which is locally homeomorphic to Euclidean spaces, i.e., for every point $p \in M$ there exists an open neighbourhood $U = U^p \subset M$ of p and a homeomorphism $\varphi : U \to \varphi(U) \subset \mathbb{R}^n$ onto an open subset of some $\mathbb{R}^n$.

We call (U, φ) a *chart* of M. For $q \in U$ we call $\varphi(q) = (x_1(q), \dots, x_n(q))$ the coordinates of q, and the functions $x_i : U \to \mathbb{R}$ are called the *coordinate functions*. If we



take $\epsilon > 0$ small so that $B^n_\epsilon(\varphi(p)) \subset \varphi(U)$, and $V = \varphi^{-1}(B^n_\epsilon(\varphi(p)))$, we have a chart (V, φ) whose image is a ball. Translating and expanding with $\tau(x) = \frac{1}{\epsilon}(x - \varphi(p))$, we have a chart $(V, \varphi' = \tau \circ \varphi)$ such that $\varphi'(V) = B_1(0)$ and $\varphi'(p) = 0$. Finally, using the homeomorphism $F : B^n_1(0) \to \mathbb{R}^n, F(x) = \frac{x}{1-||x||}$, we get a chart $(V, \phi = F \circ \varphi')$ whose image is the whole of $\mathbb{R}^n$. To have this freedom of ranges of charts is useful in what follows. A collection of charts $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that $\bigcup U_\alpha = M$ is called an *atlas*.

The number n in Definition 1.1 is called the *dimension* of M at p. The fact that n does not depend on the chart φ follows from the theorem of invariance of dimension.

Theorem 1.2 (Invariance of dimension)**.** *Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ be open subsets such that there is a homeomorphism $f : U \to V$. Then $n = m$.*

Proof. As the main focus of the course is that of surfaces (i.e., when $n = 2$), we will prove the result for $n = 2$, where it can be proved by means of the fundamental group (see section 2.1.3). For $n > 2$, it can be proved using homology groups, which we introduce in section 2.3 (see Exercise 2.20). For $n = 1$, it is proved by using arguments of connectivity (Exercise 1.2). So we assume also that $m \geq 2$.

Let $p \in U$ and $f(p) \in V$. Take $\epsilon > 0$ so that there is a ball $B^m_\epsilon(f(p)) \subset V$. Take $\delta > 0$ such that $\bar{B}^n_\delta(p) \subset f^{-1}(B^m_\epsilon(f(p)))$. There is a radial retraction $r : f^{-1}(B^m_\epsilon(f(p))) - \{p\} \to S^{n-1}_\delta(p) \cong S^{n-1} = S^1$. Therefore there is a surjection on the fundamental groups $\pi_1(f^{-1}(B^m_\epsilon(f(p))) - \{p\}) \to \pi_1(S^1) \cong \mathbb{Z}$. This implies that $f^{-1}(B^m_\epsilon(f(p))) - \{p\}$ is not simply connected. Clearly $f : f^{-1}(B^m_\epsilon(f(p))) - \{p\} \to B^m_\epsilon(f(p)) - \{f(p)\}$ is a homeomorphism, so $B^m_\epsilon(f(p)) - \{f(p)\}$ is not simply connected. The space $B^m_\epsilon(f(p)) - \{f(p)\}$ is of the homotopy type of $S^{m-1} = S^{m-1}_1(0)$, and for $m > 2$, S^{m-1} is simply connected. This is a contradiction, hence $m = 2$. □

The fact that the number n is independent of the chart readily yields that n is constant on each connected component of M, so the dimension is a well defined number for each connected topological manifold. We say that M is an n-manifold if M is a manifold with all its connected components of dimension n. We also write $n = \dim M$.

Remark 1.3. It is customary to call a topological 1-manifold a (topological) curve. Analogously, a topological 2-manifold is called a (topological) surface.

1.1.1. Categories. A large part of modern geometry consists of studying the geometric structures that a manifold can have. Each geometric structure allows one to use efficiently some mathematical tools (such as algebra, analysis, combinatorics) in the study of manifolds. The final aim is to solve the *classification problem*, i.e., to give a complete list of all different manifolds and geometric structures. The correct setting for phrasing this problem is category theory. Let us pause to introduce this.

Definition 1.4. A *category* $\mathcal{C}$ consists of a collection of *objects*, denoted $\mathrm{Obj}(\mathcal{C})$, and for every pair of objects $X, Y \in \mathrm{Obj}(\mathcal{C})$, a collection of *arrows* or *morphisms*, denoted $\mathrm{Mor}_{\mathcal{C}}(X, Y)$, satisfy the following.

- For object X there exists an arrow denoted $1_X \in \mathrm{Mor}_{\mathcal{C}}(X, X)$ (or Id if the object is clear from the context), called *identity*.
- For objects X, Y, Z we have a binary operation called *composition*,

  $$\mathrm{Mor}_{\mathcal{C}}(X, Y) \times \mathrm{Mor}_{\mathcal{C}}(Y, Z) \to \mathrm{Mor}_{\mathcal{C}}(X, Z), \quad (f, g) \mapsto g \circ f,$$

  such that
  (1) $f \circ 1_X = f = 1_Y \circ f$, $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$.
  (2) $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$, $g \in \mathrm{Mor}_{\mathcal{C}}(Y, Z)$, $h \in \mathrm{Mor}_{\mathcal{C}}(Z, W)$.

It is common to denote by $f : X \to Y$ or $X \xrightarrow{f} Y$, or even $X \to Y$, an arrow $f \in \mathrm{Mor}_{\mathcal{C}}(X, Y)$. If f, g can be composed, we say that they are composable.

Example 1.5.

- The category **Set** whose objects are sets and whose arrows are maps between sets. The composition is the usual composition of maps.
- The category **Top** whose objects are topological spaces and whose morphisms are continuous maps. The category $\mathbf{Top}_*$ whose objects are pointed topological spaces, that is pairs (X, p) with X a topological space and $p \in X$. A morphism $f : (X, p) \to (Y, q)$ is a continuous map $f : X \to Y$ with $f(p) = q$.
- The category **TMan** of topological manifolds, whose arrows are continuous maps. Recall that manifolds may be disconnected and may have components of different dimensions. In general, it is useful to specify the dimension and the compactness, so we will deal with the categories $\mathbf{TMan}^n$ and $\mathbf{TMan}^n_c$ whose objects are topological n-manifolds and compact topological n-manifolds, respectively.
- Given a field $\mathbf{k}$, the category $\mathbf{Vect}_{\mathbf{k}}$ whose objects are $\mathbf{k}$-vector spaces, and whose arrows are $\mathbf{k}$-linear maps.
- The category **Group**, whose objects are groups and whose arrows are homomorphisms of groups. The category **Abel**, whose objects are Abelian groups and arrows homomorphisms of groups. Here **Abel** is a subcategory of **Group**, that is, its objects and morphisms are contained in those of **Group**.
- For a topological space we define the category $\Pi_1(X)$ called the *fundamental groupoid* of X. The objects of this category are the points of X. The arrows between two points $x, y \in X$ are the paths

  $$\Omega_{x,y}(X) = \{\gamma : [0, 1] \to X \mid \gamma \text{ continuous}, \gamma(0) = x, \gamma(1) = y\},$$

modulo the equivalence relation of being homotopic relative to $\{x, y\}$ (Definition 2.1). The composition is the usual juxtaposition of equivalence classes of paths modulo homotopy $\Omega_{x,y}(X) \times \Omega_{y,z}(X) \to \Omega_{x,z}(X)$. The identity 1_x is the constant map with image $\{x\}$. This gives an example of arrows not being functions, and composition not being composition of functions.

- For a group $(G, \circ)$ we define the *groupoid* associated to G as the category $\widehat{G}$ such that $\mathrm{Obj}(\widehat{G}) = \{\star\}$ and $\mathrm{Mor}_{\widehat{G}}(\star, \star) = G$. For two morphisms g_1 and g_2, their composition is $g_1 \circ g_2 = g_1 \cdot g_2$, i.e., their product as elements of G.

Definition 1.6. Given a category $\mathcal{C}$, we say that an arrow $f : X \to Y$ is *invertible* if there exists $g : Y \to X$ so that $f \circ g = 1_Y$ and $g \circ f = 1_X$. We write $g = f^{-1}$ and say that g is the inverse of f.

It is an exercise to see that the inverse is unique whenever it exists (cf. Exercise 1.1). It is also easy to see that if f and h are invertible and its composition makes sense, then the composition $f \circ h$ is also invertible and $(f \circ h)^{-1} = h^{-1} \circ f^{-1}$.

If $f : X \to Y$ is invertible, we say that X and Y are *isomorphic* in the category $\mathcal{C}$, and we write $X \cong_{\mathcal{C}} Y$ (or $X \cong Y$, if there is no risk of misunderstanding). This defines an equivalence relation in $\mathrm{Obj}(\mathcal{C})$. Note however that $\mathrm{Obj}(\mathcal{C})$ may not be a set (with Zermelo-Fraenkel axioms), so we must be careful when it comes to talking of the "equivalence classes" of objects of $\mathcal{C}$ up to isomorphism.

Example 1.7.

- In **Set** the invertible maps are the bijective maps between sets $f : X \to Y$. Two sets are isomorphic if and only if they have the same cardinal.
- In $\mathbf{TMan}^n$ and $\mathbf{TMan}^n_c$ the invertible maps are the homeomorphisms, and two manifolds are isomorphic when they are homeomorphic. Observe that, in $\mathbf{TMan}^n_c$ a continuous bijective map is automatically a homeomorphism (Proposition 1.28).
- In $\mathbf{Vect}_k$ the invertible maps are the bijective linear maps. Two objects are isomorphic when their dimensions have the same cardinal.
- In $\Pi_1(X)$ all arrows are invertible. Indeed it is enough to define $\gamma^{-1}(t) = \gamma(1-t)$ for any arrow γ. Two objects are isomorphic if and only if there exists an arrow between them, that is they are in the same path connected component of X. In general, a *groupoid* is a category whose morphisms are always invertible. So $\Pi_1(X)$ is a groupoid, and the category $\widehat{G}$ of Example 1.5 is also a groupoid.

With this language, solving the classification problem in a given category $\mathcal{C}$ consists of giving a list $\mathbb{L}_{\mathcal{C}}$ which has exactly one element of each isomorphism class in $\mathcal{C}$. In other words, we find $\mathbb{L}_{\mathcal{C}} \subset \mathrm{Obj}(\mathcal{C})$ such that:

(1) if $X, Y \in \mathbb{L}_{\mathcal{C}}$, then $X \cong Y$ if and only if $X = Y$.

(2) for each $X \in \mathrm{Obj}(\mathcal{C})$ there exists an $Y \in \mathbb{L}_{\mathcal{C}}$ such that $X \cong Y$.

Example 1.8. Examples of lists are $\mathbb{L}_{\mathbf{Set}} = \{\text{cardinals}\}$ and $\mathbb{L}_{\mathbf{Vect}_\mathbf{k}} = \{\mathbf{k}^\alpha \,|\, \alpha \in \{\text{cardinals}\}\}$. For manifolds, we usually restrict the classification lists to connected manifolds, following the notation in (1.2). We have $\mathbb{L}^{co}_{\mathbf{TMan}^0} = \{\star\}$, where $\star$ is the set with one point (called *singleton*), whereas $\mathbb{L}_{\mathbf{TMan}^0} = \mathbb{N} \cup \{\infty\}$, where $k \in \mathbb{N}$ indicates the set of k points and ∞ the set of countable many points (with the discrete topology). Also $\mathbb{L}^{co}_{\mathbf{TMan}^1} = \{S^1, \mathbb{R}\}$, $\mathbb{L}^{co}_{\mathbf{TMan}^1_c} = \{S^1\}$ (cf. Exercise 1.3).

Maps between categories are defined by the concept of functor, which will be useful throughout this course.

Definition 1.9.

(1) A *covariant functor* $F : \mathcal{C}_1 \to \mathcal{C}_2$ between two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ consists of a map $F : \mathrm{Obj}(\mathcal{C}_1) \to \mathrm{Obj}(\mathcal{C}_2)$ and for all $X, Y \in \mathrm{Obj}(\mathcal{C}_1)$, a map $F : \mathrm{Mor}_{\mathcal{C}_1}(X, Y) \to \mathrm{Mor}_{\mathcal{C}_2}(F(X), F(Y))$. It is usually denoted $f_* = F(f)$. These satisfy the following.
   - $F(f \circ g) = F(f) \circ F(g)$, for composable morphisms f, g. In other words, $(f \circ g)_* = f_* \circ g_*$.
   - It satisfies $F(1_X) = 1_{F(X)}$, for all X.

(2) A *contravariant functor* $F : \mathcal{C}_1 \to \mathcal{C}_2$ between two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ consists of a map $F : \mathrm{Obj}(\mathcal{C}_1) \to \mathrm{Obj}(\mathcal{C}_2)$ and for all $X, Y \in \mathrm{Obj}(\mathcal{C}_1)$, a map $F : \mathrm{Mor}_{\mathcal{C}_1}(X, Y) \to \mathrm{Mor}_{\mathcal{C}_2}(F(Y), F(X))$. It is usually denoted $f^* = F(f)$. These satisfy the following.
   - It is reverse compatible with the compositions, i.e., $F(f \circ g) = F(g) \circ F(f)$. In other words, $(f \circ g)^* = g^* \circ f^*$.
   - It satisfies $F(1_X) = 1_{F(X)}$, for all X.

Example 1.10. The fundamental group is a covariant functor $\pi_1 : \mathbf{Top}_* \to \mathbf{Group}$, $(X, x) \to \pi_1(X, x)$ which maps each $f : (X, x) \to (Y, y)$ to $f_* : \pi_1(X, x) \to \pi_1(Y, y)$, the induced map on loops (see section 2.1).

A functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ preserves isomorphisms:

$$X \cong_{\mathcal{C}_1} Y \implies F(X) \cong_{\mathcal{C}_2} F(Y). \tag{1.1}$$

This is an important property with an easy proof. Let us check in the case that F is covariant. Let $f : X \to Y$, $g : Y \to X$ with $f \circ g = 1_Y$, $g \circ f = 1_X$. Apply F to get $F(f) \circ F(g) = F(f \circ g) = F(1_Y) = 1_{F(Y)}$ and $F(g) \circ F(f) = F(g \circ f) = F(1_X) = 1_{F(X)}$. Therefore $F(f) : F(X) \to F(Y)$ is an isomorphism, with inverse $F(g)$. Note that this can be written as $F(f^{-1}) = F(f)^{-1}$. The property (1.1) is usually used as

$$F(X) \not\cong_{\mathcal{C}_2} F(Y) \implies X \not\cong_{\mathcal{C}_1} Y,$$

to distinguish objects in $\mathcal{C}_1$.

As a consequence, F induces a map between the classifying lists of the categories,

$$F : \mathbb{L}_{\mathcal{C}_1} \to \mathbb{L}_{\mathcal{C}_2},$$

which can serve to help in classification problems, as will be clear in later chapters.

1.1.2. Smooth structures. In this section we introduce an extra structure into topological manifolds, the so-called smooth structure, giving rise to differentiable manifolds. This extra piece of data will allow us to transfer concepts from mathematical analysis, such as differentiable functions and differential equations, to manifolds. This opens the door to new and powerful techniques for tackling classification problems unavailable with raw topological structure. These techniques will appear continuously within the following chapters with a particularly prominent role in section 2.5 and Chapter 5.

For this purpose, we need a further concept, sheaves on topological spaces. These are defined as follows.

Definition 1.11. For a topological space X we define the category $\mathbf{Open}(X)$ whose objects are the open sets of X. For any open sets U, V of X, $\mathrm{Mor}(U, V) = \{i : U \to V\}$ is the natural inclusion if $U \subset V$, and $\mathrm{Mor}(U, V) = \emptyset$ otherwise.

Definition 1.12. Let X be a topological space. A *presheaf* on X is a contravariant functor $F : \mathbf{Open}(X) \to \mathbf{Set}$. To be more specific, for each open set $U \subset X$, we have a set $F(U)$. It is customary to call the elements $s \in F(U)$ the *sections* on U. This satisfies the following.

- If $V \subset U$ and $i_{V,U}$ denotes the inclusion, we have a map
$$r_{U,V} = F(i_{V,U}) : F(U) \to F(V)$$
called *restriction*. It is usually denoted $s|_V = r_{U,V}(s)$.
- If $W \subset V \subset U$, then $r_{V,W} \circ r_{U,V} = r_{U,W}$. In other words, $(s|_V)|_W = s|_W$, for $s \in F(U)$.
- For the identity $U = U$, we have $r_{U,U} = 1_{F(U)}$.

The most interesting presheaves arise from functions. With this in mind we have the following definition.

Definition 1.13. Let X be a topological space. A *sheaf* on X is a presheaf $F : \mathbf{Open}(X) \to \mathbf{Set}$ that satisfies two additional properties. Suppose that $U_\alpha \subset X$ is a collection of open sets, and denote $U = \bigcup U_\alpha$. Then,

(1) *Local property*. Given sections $s_1, s_2 \in F(U)$, if $s_1|_{U_\alpha} = s_2|_{U_\alpha}$ for all α, then $s_1 = s_2$.
(2) *Gluing property*. Given sections $s_\alpha \in F(U_\alpha)$, if $s_\alpha|_{U_\alpha \cap U_\beta} = s_\beta|_{U_\alpha \cap U_\beta}$ for all α, β, it follows that there exists $s \in F(U)$ such that $s|_{U_\alpha} = s_\alpha$ for all α.

Example 1.14.

- The most basic example is the sheaf C^0 of continuous functions on a space X, i.e., $C^0(U) = \{f : U \to \mathbb{R} \text{ continuous}\}$. The restrictions are the restriction of functions.
- A subsheaf of F is a sheaf F' such that $F'(U) \subset F(U)$ for all U, and the restriction maps are the same.

- For any topological space, we have the sheaf $C^0(-,Y)$ of continuous maps with values in Y. In particular, if Y has the discrete topology $C^0(-,Y)$ is the sheaf of locally constant functions with values in Y.
- Consider the presheaf given by assigning to U the injective functions $U \to \mathbb{R}$. It is clearly a presheaf with the restrictions of functions. However, it is easy to verify that it is not a sheaf because condition (2) of Definition 1.13 fails in general.

Definition 1.15. A *smooth* (*or differentiable*) *manifold* is a topological manifold M endowed with a subsheaf $\mathrm{S} \subset C^0_M$ called the sheaf of *smooth* or *differentiable functions*. The sheaf S must satisfy that for every $p \in M$ there exists a chart $\varphi : U = U^p \to \varphi(U) \subset \mathbb{R}^n$ which induces a bijection $\varphi_* : \mathrm{S}(U) \to C^\infty(\varphi(U))$, $s \mapsto s \circ \varphi^{-1}$, being $C^\infty(\varphi(U))$ the set of C^∞ functions in the classical sense for open sets of $\mathbb{R}^n$. The inverse of this bijection is given by $\varphi^* : C^\infty(\varphi(U)) \to \mathrm{S}(U)$, $f \mapsto f \circ \varphi$.

When a topological manifold M can be given a sheaf S as above, one says that M admits a *smooth structure* (M, S).

Theorem 1.16. *Let M be a topological manifold. There exists a sheaf of differentiable functions $\mathrm{S} \subset C^0$ if and only if we can find an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ on M such that the* changes of coordinates (*also called* changes of charts) $\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(U_\beta \cap U_\alpha) \to \varphi_\alpha(U_\beta \cap U_\alpha)$ *are C^∞ diffeomorphisms between open subsets of $\mathbb{R}^n$, for all α, β. The correspondence is given by saying that $f \in \mathrm{S}(U)$ if and only if $f \circ \varphi_\alpha^{-1} \in C^\infty(\varphi_\alpha(U \cap U_\alpha))$ for all α.*

Proof. Suppose that S exists. For each $p \in M$ there exists a chart (U^p, φ_p) so that φ_p^* induces a bijection between $C^\infty(\varphi_p(U^p))$ and $\mathrm{S}(U^p)$. We can cover $M = \bigcup_{p \in M} U^p$ with such open sets. We claim that this is the required atlas with C^∞ change of coordinates. Indeed, let $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ be two open sets of the cover and its respective charts, and suppose they have non-empty intersection. Consider the function $y_i : \varphi_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}, (y_1, \ldots, y_n) \mapsto y_i$, which is C^∞. Therefore $y_i \circ \varphi_\beta \in \mathrm{S}(U_\alpha \cap U_\beta)$, which implies that $y_i \circ \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \mathbb{R}$ is C^∞. Since this is true for $i = 1, \ldots, n$, it follows that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^∞. An analogous argument shows that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is C^∞. Now suppose we have an atlas $\{(U_\alpha, \varphi_\alpha)\}$ with C^∞ changes of coordinates, and we want to define the sheaf S. We put $\mathrm{S}(U_\alpha) = \varphi_\alpha^*(C^\infty(\varphi_\alpha(U_\alpha)))$. If U is an open set contained in some U_α, we also put $\mathrm{S}(U) = \varphi_\alpha^*(C^\infty(\varphi_\alpha(U)))$. This does not depend on the choice of α, because the changes of coordinates are smooth. So we have succeeded in defining S in sufficiently small open sets. For general $U \subset X$ we define $\mathrm{S}(U) = \{f \in C^0(U) \mid f|_{U \cap U_\alpha} \in \mathrm{S}(U \cap U_\alpha)$, for all $\alpha\}$. It is easy to verify that S satisfies all the properties of sheaves because C^∞ functions do. □

Remark 1.17. The last proposition opens the door for an equivalent definition of smooth manifold. Namely, a smooth manifold is a topological manifold X endowed with a *maximal smooth atlas*, which is an atlas $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ such that the following hold.

(1) For every non-empty intersection $U_\alpha \cup U_\beta \neq \emptyset$, the change of coordinates

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

is a C^∞ diffeomorphism between open sets of $\mathbb{R}^n$. Recall that, by the inverse function theorem, a C^∞ diffeomorphism is the same as a bijection between open sets of $\mathbb{R}^n$ whose differential at every point is an invertible linear map.

(2) The atlas $\mathcal{A}$ is maximal, in the following sense. If there exists a homeomorphism $\varphi : U \to V$ between open sets $U \subset X$ and $V \subset \mathbb{R}^n$, such that for every non-empty intersection $U \cap U_\alpha \neq \emptyset$, the map $\varphi \circ \varphi_\alpha^{-1} : \varphi_\alpha(U \cap U_\alpha) \to \varphi(U \cap U_\alpha) \subset V$ is a diffeomorphism between open sets of Euclidean spaces, then it follows that $(U, \varphi) \in \mathcal{A}$.

Note that any atlas $\mathcal{A}'$ of M has its own maximal atlas $\mathcal{A}$, which is obtained from $\mathcal{A}'$ by adding to it all the homeomorphisms φ as in condition (2) above. Therefore, a smooth manifold is completely determined by a topological manifold M and a (not necessarily maximal) smooth atlas on it, since this automatically yields a maximal smooth atlas.

If two different smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$ on a topological manifold M have the same maximal atlas, then they define the same smooth structure on M, i.e., the sheaf of differentiable functions is the same for the two atlases.

Now we define the category $\mathbf{DMan}^n$ of differentiable n-manifolds. The objects are pairs (M, S_M) of smooth n-manifolds and their sheaf of differentiable functions. Recall that by our definitions, these have all connected components of dimension n. The arrows are continuous functions $f : M \to N$ such that for each open set $V \subset N$ we have $f^*(\mathrm{S}_N(V)) \subset \mathrm{S}_M(f^{-1}(V))$. In other words, if $g : V \to \mathbb{R}$ is smooth, then $g \circ f : f^{-1}(V) \to \mathbb{R}$ is smooth. Such functions f are called differentiable (or smooth) maps, and we write $f \in C^\infty(M, N)$. It is easy to check that $f \in C^\infty(M, N)$ if and only if $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is a C^∞ map between open subsets of Euclidean spaces, where $(U_\alpha, \varphi_\alpha)$ is a smooth chart of M and (V_β, ψ_β) is a smooth chart of N.

The isomorphisms in the category $\mathbf{DMan}^n$ are called *diffeomorphisms*. Note that if we endow the same topological manifold M with two different smooth atlases $\mathcal{A}_1$ and $\mathcal{A}_2$, then the two atlases define the same smooth structure if and only if $\mathrm{Id} : (X, \mathcal{A}_1) \to (X, \mathcal{A}_2)$ is an isomorphism in the category $\mathbf{DMan}^n$.

We also consider the subcategory $\mathbf{DMan}_c^n$ of compact differentiable n-manifolds. And we denote the category $\mathbf{DMan}$ of all smooth manifolds, which may have components of different dimensions.

Remark 1.18. It is important to note the difference between being isomorphic and being equal in the category $\mathbf{DMan}^n$. For example, the real line $\mathbb{R}$ can be endowed, as a topological manifold, with *different* smooth structures, which nevertheless are isomorphic in $\mathbf{DMan}^1$. For instance, we have the standard smooth structure in $\mathbb{R}$ given by the chart $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$, $t \mapsto t$. Another smooth structure is given by the chart $\phi : \mathbb{R} \to \mathbb{R}$, $t \mapsto t^3$. These two smooth structures are not the same, since ϕ does not belong to the atlas $(\mathbb{R}, \mathrm{Id})$. Also, $\mathrm{Id} : \mathbb{R} \to \mathbb{R}$ is not even differentiable for $(\mathbb{R}, \phi)$, since $\mathrm{Id} \circ \phi^{-1} : \mathbb{R} \to \mathbb{R}$ is not C^∞ at $0 \in \mathbb{R}$. On the other hand we do have a diffeomorphism $F : (\mathbb{R}, \phi) \to (\mathbb{R}, \mathrm{Id})$, $F(x) = x^{1/3}$ which expressed in charts is just the identity map $\mathbb{R} \to \mathbb{R}$. This proves that $(\mathbb{R}, \phi) \cong (\mathbb{R}, \mathrm{Id})$ in $\mathbf{DMan}^1$.

Remark 1.19. Recall that an *embedding* between two topological spaces, $f : X \to Y$, is a continuous map which is a homeomorphism onto the image $f(X) \subset Y$ (with subspace topology). If M, N are (topological) manifolds and $N \subset M$, we will say that N is a *submanifold* if the inclusion map $i : N \to M$ is a smooth map that is an embedding.

Moreover, if M and N are also smooth, we can also consider a weaker notion called an *immersion*. This is a smooth map $f : N \to M$ such that its differential (see Remark 3.5) $d_p f : T_p N \to T_{f(p)} M$ is a linear monomorphism for all $p \in N$. This implies that f is locally invertible onto its image, but it may happen that globally it is not injective or it is not open onto its image for subspace topology on $f(N)$ (see [Boo]).

Remark 1.20. There is a notion of C^r structure, $r \geq 1$. This is given by an atlas in which the changes of charts are given by C^r differentiable maps. A C^r manifold is a manifold with a C^r structure. It is a fact that a manifold with a C^r structure admits a compatible C^s atlas for any $s \geq r$, including $s = \infty$ (see [Hir]). There is also the notion of analytic manifold, in which the changes of charts are given by (real) analytic maps, usually denoted C^ω. It is a deep theorem of Whitney that a compact smooth manifold admits a compatible real analytic structure [Wh1].

Remark 1.21. One of the main concerns of differential geometry is the classification problem. Suppose that $\mathcal{C}$ is a subcategory of locally connected spaces of **Top**, and we want to study $\mathbb{L}_\mathcal{C}$. Without loss of information, we can restrict our attention to

$$\mathbb{L}^{co}_\mathcal{C} = \{X \in \mathbb{L}_\mathcal{C} | X \text{ is connected}\}. \tag{1.2}$$

From this we recover $\mathbb{L}_\mathcal{C}$ by taking disjoint unions of elements of $\mathbb{L}^{co}_\mathcal{C}$. Observe that, if $\mathcal{C}$ is only composed of compact topological spaces, we take finite disjoint unions up to reordering.

Remark 1.22. It is interesting to study the *forgetful functor* $\mathbf{DMan}^n \to \mathbf{TMan}^n$ which ignores the smooth structure. It is important to study whether this functor induces a bijection or not,

$$\mathbb{L}^{co}_{\mathbf{DMan}^n_c} \to \mathbb{L}^{co}_{\mathbf{TMan}^n_c}. \tag{1.3}$$

Note that (1.3) is surjective if and only if every topological n-manifold admits some smooth structure. It is injective if and only if every topological manifold admits at most one smooth structure up to diffeomorphism. Said otherwise, if two smooth n-manifolds are homeomorphic, then they are diffeomorphic.

(a) In dimension 0, $\mathbb{L}^{co}_{\mathbf{DMan}^0_c} = \{\star\}$ (see Example 1.8).

(b) In dimension 1, $\mathbb{L}^{co}_{\mathbf{DMan}^1_c} = \{S^1\}$ and $\mathbb{L}^{co}_{\mathbf{DMan}^1} = \{S^1, \mathbb{R}\}$ (see Exercise 1.4). The forgetful map (1.3) is therefore bijective.

(c) In dimension 2 we will give a complete classification of compact surfaces in Chapters 1 and 2 (Theorem 2.29), and we will see that all of them admit a unique smooth structure (Theorem 6.56). Therefore $\mathbb{L}^{co}_{\mathbf{DMan}^2_c} \to \mathbb{L}^{co}_{\mathbf{TMan}^2_c}$ is bijective.

(d) In dimension 3, a result of Moise proves that $\mathbb{L}^{co}_{\mathbf{DMan}^3_c} \to \mathbb{L}^{co}_{\mathbf{TMan}^3_c}$ is bijective [Moi]. However, the classification of compact 3-manifolds has not been completed.

(e) In dimensions $n \geq 5$, the forgetful map $\mathbb{L}^{co}_{\mathbf{DMan}^n_c} \to \mathbb{L}^{co}_{\mathbf{TMan}^n_c}$ is neither injective nor surjective. This failure is controlled by homotopy theoretic invariants, so it is known for a topological n-manifold how many smooth structures it admits (in particular, there are finitely many for a given topological compact n-manifold). The most remarkable achievement was the discovery by Milnor [Mi1] of a differentiable manifold homeomorphic to S^7 but not diffeomorphic to the standard round sphere $S^7 \subset \mathbb{R}^8$. This gave rise to a collection of examples of *exotic* differentiable structures on spheres S^n, which happen in some dimensions $n \geq 7$.

(f) In dimension 4, the forgetful map $\mathbb{L}^{co}_{\mathbf{DMan}^4_c} \to \mathbb{L}^{co}_{\mathbf{TMan}^4_c}$ is neither injective nor surjective. The classification of compact topological *simply connected* 4-manifolds was given by Freedman [Fre]. Also strong restrictions for a 4-manifold to admit a smooth structure have been determined, initially by Donaldson [Do1], although they are not of homotopy theoretic type and depend on tools of global analysis (i.e., partial differential equations on manifolds). About the failure of injectivity, this is known by means of invariants of the smooth structure (Donaldson and Seiberg-Witten invariants), and there are instances of infinitely many (countable) smooth structures on a given compact connected topological 4-manifold.

(g) The non-compact case is by far more involved. The following result is remarkable: $\mathbb{R}^n$ admits a unique smooth structure for $n \neq 4$, and $\mathbb{R}^4$ admits a non-countable number of non-diffeomorphic smooth structures [Sco].

Remark 1.23. In algebraic geometry, the main objects of study are the *algebraic varieties*. It is remarkable that we can give a definition in large analogy with that of smooth manifold. For the general theory we refer to [Har] or to Chapter 5. An (abstract) *algebraic variety* over the complex numbers is a topological space X, endowed with a sheaf $\mathcal{O}_X^{\mathrm{alg}}$, called the sheaf of *regular functions* or *algebraic functions*, such that every point $p \in X$ has an open neighbourhood U^p and a homeomorphism $\varphi : U^p \to V(I) = \{x \in \mathbb{C}^n | F(x) = 0, \text{ for all } F \in I\} \subset \mathbb{C}^n$, where $I = I(F_1, \ldots, F_d)$ is the ideal generated by some polynomials with complex coefficients $F_i(x_1, \ldots, x_n) \in \mathbb{C}[x_1, \ldots, x_n]$. Moreover, for a function $f : U^p \to \mathbb{C}$, we require that $f \in \mathcal{O}_X^{\mathrm{alg}}(U^p)$ if and only if the function $f \circ \varphi^{-1} : V(I) \to \mathbb{C}$ is a well defined rational function, i.e.,

$$f \circ \varphi^{-1} \in \Big\{ \frac{p}{q} \,\Big|\, p, q \in \mathbb{C}[x_1, \ldots, x_n], q \text{ never vanishes on } V(I) \Big\}.$$

In other words, in the category of algebraic varieties, we want to be able to define the sheaf of rational functions on X.

We have the following alternative characterization. An algebraic variety X is a topological space with an atlas $\{(U_\alpha, \varphi_\alpha)\}$ with charts $\varphi_\alpha : U_\alpha \to V(I_\alpha) = \varphi_\alpha(U_\alpha)$ such that for each non-empty intersection $U_\alpha \cap U_\beta \neq \emptyset$, the change of charts $\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ can be expressed as the restriction of a map whose components are rational functions, i.e.,

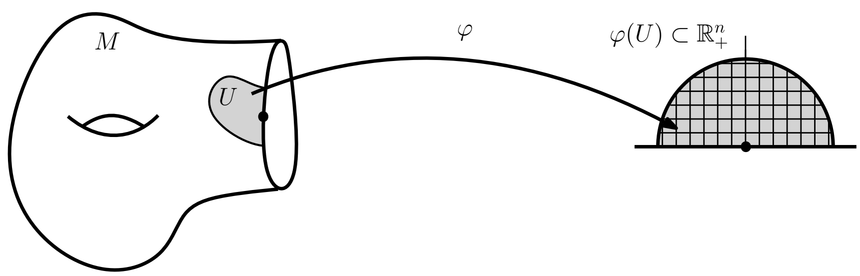
$$\varphi_{\alpha\beta}(x_1, \ldots, x_n) = \Big(\frac{p_1}{q_1}(x_1, \ldots, x_n), \ldots, \frac{p_n}{q_n}(x_1, \ldots, x_n) \Big),$$

being $p_i, q_i \in \mathbb{C}[x_1, \ldots, x_n]$ polynomials.

A couple of comments are relevant: The varieties $X = V(I)$ are called *affine varieties* (these are algebraic subvarieties of $\mathbb{C}^n$, see Remark 5.47). These are not open subsets of $\mathbb{C}^n$, and the dimension of X is not n in general. The second remark is that the topology used in algebraic geometry for X is highly non-Hausdorff. It is called the *Zariski topology*, and its closed subsets are given by the algebraic subvarieties of X.

1.1.3. Manifolds with boundary. Now we come back to topological manifolds and introduce the concept of *manifold with boundary*.

Definition 1.24. A *manifold with boundary* M is a topological space which is Hausdorff, second countable, and such that each $p \in M$ has a neighbourhood $U = U^p$ which is homeomorphic to an open set of $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n | x_1 \geq 0\}$ via $\varphi : U \to \varphi(U) \subset \mathbb{R}^n_+$.



It is easy to see that in the previous definition, one can reduce the open set U^p so that $\varphi(U)$ is either a ball $B_r(0)$ or a semiball $B_r^+(0) = B_r(0) \cap \mathbb{R}^n_+$, with $\varphi(p) = 0$. In the first case the point p is called an *interior point*, and we write $p \in \mathrm{Int}(M)$; in the second case p is called a *boundary point*,[1] and we write $p \in \partial M$. We have the following.

Proposition 1.25. *The definition of being an interior and boundary point does not depend on the chosen chart* φ.

Proof. We shall prove this for our most relevant case $n = 2$, where we use the fundamental group. The general case can be treated with the aid of homology groups (cf. Exercise 2.21).

Suppose that there exists a point $p \in M$ that has neighbourhoods U^p, V^p with homeomorphisms $f : U \to B_\epsilon(0)$ and $g : V \to B_\eta^+(0)$ with $f(p) = 0$, $g(p) = 0$. Take $0 < \delta < \eta$ such that $g^{-1}(B_\delta^+(0)) \subset U \cap V$, and take $0 < \nu < \epsilon$ such that $\bar{B}_\nu(0) \subset f(g^{-1}(B_\delta^+(0)))$. Then $f \circ g^{-1} : B_\delta^+(0) - \{0\} \to f(g^{-1}(B_\delta^+(0))) - \{0\}$ is a homeomorphism. The first space is contractible. The second space is contained in $B_\epsilon(0) - \{0\}$ and retracts radially to $S_\nu(0) = \partial \bar{B}_\nu(0) \cong S^1$. Therefore it must be that $\pi_1(B_\delta^+(0) - \{0\}) = \{1\}$ surjects to $\pi_1(S_\nu(0)) = \mathbb{Z}$, which is impossible. □

We conclude that $M = \mathrm{Int}(M) \bigsqcup \partial M$ as a disjoint union of sets. Since $\mathrm{Int}(M)$ is clearly open, ∂M is closed. (A comment in terminology: Technically, a topological

[1] We use the notation ∂M for the boundary of M, whereas the notation ∂A is also used for the topological boundary of a set $A \subset X$ of a topological space. We hope this causes no confusion as the meaning should be clear from the context in each case.

manifold M is also a manifold with boundary such that $\partial M = \emptyset$. We sometimes stress this as saying that M is a manifold without boundary.)

As in the case of manifolds, a manifold with boundary is a union of connected components and each of them has a well defined dimension. An n-manifold with boundary is a manifold with boundary (or without!) such that all connected components have dimension n. If M is an n-manifold with boundary, then $\mathrm{Int}(M)$ is an n-manifold without boundary. Moreover, ∂M is a manifold of dimension $n-1$ without boundary (maybe not connected). To see this, take $p \in \partial M$ and a chart $\varphi : U^p \to B_\epsilon^+(0)$, $\varphi(p) = 0$. Then $(\partial M) \cap U = \varphi^{-1}(B_\epsilon^+(0) \cap \{x_1 = 0\})$, and $\varphi : (\partial M) \cap U \to B_\epsilon^+(0) \cap \{x_1 = 0\} = B_\epsilon^{n-1}(0)$ is an $(n-1)$-dimensional chart.

We can also define the concept of *smooth manifold with boundary*. To do this, first we define what it means to be a C^∞ function $f : B_\epsilon^+(0) \to \mathbb{R}$. This simply means that f has an extension $F \in C^\infty$ to some open set of $\mathbb{R}^n$ containing $B_\epsilon^+(0)$. A map $\varphi : B_\epsilon^+(0) \to B_r^+(0)$ is C^∞ if all its coordinate functions are C^∞, and it is a C^∞ diffeomorphism if it is a homeomorphism that is C^∞ and its inverse is also C^∞. Having done this, we say that a manifold with boundary M is *smooth* if it has an atlas whose change of coordinates are C^∞ maps between open sets of $\mathbb{R}^n_+$. In analogy with the case without boundary, this definition is equivalent to defining a sheaf S of differentiable functions on M. If a manifold with boundary M is smooth, then its boundary ∂M is also smooth, and it has a canonical smooth structure inherited from that of M by restricting the charts to $\partial \mathbb{R}^n_+ = \{x_1 = 0\}$.

The categories of n-manifolds with boundary, compact n-manifolds with boundary, and the smooth counterparts are $\mathbf{TMan}^n_\partial$, $\mathbf{TMan}^n_{\partial,c}$, $\mathbf{DMan}^n_\partial$, and $\mathbf{DMan}^n_{\partial,c}$, respectively. There are corresponding classification problems (cf. Exercise 1.7).

1.2. PL structures

1.2.1. Quotient topologies. When we have a topological space, any equivalence relation on it yields a natural topology on the quotient space, as follows.

Definition 1.26. Let X be a topological space, and suppose that we have an equivalence relation $\sim$ on X. Let $\pi : X \to X^\star = X/\!\sim$ be the map passing to the quotient. The *quotient topology* on $X^\star$ is the topology where $U^\star \subset X^\star$ is open if and only if $\pi^{-1}(U^\star) \subset X$ is open.

Accordingly with the notation above, open and closed subsets of $X^\star$ will be denoted with the superscript $\star$. Given an element $x \in X$, its equivalence class under $\sim$ will be denoted by $[x]$. It is easy to see that the open subsets of $X^\star$ are the images by π of the open sets of X which are the union of equivalence classes. These open sets are called *saturated* for the equivalence relation $\sim$.

Since it is always uncomfortable (and usually ineffective) to work with equivalence classes of objects, it is useful to have criteria for knowing whether a space is homeomorphic to a quotient $X/\!\sim$. For this purpose the following is defined.

Definition 1.27. Let X and Y be topological spaces. A *quotient map* $q : X \to Y$ is a continuous, surjective map, such that if we define the equivalence relation $\sim$ on X by

$x \sim x'$ if and only if $q(x) = q(x')$, then the natural map $\bar{q}$ that makes the following diagram commutative,

$$\begin{array}{ccc} X & \xrightarrow{q} & Y \\ {\scriptstyle\pi}\downarrow & \nearrow_{\bar{q}} & \\ X^\star & & \end{array}$$

where $X^\star = X/{\sim}$ is the quotient space, is a homeomorphism $\bar{q} : X^\star \xrightarrow{\cong} Y$.

Obviously the quotient map $q : X \to X^\star = X/{\sim}$ satisfies the above definition for any equivalence relation $\sim$, by the sheer definition of the quotient topology in $X^\star$.

Proposition 1.28. *Let $q : X \to Y$ be a continuous, surjective map between topological spaces X, Y. The following are true:*

(1) *If q is an open map, then it is a quotient map.*
(2) *If q is a closed map, then it is a quotient map.*
(3) *If X is compact and Y is Hausdorff, then q is a quotient map.*

Proof. Let $\sim$ be the equivalence relation $x \sim x'$ if and only if $q(x) = q(x')$, let $\pi : X \to X^\star = X/{\sim}$ be the quotient map given by $\sim$, and let $\bar{q} : X^\star \to Y$ be the induced map.

To see (1) and (2) it is enough to note that for an open (resp., closed) set $A^\star \subset X^\star$, we have that $\bar{q}(A^\star) = q(\pi^{-1}(A^\star))$ is an open (resp., closed) subset of Y since q is an open (resp., closed) map and π is continuous. We conclude that $\bar{q}$ is continuous, bijective, and open (resp., closed), and therefore is a homeomorphism and q is a quotient map.

Let us see (3) by proving that q is a closed map. If $F \subset X$ is closed, then it is compact. Therefore $q(F) \subset Y$ is compact, and then it is closed since Y is Hausdorff. □

Definition 1.29. Let X be a topological space, and let $Z \subset X$ be a subspace. We denote X/Z the space obtained by *collapsing* Z, that is, the quotient $X/{\sim}$, where $\sim$ is defined by $z \sim z'$, for all $z, z' \in Z$.

Example 1.30. Let us prove that the closed disc D^n collapsing ∂D^n, $D^n/\partial D^n$, is homeomorphic to the sphere S^n. The equivalence relation in D^n is $x \sim x' \Leftrightarrow x, x' \in \partial D^n$. First define the map $x = (x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, 2||x||^2 - 1)$, which puts the points of the disc on a paraboloid. Now we expand the meridians (that is, the n first coordinates $(x_1, \dots, x_n)$) by a factor of λ while leaving the height (the last coordinate) fixed, so as to make them lie in S^n. Substituting in the equation of S^n, we get $||\lambda x||^2 + (2||x||^2 - 1)^2 = 1$, which yields $\lambda = 2\sqrt{1 - ||x||^2}$. We obtain the map

$$q : D^n \to S^n,$$
$$q(x_1, \dots, x_n) = \left(2\sqrt{1 - ||x||^2}\, x_1, \dots, 2\sqrt{1 - ||x||^2}\, x_n, 2||x||^2 - 1\right).$$

The map q is continuous. Moreover, q maps $\partial D^n = \{||x|| = 1\}$ to the north pole $N = (0, \dots, 0, 1)$, and it is a bijection between $\mathrm{Int}(D^n)$ and $S^n - \{N\}$, so q induces the desired equivalence relation. Finally, since D^n is compact and S^n is Hausdorff, Proposition 1.28(3) yields that the quotient $D^n/{\sim}$ is homeomorphic to the sphere S^n, as required.

Example 1.31. The (*real*) *projective space* $\mathbb{R}P^n$ is defined as the quotient space $(\mathbb{R}^{n+1} - \{0\})/\mathbb{R}^*$, where $\mathbb{R}^* = \mathbb{R} - \{0\}$ acts by $\lambda \cdot (x_0, \dots, x_n) = (\lambda x_0, \dots, \lambda x_n)$, with its quotient topology. The equivalence relation is thus $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$, for $\lambda \in \mathbb{R}^*$. The quotient is denoted as $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ and points in the quotient as $[x_0, \dots, x_n] = \pi(x_0, \dots, x_n)$.

- $\mathbb{R}P^n$ is compact. Take $S^n \subset \mathbb{R}^{n+1} - \{0\}$. As any vector $v \sim v/||v||$, we have that $\pi : S^n \to \mathbb{R}P^n$ is surjective, and since S^n is compact, so is $\mathbb{R}P^n$. By the same argument, $\mathbb{R}P^n$ is connected.
- The projective space is also Hausdorff. If $[v] \neq [w]$, take representants $v, w \in S^n$. Then $v \neq \pm w$, so we can take neighbourhoods U^v, V^w such that $U \cap V = \emptyset$ and $U \cap (-V) = \emptyset$. So $U^\star = \pi(U \cup (-U))$ and $V^\star = \pi(V \cup (-V))$ are disjoint neighbourhoods of $[v], [w]$, respectively. By Proposition 1.28, π is a quotient map. Therefore $\mathbb{R}P^n \cong S^n/\!\sim$, where $v \sim -v$.
- $\mathbb{R}P^n$ is a topological n-manifold. Let $\pi : S^n \to \mathbb{R}P^n$ be the quotient map. Take $p = \pi(x) \in \mathbb{R}P^n$. Consider an open set $U = U^x \subset S^n$, and a chart $\varphi : U \to B_\epsilon(0) \subset \mathbb{R}^n$, small enough so that $U \cap (-U) = \emptyset$. Then $\pi|_U : U \to U^\star = \pi(U) = \pi(U \cup (-U))$ is a homeomorphism. Therefore $\bar{\varphi} = \varphi \circ (\pi|_U)^{-1} : U^\star \to B_\epsilon(0)$ is a chart for $\mathbb{R}P^n$.
- Affine charts. When we remove a hyperplane of $\mathbb{R}P^n$, we have an affine space; e.g., take $H = \{x_0 = 0\} \subset \mathbb{R}P^n$, and accordingly in the previous item we can take $U = \{x_0 > 0\} \subset S^n$. As a chart, we consider $\varphi : U \to \mathbb{R}^n$, $\varphi(x_0, x_1, \dots, x_n) = (x_1/x_0, \dots, x_n/x_0)$. This gives a chart for $U^\star = \mathbb{R}P^n - H$, defined as $\bar{\varphi}([x_0, x_1, \dots, x_n]) = (x_1/x_0, \dots, x_n/x_0)$. This is the same as the affine structure for $U^\star$, given as $\bar{\varphi} : U^\star \to \mathbb{R}^n$.
- $\mathbb{R}P^n$ is a smooth manifold. The affine charts give a smooth atlas. Let us see this. For each $i = 0, 1, \dots, n$, take the open set $U_i = \{[x_0, \dots, x_n] | \, x_i \neq 0\} \subset \mathbb{R}P^n$. Define the chart $\varphi_i : U_i \to \mathbb{R}^n$,
$$\varphi_i([x_0, \dots, x_n]) = \left(\frac{x_0}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right),$$
where the symbol $\widehat{\ }$ means that the entry has been deleted. The inverse is $\varphi_i^{-1}(y_1, \dots, y_n) = [y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n]$. The changes of charts are given, for $i < j$ as
$$\varphi_i \circ \varphi_j^{-1}(y_1, \dots, y_n) = \left(\frac{y_1}{y_i}, \dots, \widehat{\frac{y_i}{y_i}}, \dots, \frac{y_j}{y_i}, \frac{1}{y_i}, \frac{y_{j+1}}{y_i}, \dots, \frac{y_n}{y_i} \right),$$
defined for $(y_1, \dots, y_n)$, $y_i \neq 0$. This is a diffeomorphism onto its image, so $\mathcal{A} = \{(U_i, \varphi_i) | 0 \leq i \leq n\}$ is a smooth atlas.

The following topological construction is very useful when it comes to constructing new topological spaces from old ones.

Definition 1.32. Let X, Y be topological spaces, let $A \subset X$, and let $f : A \to Y$ be a continuous map. We define the topological space
$$X \cup_f Y = (X \sqcup Y)/\!\sim,$$

where the identifications are for each $a \in A$, $a \sim f(a)$. We endow it with the quotient topology.

Example 1.33.

(1) Wedge (pointed union) of spaces. Consider $(X, p), (Y, p') \in \mathrm{Obj}(\mathbf{Top}_*)$. We define the wedge $X \vee Y$ as $X \cup_f Y$, where $A = \{p\}$, $f(p) = p'$. Let $[p] = \pi(p) = \pi(p')$, where $\pi : X \sqcup Y \to X \vee Y$ is the quotient map. Then $(X \vee Y, [p]) \in \mathrm{Obj}(\mathbf{Top}_*)$. The neighbourhoods of $[p]$ in $X \vee Y$ are given as $\pi(U^p \sqcup V^{p'})$, where U^p and $V^{p'}$ are neighbourhoods of p and p', respectively.

(2) Attaching n-cells. Given a topological space X and $f : \partial D^n \to X$ a continuous map, we define $X' = X \cup_f D^n$, and we say that X' is obtained from X by *attaching an n-cell*.

(3) Mapping torus. Let $f : X \to X$ be a homeomorphism, We define the *mapping torus* as
$$T_f = X \times [0,1]/\!\sim,$$
where $(x, 0) \sim (f(x), 1)$, so we have the quotient map $q : X \times [0,1] \to T_f$. If X is a topological n-manifold, then T_f is a topological $(n+1)$-manifold. Also if X is an n-manifold with boundary, T_f is a $(n+1)$-manifold with boundary (Exercise 1.12).

In general the manifold T_f depends crucially on f. When the map $f : X \to X$ is the identity, we get $T_{\mathrm{Id}} = X \times S^1$. For instance, if $X = [0,1]$ is a segment, $T_{\mathrm{Id}} = [0,1] \times S^1$ is the *cylinder*. When $X = [0,1]$ and $f(t) = 1 - t$, the manifold T_f is called the *Möbius band*, denoted *Mob* (see Figure 1.1).

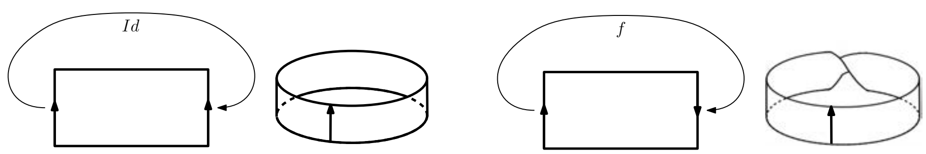


Figure 1.1. Cylinder and Möbius band.

(4) For $X = S^1$ a circle, $T_{\mathrm{Id}} = S^1 \times S^1$ is the *torus* (see Figure 1.2). If we consider the map $f : S^1 \to S^1$, $f(x,y) = (x, -y)$, we have the *Klein bottle*[2] Kl as the mapping torus of f. Equivalently, $Kl = S^1 \times [0,1]/\sim$, where $(e^{2\pi i t}, 0) \sim (e^{2\pi i(1-t)}, 1)$. This surface cannot be depicted in $\mathbb{R}^3$ without self-intersections because the identification f forces one of the boundaries to be inserted inside the "tube" to glue with the other boundary with the right orientation. This is due to the fact that Kl is non-orientable (Example 1.75) and non-orientable compact surfaces without boundary cannot be embedded in $\mathbb{R}^3$ (Exercise 2.36).

[2] Anecdotally, the Klein bottle was first described by Klein in 1882. It was given the name Kleinsche Fläche ("Klein surface" in German), and then misinterpreted as Kleinsche Flasche ("Klein bottle"), which ultimately led to the adoption of this name. Probably the picture of Figure 1.2 helped for this.

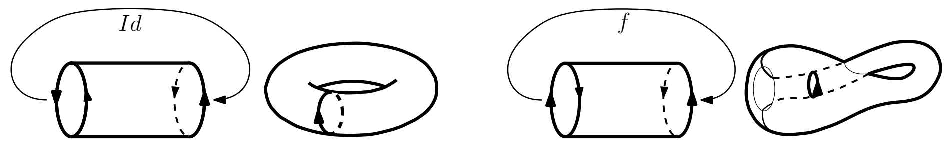


Figure 1.2. Torus and Klein bottle.

1.2.2. Connected sum. Let M_1 and M_2 be two topological connected n-manifolds. Choose points $p_i \in M_i$ and neighbourhoods $U^{p_i} = U_i \subset M_i$ with homeomorphisms $\varphi_i : U_i \to B_2(0)$. Set $W_i = \varphi_i^{-1}(B_1(0))$, so that $M_i^o = M_i - W_i$ are manifolds with boundary, being the boundary homeomorphic to the $(n-1)$-sphere via $\varphi_i : \partial M_i^o \to S_1(0) = S^{n-1}$.

Therefore $f = \varphi_2^{-1} \circ \varphi_1 : \partial M_1^o \to \partial M_2^o \subset M_2^o$ can be used as a gluing map to form the connected sum (see Figure 1.3),

$$M_1 \#_f M_2 = M_1^o \cup_f M_2^o.$$

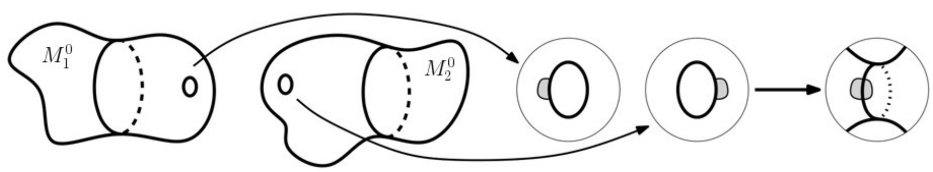


Figure 1.3. Connected sum.

It is easy to see that $M = M_1 \#_f M_2$ is a connected topological n-manifold. To see the existence of charts for M, consider the quotient map $\pi : M_1^o \sqcup M_2^o \to M$, and let $C = \pi(\partial M_1^o) = \pi(\partial M_2^o)$ be the "core" of the connected sum. If $p \in M - C$, then p lies in the interior of M_1^o or of M_2^o. In either case, a chart inside it serves as a chart for M. If $p \in C$, then $p = \pi(p_1) = \pi(p_2)$, with $p_j \in \partial M_j^o$, $j = 1, 2$, and $f(p_1) = p_2$. Pick neighbourhoods $V_j^{p_j} \subset U_j \cap M_j^o$, $j = 1, 2$, homeomorphic to semiballs of $\mathbb{R}^n_+$, and such that $f(\partial M_1^o \cap V_1) = \partial M_2^o \cap V_2$. Therefore $V_1 \cup_f V_2 = \phi(V_1 \sqcup V_2)$ is a neighbourhood of p and it is homeomorphic to a ball in $\mathbb{R}^n$.

The following concept is important in order to understand the connected sum. Two homeomorphisms $f_0, f_1 : X \to Y$ are called *isotopic* if they are homotopic via a homotopy $H : X \times [0,1] \to Y$ with the maps $f_s : X \to Y$, $f_s(x) = H(x,s)$, $s \in [0,1]$, being homeomorphisms. The set of homeomorphisms $X \xrightarrow{\cong} Y$ under the equivalence relation of being isotopic is called the *mapping class set* of (X, Y), and it is denoted $\mathrm{MCS}(X,Y)$. When $X = Y$ it is called the *mapping class group* and is denoted $\mathrm{MCG}(X)$. In the latter case, it is a group under composition.

Lemma 1.34.

(1) *For intervals* $[a,b]$ *and* $[c,d]$, $\mathrm{MCS}([a,b],[c,d]) = \{l_1, l_2\}$, *where* l_1 *and* l_2 *are the two affine homeomorphisms between the intervals, one of them increasing and the other decreasing.*

(2) *For the circle* S^1, $\mathrm{MCG}(S^1) = \{\mathrm{Id}, r\}$ *where* $r(x,y) = (x,-y)$. *In other words, any homeomorphism of* S^1 *is isotopic either to the identity or a reflection.*

Proof. Any homeomorphism $f : [a,b] \to [c,d]$ is either an increasing or a decreasing map. Two increasing maps f_0, f_1 are isotopic via $H_s = sf_0 + (1-s)f_1$ which is also increasing and therefore a homeomorphism. Analogously, two decreasing maps are isotopic. Finally, a decreasing and an increasing map are not isotopic because their restrictions to $\{a,b\}$ are different.

A homeomorphism $f : S^1 \to S^1$ is equivalent to a homeomorphism of the interval $\tilde{f} : [0,2\pi] \longrightarrow [\theta_0, \theta_0 + 2\pi]$ via

$$f(\cos(t), \sin(t)) = (\cos(\tilde{f}(t)), \sin(\tilde{f}(t))).$$

If $\tilde{f}$ is increasing, then f is isotopic to the identity via

$$H(t,s) = (\cos(s\tilde{f}(t) + (1-s)t), \sin(s\tilde{f}(t) + (1-s)t)).$$

Note that $H(t,s)$ are homeomorphisms for s fixed because the linear interpolation of increasing maps is increasing. If $\tilde{f}$ is decreasing, then f is isotopic to the reflection r via

$$H(t,s) = (\cos(s\tilde{f}(t) + (1-s)(2\pi - t)), \sin(s\tilde{f}(t) + (1-s)(2\pi - t))).$$ □

It is also true that $\mathrm{MCG}(S^n) = \{\mathrm{Id}, r\}$, for $n \geq 2$, where r is a reflection on a hyperplane, although this is harder to prove.

Theorem 1.35. *The connected sum* $M_1 \#_f M_2$ *does not depend on the points* p_1, p_2, *up to homeomorphism. With respect to the chosen charts, suppose that we have a fixed chart* φ_2 *and two charts* φ_1, φ_1'. *Let* $f = \varphi_2^{-1} \circ \varphi_1 : \partial M_1^o \to \partial M_2^o$ *and* $f' = \varphi_2^{-1} \circ \varphi_1' : \partial (M_1^o)' \to \partial M_2^o$. *There exists a homeomorphism* $\phi : \partial M_1^o \to \partial (M_1^o)'$ *such that if* f *is isotopic to* $f' \circ \phi$, *then the connected sums with the chart* φ_1 *and with the chart* φ_1' *are homeomorphic.*

Proof. This is a deep result that requires hard techniques. We shall sketch the case $n = 2$, where we use Lemma 1.34. Let $p_1, p_1' \in M_1$ be two different points, and let $p_2 \in M_2$. By Exercise 1.14, there is a homeomorphism $\phi : M_1 \to M_1$ with $\phi(p_1) = p_1'$. Thus doing the connected sum with the charts φ_1, φ_2 (centered at the points p_1, p_2) gives a surface homeomorphic to the surface obtained by doing a connected sum with the charts $\varphi_1 \circ \phi^{-1}, \varphi_2$ (centered at the points p_1', p_2). This proves the invariance with respect to the choice of points.

Now suppose that we have two charts $\varphi_1 : U^{p_1} \to B_2(0)$, $\varphi_1' : V^{p_1} \to B_2(0)$ both centered at $p_1 \in M_1$, and a chart $\varphi_2 : W^{p_2} \to B_2(0)$ centered at $p_2 \in M_2$. Take a small ball $B_\epsilon(0)$ such that $\varphi_1^{-1}(\bar{B}_\epsilon(0)) \subset U \cap V$. The image $D = \varphi_1'(\varphi_1^{-1}(\bar{B}_\epsilon(0))) \subset B_2(0) \subset \mathbb{R}^2$ is a closed disc. We use the following result, whose proof can be found in [Cai].

Theorem 1.36 (Jordan-Schönflies). *If* $C \subset \mathbb{R}^2$ *is a* Jordan curve (*a subspace homeomorphic to* S^1), *then there is a homeomorphism* $\mathbb{R}^2 \to \mathbb{R}^2$ *that sends* C *to* $S^1 = \partial B_1(0) \subset \mathbb{R}^2$.

With Theorem 1.36, we can prove that $\bar{B}_2(0)-\mathrm{Int}(D)$ is homeomorphic to the annulus $\bar{B}_2(0)-B_1(0)$. Certainly, take two non-intersecting paths joining points of $\partial B_2(0)$ to ∂D. This divides the region into two parts bounded by a Jordan arc. So these two regions are homeomorphic to discs, and their union is homeomorphic to the annulus. This can be arranged to be the identity on $\partial B_2(0)$. There are homeomorphisms

$$\begin{aligned} M_1-\varphi_1^{-1}(B_2(0)) &\cong M_1-\varphi_1^{-1}(B_\epsilon(0)) \cong M_1-(\varphi_1')^{-1}(\mathrm{Int}\,D) \\ &= (M_1-(\varphi_1')^{-1}(B_2(0)))\cup(\varphi_1')^{-1}(\bar{B}_2(0)-\mathrm{Int}(D)) \\ &\cong (M_1-(\varphi_1')^{-1}(B_2(0)))\cup(\varphi_1')^{-1}(\bar{B}_2(0)-B_1(0)) \\ &= M_1-(\varphi_1')^{-1}(B_1(0)) \cong M_1-(\varphi_1')^{-1}(B_2(0)). \end{aligned}$$

Let $\phi : M_1^o \to (M_1^o)'$ be the induced homeomorphism by this identification. Taking the union with $M_2-\varphi_2^{-1}(B_1(0))$, we have a homeomorphism between the connected sum with the chart φ_1 and with the chart $\varphi_1'\circ\phi$.

To see that the gluing f is only important up to isotopy, fix homeomorphisms $S^1\cong\partial M_1^o$ and $S^1\cong\partial M_2^o$ (for instance, through the charts φ_1 and φ_2). Now we can see f as a map $f : S^1\to S^1$. Suppose that f_0, f_1 are two isotopic homeomorphisms, with $H : S^1\times[0,1]\to S^1$ their isotopy. There is a homeomorphism $M_1-\varphi_1^{-1}(B_1(0))\cong(M_1-\varphi_1^{-1}(B_2(0)))\cup_{f_0}(S^1\times[0,1])$, where the map $h : \bar{B}_2(0)-B_1(0)\to S^1\times[0,1]$ is defined as $h(x)=\left(H\left(\frac{x}{||x||},2-||x||\right),2-||x||\right)$. Hence

$$\begin{aligned} &(M_1-\varphi_1^{-1}(B_1(0)))\cup_{f_1}(M_2-\varphi_2^{-1}(B_2(0))) \\ &\quad\cong (M_1-\varphi_1^{-1}(B_2(0)))\cup_{f_0}(S^1\times[0,1])\cup_{f_1}(M_2-\varphi_2^{-1}(B_2(0))) \\ &\quad\cong (M_1-\varphi_1^{-1}(B_2(0)))\cup_{f_0}(M_2-\varphi_2^{-1}(B_1(0))), \end{aligned}$$

doing the same trick on the M_2 side. At this point the result follows from the obvious homeomorphisms that allow us to change the radii of the chosen holed balls. □

Remark 1.37.

- Take $\psi=\varphi_1'\circ\phi\circ\varphi_1^{-1} : S^{n-1}\to S^{n-1}$ so that gluing with the chart $\varphi_1'\circ\phi$ is the same as gluing with the chart $\psi\circ\varphi_1$. Observe that f is isotopic to f' if and only if $\psi : S^{n-1}\to S^{n-1}$ is isotopic to the identity. In this way, we can reduce the indeterminacy to the mapping class group of the standard sphere. However, there are two isotopy classes of homeomorphisms $S^{n-1}\to S^{n-1}$, so Theorem 1.35 implies that there are at most two possible connected sums.
- The indeterminacy in Theorem 1.35 can be sometimes reduced. Let $X\#_{\mathrm{Id}}Y$ and $X\#_rY$ denote the two connected sums (with ψ the identity, and with ψ the antipodal map). Then if X is non-orientable, $X\#_{\mathrm{Id}}Y\cong X\#_rY$ (cf. Exercise 1.26). If X, Y are orientable and there exists an orientation reversing homeomorphism $f : X\to X$, then $X\#_{\mathrm{Id}}Y\cong f(X)\#_rY=X\#_rY$ (see section 1.4 for the notion of orientability). This happens in dimension 2 (Remark 1.81), so it is customary to denote just $X\#Y$. The first dimension where this does not occur is 4 (see Exercise 5.19).

Remark 1.38. The connected sum operation can be performed on connected n-manifolds with boundary. One only has to take care that the points chosen for removing the balls should be interior. Note that $\partial(X\#Y) = (\partial X) \sqcup (\partial Y)$.

Also, the connected sum can be performed on non-connected n-manifolds by choosing a connected component on each of the summands, but it will depend on the choice.

Now we discuss the connected sum in the smooth category $\mathbf{DMan}^n$. Let M_1, M_2 be two smooth connected n-manifolds and pick interior points $p_i \in M_i$. Choose C^∞ charts $\varphi_1 : U_1 = U_1^{p_1} \to B_2(0)$ and $\varphi_2 : U_2 = U_2^{p_2} \to B_2(0)$. Put also

$$A_i = \varphi_i^{-1}(B_{1+\varepsilon}(0) - \bar{B}_{1/(1+\varepsilon)}(0)),$$

$\varepsilon > 0$ small, which is an open subset of M_i, and $M_i' = M_i - \varphi_i^{-1}(\bar{B}_{1/(1+\varepsilon)}(0))$ for $i = 1, 2$, which is an open n-manifold. The diffeomorphism $f = \varphi_2^{-1} \circ \varphi_1 : \partial M_1^o \to \partial M_2^o$ can be extended to a diffeomorphism $F = \varphi_2^{-1} \circ \varphi_1 : A_1 \to A_2$. Note that f is the identity when expressed in the charts.

Definition 1.39. We define the *smooth connected sum* as $M_1 \#_f M_2 = M_1' \cup_F M_2'$.

Since $F : A_1 \to A_2$ is a diffeomorphism between open subsets $A_i \subset M_i'$, $i = 1, 2$, it is easy to see that $M_1 \#_f M_2$ has a canonical smooth structure inherited from the structures of M_1 and M_2. Indeed, the atlas for $M_1 \#_f M_2$ is obtained from an atlas of M_1' and M_2', where the changes of coordinates involve the diffeomorphism F for open sets contained in A_1 or A_2.

The manifold M is topologically the same as the topological connected sum $M_1 \#_f M_2$. This is seen as follows. The inclusion $\imath : M_1^o \sqcup M_2^o \to M_1' \sqcup M_2'$ is compatible with the equivalence relations, so it induces a continuous map $F : (M_1^o \sqcup M_2^o)/\sim \to (M_1' \sqcup M_2')/\sim$. This is clearly injective, and it is easily seen to be surjective since the sets $\pi(M_1' - M_1^o) = \pi(\varphi_1^{-1}(B_{1+\varepsilon}(0) - B_1(0))) = \pi(\varphi_2^{-1}(B_1(0) - B_{1/(1+\varepsilon)}(0))) \subset \pi(M_2^o)$ and $\pi(M_2' - M_2^o) \subset \pi(M_1^o)$. It is also seen to be open, since a basis of open sets has been constructed for the charts of $(M_1^o \sqcup M_2^o)/\sim$, and its image under F is open.

In analogy with the continuous case, we say that two diffeomorphisms $f_0, f_1 : X \to Y$ between smooth manifolds are diffeotopic if there exists a differentiable isotopy $H : X \times [0, 1] \to Y$ between f_0, f_1, where all maps $f_s : X \to Y$, $f_s(x) = H(x, s)$, are diffeomorphisms.

Theorem 1.40. *The smooth connected sum $M_1 \#_f M_2$ does not depend on the points p_1, p_2, up to diffeomorphism. With respect to the chosen charts, suppose that we have a fixed chart φ_2 and two charts φ_1, φ_1'. Let $f = \varphi_2^{-1} \circ \varphi_1 : \partial M_1^o \to \partial M_2^o$ and let $f' = \varphi_2^{-1} \circ \varphi_1' : \partial(M_1^o)' \to \partial M_2^o$. There exists a diffeomorphism $\phi : \partial M_1^o \to \partial(M_1^o)'$ such that if f is diffeotopic to $f' \circ \phi$, then the smooth connected sums with the chart φ_1 and with the chart φ_1' are diffeomorphic.*

Proof. This result (not being completely trivial) is simpler than that of Theorem 1.35. The case $n = 2$ appears in Exercise 1.15. □

Remark 1.41. As in Remark 1.37, the indeterminacy on f is controlled by a diffeomorphism $\psi : S^{n-1} \to S^{n-1}$. As the charts are defined on balls, ψ is the restriction of a diffeomorphism between open balls. Up to diffeotopy, there are only two choices for this type of diffeomorphism, given by the identity and the antipodal map.

There is a notion of *twisted connected sum* given by using a diffeomorphism $\psi : S^{n-1} \to S^{n-1}$ which cannot be extended to the ball. In this case we may obtain many non-diffeomorphic twisted connected sums $M_1 \#_f M_2$. For instance, the Milnor spheres (cf. Remark 1.22(e)) can be obtained by gluing two discs D^7 of dimension 7 via an exotic diffeomorfism of the sphere $S^6 = \partial D^7$. This feature is in sharp contrast with the case of the topological connected sum.

1.2.3. Piecewise linear manifolds. In order to better understand the differences between the world of topological manifolds and that of smooth manifolds, it is convenient to introduce an intermediate kind of structure on manifolds, the so-called *PL structures* (PL stands for piecewise linear). This is a structure of a combinatorial nature and provides useful algebraic tools for topological computations. The strategy consists of putting *triangulations* on manifolds. We need first a technical definition.

Definition 1.42. Let X be a topological space, and let $\mathcal{C} = \{C_i \mid i \in I\}$ be a covering of X (this means that $C_i \subset X$ and $\bigcup_{i\in I} C_i = X$). We say that X has a *coherent topology* with respect to the cover $\mathcal{C}$ if the natural inclusion map

$$q : \bigsqcup_{i\in I} C_i \longrightarrow X$$

is a quotient map. In other words, $U \subset X$ is open if and only if $U \cap C_i \subset C_i$ is open as a subset of C_i for each $i \in I$.

Example 1.43.

(1) The topology of $Y = \bigsqcup_{i\in I} C_i$ is the *disjoint union topology*, that is, $U \subset Y$ is open if $U = \bigsqcup_{i\in I} U_i$, with $U_i \subset C_i$ open for all $i \in I$.

(2) If all the C_i are open subsets of X, then it is clear that X has a coherent topology with respect to $\mathcal{C}$. Indeed, if $A \cap C_i \subset C_i$ is open as a subset of C_i, since $C_i \subset X$ is open, then $A \cap C_i \subset X$ also is open as a subset of X, so $A = \bigcup_{i\in I}(A \cap C_i)$ is open in X.

(3) If the C_i are compact, $|I| < \infty$, and X is Hausdorff, then X has a coherent topology with respect to the C_i. Here we denote by $|I|$ the cardinal of I. To see it, simply note that $q : \bigsqcup_{i\in I} C_i \to X$ is continuous and surjective between a compact and a Hausdorff space, so it is a quotient map.

Recall that a cover $\mathcal{C} = \{C_i\}$ is *locally finite* if for every $x \in X$ there exists an open neighbourhood U^x of x in X such that U^x intersects only a finite number of the sets C_i.

Lemma 1.44. *Let $\mathcal{C} = \{C_i\}$ be a covering of a topological space X. If $\mathcal{C}$ is a closed and locally finite covering, then the topology of X is coherent with respect to $\mathcal{C}$.*

Proof. Let $A \subset X$ such that $A \cap C_i \subset C_i$ is open for all $i \in I$. We want to see that A is open in X. For this, take $x \in A$, and let us find a suitable open neighbourhood of x contained in A. Since $\mathcal{C}$ is locally finite, there exists an open neighbourhood $U = U^x$ of x in X such that U only intersects a finite number of the C_i's. Suppose that U only intersects $C_{i_1}, \ldots, C_{i_m}$. Among these, x will only be contained in some of them, say that $x \in C_{i_1} \cap \cdots \cap C_{i_s}$ and $x \notin E = C_{i_{s+1}} \cup \cdots \cup C_{i_m}$. Put $U' = U - E$, which is also an open neighbourhood of x, since E is closed and $x \notin E$. On the other hand, by hypothesis, $A \cap C_i = V_i \cap C_i$ for some open set V_i of X.

Now put $\tilde{V} = U' \cap V_{i_1} \cdots \cap V_{i_s}$ which clearly contains x and is open in X. We claim that $\tilde{V}$ is contained in A. To see it, it suffices to see that $\tilde{V} \cap C_i \subset A$ for all $i \in I$, since $\tilde{V} = \bigcup_{i \in I} (\tilde{V} \cap C_i)$. By definition of U' it is clear that for $i \notin \{i_1, \ldots, i_s\}$, $\tilde{V} \cap C_i = \emptyset \subset A$, and if $i \in \{i_1, \ldots, i_s\}$, then $\tilde{V} \cap C_i \subset V_i \cap C_i = A \cap C_i \subset A$. We conclude that A is open in X, and therefore the topology of X is coherent with $\mathcal{C}$. □

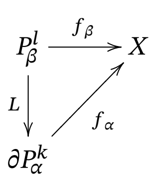
Remark 1.45. One of the consequences of the fact that a topological space X has a coherent topology with respect to a cover are the so-called gluing lemmas. Indeed, the following are equivalent:

(1) X has a coherent topology with respect to the cover $\{C_i | i \in I\}$.
(2) For each topological space Y and any map $f : X \to Y$, f is continuous if and only if all the restrictions $f|_{C_i} : C_i \to Y$ are continuous, for all $i \in I$.

A PL structure is based on the concept of a triangulation of a topological manifold. This consists of expressing an n-manifold as a union of n-polyhedra which patches nicely. A (convex) *k-polyhedron* P^k is the convex hull of finitely many points in $\mathbb{R}^k$ with non-empty interior. An l-face of P^k is a subset $P^l \subset \partial P^k$ which is the intersection of P^k with a hyperplane (not passing through the interior of P^k) and which is affinely isomorphic to an l-polyhedron (that is, it has dimension l).

Definition 1.46. A *triangulation* of a topological space X, denoted $\tau = \{(P^k_\alpha, f_\alpha) | \, 0 \leq k \leq n, \alpha \in \Lambda\}$, consists of the following.

(1) A family of polyhedra P^k_α with dimensions varying between 0 and n.
(2) For each polyhedron, a continuous map $f_\alpha : P^k_\alpha \to X$. Its image $C^k_\alpha = f_\alpha(P^k_\alpha)$ is called a *k-cell*.
(3) The maps $f_\alpha : \operatorname{Int} P^k_\alpha \to f_\alpha(\operatorname{Int} P^k_\alpha) \subset X$ are homeomorphisms.
(4) The sets $C^k_\alpha \subset X$ form a locally finite covering of X by closed sets.
(5) The images $f_\alpha(\operatorname{Int} P^k_\alpha) \subset X$ form a disjoint covering of X.
(6) For each l-face $P^l \subset \partial P^k_\alpha$ there exists a map in the triangulation $f_\beta : P^l_\beta \to X$ such that the following diagram commutes



where $L : P^l_\beta \to P^l \subset P^k_\alpha$ is an affine isomorphism followed by the inclusion.

In general, 0-cells are called *vertices*, 1-cells are called *edges*, and 2-cells are called *faces*. The dimension of X is $n = \dim X$, the maximum of the dimensions of the polyhedra of the triangulation.

Remark 1.47.

- The topology of X is the coherent topology with respect to the cells $\{C^k_\alpha\}$ by Lemma 1.44.
- Condition (6) is a patching condition. If two polyhedra P^k_α and P^t_β have intersecting images, then the intersection is formed by l-faces C^l_γ, where $i_1 : P^l_\gamma \hookrightarrow \partial P^k_\alpha$ and $i_2 : P^l_\gamma \hookrightarrow \partial P^t_\beta$, and moreover $f_\alpha \circ i_1 = f_\beta \circ i_2 = f_\gamma : P^l_\gamma \to C^l_\gamma$.
- We allow for f_α to send two (or more) l-faces $P^l \subset \partial P^k_\alpha$ to the same l-cell $C^l_\gamma \subset X$ of the triangulation. A stronger definition of triangulation can be made with the following requirements.
  (1) All $f_\alpha : P^k_\alpha \to C^k_\alpha$ are homeomorphisms.
  (2) The P^k_α are k-simplices. A *k-simplex* is the convex hull of $k+1$ affinely independent points in $\mathbb{R}^k$
  (3) Any two of the k-cells cannot share all their vertices.

  We call this a *regular triangulation*. A triangulation can always be subdivided (by subdividing each polyhedron into smaller ones) to achieve this (see Remark 1.49).
- Given a regular triangulation, take S to be the set of vertices. Any k-cell is determined by a finite collection $v_0, \dots, v_k \in \mathrm{S}$, so the triangulation is given by a collection σ of finite subsets of S. Here σ satisfies that if $A \in \sigma$ and $B \subset A$, then $B \in \sigma$.

  If X is compact, then $L = |\mathrm{S}| - 1$ is finite. Take a map $f : \mathrm{S} \to \{w_0, \dots, w_L\} \subset \mathbb{R}^L$, where $w_0, \dots, w_L$ are affinely independent points. For any $A \in \sigma$, let C_A be the convex hull of $f(A)$. Then X is homeomorphic to the *simplicial complex* $\mathbf{C}_\mathrm{S} = \bigcup_{A \in \mathrm{S}} C_A$.
- The definition of triangulation can also be made more flexible by allowing curvilinear polyhedra. This is given by a k-disc $D^k \subset \mathbb{R}^k$ where the boundary $\partial D^k = \bigcup P_i$, for finitely many closed subsets P_i, each of which is homeomorphic to a curvilinear $(k-1)$-polyhedron (defined by induction), and the polyhedra in the boundaries of the different P_i agree. We call this a *pseudo-triangulation*. A curvilinear 2-polyhedron is a disc in which the boundary S^1 has been divided into edges by several vertices.

Example 1.48. Let us give examples of triangulations of surfaces.

(1) Take the disc D^2. Choose $Q = (0,1)$ and $P = (0,-1)$. These vertices determine two edges a, a', which are two semicircles on ∂D^2. Now we identify each point $(x,y) \in a$ with $(-x,y) \in a'$. The quotient $D^2/\!\sim$ is homeomorphic to S^2. Therefore S^2 admits a pseudo-triangulation with one face D^2, one edge $a = a'$, and two vertices P and Q.

In this triangulation, there is only one edge which is embedded into the 2-cell (in this case, the face D^2) in two different ways. To denote this, we label all the images of the embedding with the same letter. As the embeddings are determined only by the orientation (cf. Lemma 1.34), we just depict it with an arrow (see Exercise 1.29).

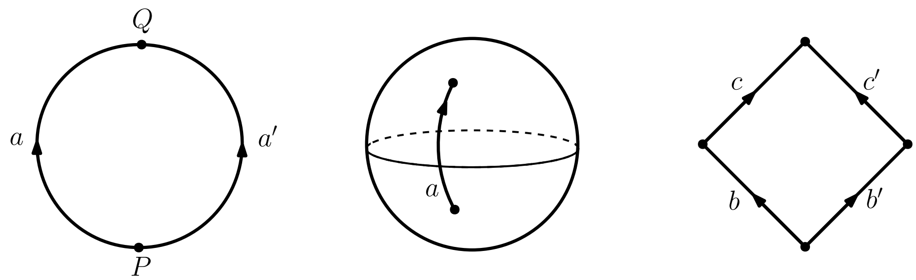


Figure 1.4. Pseudo-triangulation and triangulation of S^2.

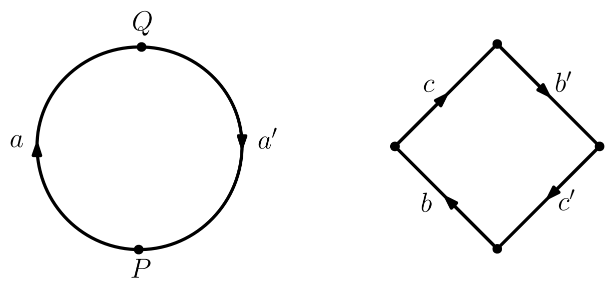


Figure 1.5. Pseudo-triangulation and triangulation of $\mathbb{R}P^2$.

To get a triangulation, it is enough to divide each edge into two consecutive edges, as $a = bc$. We get a square, as in Figure 1.4.

(2) Let us describe a pseudo-triangulation for $\mathbb{R}P^2$ (see Figure 1.5). Recall that $\mathbb{R}P^2 = S^2/\!\sim$, where $(x, y, z) \sim (-x, -y, -z)$. We consider the hemisphere

$$S^2_+ = \{(x, y, z) \in S^2 | z \geq 0\}$$

and the induced equivalence relation, given by $(x, y, 0) \sim (-x, -y, 0)$, for $(x, y, 0) \in \partial S^2_+$. The inclusion $\imath : S^2_+ \to S^2$ induces a map $\bar{\imath} : S^2_+/\sim \to S^2/\!\sim$, which is clearly continuous and bijective. As the first space is compact and the second is Hausdorff, it is a homeomorphism. Hence $\mathbb{R}P^2 \cong S^2_+/\!\sim$. Now $D^2 \cong S^2_+$ via $(x, y) \mapsto (x, y, \sqrt{1 - x^2 - y^2})$. So the induced equivalence relation in D^2 is $(x, y) \sim (-x, -y)$ for $(x, y) \in \partial D^2$. Therefore

$$q : D^2 \longrightarrow \mathbb{R}P^2, \qquad (x, y) \mapsto \left[x, y, \sqrt{1 - x^2 - y^2}\right]$$

is a quotient map, and $\mathbb{R}P^2 \cong D^2/\!\sim$. This produces a pseudo-triangulation: take the disc D^2 with vertices P and Q and edges a and a' as in (1). We identify each point $(x, y) \in a$ with the point $(-x, -y) \in a'$. For a triangulation, subdivide $a = bc$ in the two edges.

(3) Take the square $P = [0, 1] \times [0, 1]$ and the map $f : P \to T^2 = S^1 \times S^1$, $f(x, y) = (e^{2\pi i x}, e^{2\pi i y})$. This gives a triangulation of the torus (see Figure 1.6).

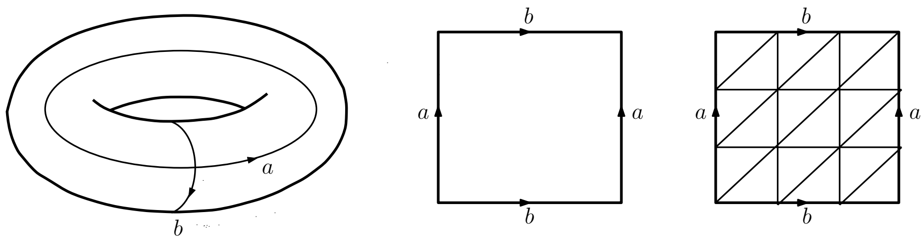


Figure 1.6. Triangulation and regular triangulation of T^2.

To obtain a regular triangulation, we just make a subdivision on the square. See the right picture of Figure 1.6.

The Klein bottle can also be triangulated with the square $P = [0,1] \times [0,1]$, with two edges a, b and one vertex. Recall the definition of the Klein bottle as the mapping torus of the map $f : S^1 \to S^1$, $f(x,y) = (-x,y)$, given in Example 1.33(8). This means that $Kl = (S^1 \times [0,1])/\sim$, $(e^{2\pi i t}, 0) \sim (e^{2\pi i(1-t)}, 1)$, $t \in [0,1]$. This is equivalent to $Kl \cong P/\sim$, with $(t,0) \sim (1-t,1)$ and $(0,y) \sim (1,y)$, $t \in [0,1]$, $y \in [0,1]$, giving the triangulation in Figure 1.7.

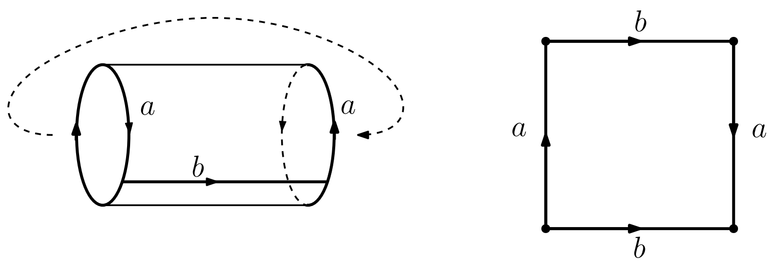


Figure 1.7. Triangulation of the Klein bottle.

Remark 1.49. If we have a triangulation (or pseudo-triangulation) τ on a space X, we can obtain another triangulation τ' by subdividing the polyhedra. The typical way of doing this is by taking the barycenters of all the polyhedra of the triangulation as new vertices of τ', and form the other l-polyhedra as convex hulls of successive barycenters. This procedure is called the *barycentric subdivision* of τ. After one barycentric subdivision, all polyhedra in the triangulation are k-simplices. If we repeat the barycentric subdivision twice, then we get a regular triangulation even if τ was a pseudo-triangulation.

Note that the sizes of the l-polyhedra are divided at least by two after a barycentric subdivision. So if we repeat this procedure a high number of times, we can make all the polyhedra of the triangulation as small as we want. In particular, by doing barycentric subdivision, we can subordinate the triangulation τ to any open covering of X, in the sense that any cell is contained in at least one open set of the cover.

Definition 1.50. A *piecewise linear (PL) n-manifold* is a topological n-manifold M endowed with a triangulation $\tau = \{(P_\alpha^k, f_\alpha)\}$. The triangulation τ is called a PL structure on M.

For a triangulation τ on an n-manifold, all k-cells with $k < n$ are contained as a face of an n-cell. Therefore $q : \bigsqcup P_\alpha^n \to M$ is also a quotient map. This is proved as follows: suppose that there is a k-cell $f_\beta(P_\beta^k)$ not contained in the closure of an l-cell with $l > k$. Then $f_\beta(\operatorname{Int} P_\beta^k)$ is an open subset (since its complement is the closure of the rest of the cells), and so any point of it has a neighbourhood homeomorphic to an open subset of $\mathbb{R}^k$. This contradicts Theorem 1.2.

PL manifolds can be put into a category, but in this case it is somewhat involved to formalize the adequate definitions. Let $\big(M_1, \tau_1 = \{(P_\alpha^k, f_\alpha)\}\big)$, $\big(M_2, \tau_2 = \{(P_\beta^k, g_\beta)\}\big)$ be triangulated spaces (with regular triangulations). We say that a continuous map $h : M_1 \to M_2$ is *simplicial* if for every $P_\alpha^l \in \tau_1$ there exists a $P_\beta^m \in \tau_2$ so that $h(f_\alpha(P_\alpha^l)) \subset g_\beta(P_\beta^m)$ and an affine map $L_{\alpha\beta} : P_\alpha^l \to P_\beta^m$ sending vertices to vertices, such that $g_\beta \circ L_{\alpha\beta} = h \circ f_\alpha$. The weak point about this notion is the scarcity of simplicial maps. So we allow as a PL map a continuous map that is simplicial after a polyhedral subdivision.

It is thus natural to define a *PL structure* on a manifold M as an equivalence class of triangulations τ up to polyhedral subdivisions, in the same vein as a smooth structure is defined as a smooth atlas up to compatibility of atlas. Note that there are tricky situations to have in mind: take $M = [0,1] \times [-1,1]$, and let τ be a triangulation with the edges of ∂M and adding the edge $[0,1] \times \{0\}$, and let τ' be the triangulation adding the edge $\{(x, x\sin(1/x)) | x \in [0,1]\}$. Then $(M,\tau), (M,\tau')$ are different PL structures ($\mathrm{Id} : (M,\tau) \to (M,\tau')$ is not a PL map) but there is a homeomorphism $f : (M,\tau) \to (M,\tau')$ which is a PL isomorphism.

Definition 1.51. We define the category of PL n-manifolds $\mathbf{PLMan}^n$, by taking as objects pairs (M,τ), where M is a topological n-manifold and τ is a triangulation of M, and as morphisms the PL maps. The category $\mathbf{PLMan}_c^n$ has objects compact PL n-manifolds.

Remark 1.52. We can define the connected sum in $\mathbf{PLMan}^n$. Let $(M_1,\tau_1), (M_2,\tau_2)$ be two connected PL manifolds, and subdivide the triangulations until they are regular. Take two n-polyhedra $P_\alpha^n \in \tau_1$, $P_\beta^n \in \tau_2$, which are n-simplices. Then there is an isomorphism $f : \partial P_\alpha^n \to \partial P_\beta^n$. We take $X_1^o = X_1 - f_\alpha(\operatorname{Int} P_\alpha^n)$ and $X_2^o = X_2 - f_\beta(\operatorname{Int} P_\beta^n)$. Then we define $X_1 \#_f X_2 = X_1^o \cup_f X_2^o$.

Remark 1.53. Let us briefly discuss the classification problem in the PL category. For low dimensions, we have $\mathbb{L}^{co}_{\mathbf{PLMan}^0} = \{\star\}$, $\mathbb{L}^{co}_{\mathbf{PLMan}^1} = \{S^1, \mathbb{R}\}$. In general, any differentiable manifold admits a triangulation unique up to PL isomorphism (proved by Whitehead). The existence appears in Exercise 3.16 for the case $n = 2$. Therefore we have maps

$$\mathbb{L}_{\mathbf{DMan}^n} \longrightarrow \mathbb{L}_{\mathbf{PLMan}^n} \longrightarrow \mathbb{L}_{\mathbf{TMan}^n}\,.$$

For low dimensions $n = 1, 2, 3$, both maps are bijections. For $n \geq 4$ these maps are in general neither injective nor surjective. That means that not every PL manifold admits a smooth structure, or that there are topological manifolds which do not admit a triangulation. The differences between these three classes is controlled by some *classifying spaces* denoted **BTop**, **BPL**, and **BDiff**, defined in homotopy theoretic terms.

Remark 1.54. Suppose that M is a PL n-manifold (with or without boundary). Let $\tau = \{(P^k_\alpha, f_\alpha)\}$ be the triangulation of M, so $q : \bigsqcup P^n_\alpha \to M$ is a quotient map. Let us see how the n-cells glue. Pick a $(n-1)$-cell $C^{n-1}_\beta \subset M$. As we have remarked above, this appears in the boundary of some (or several) n-polyhedra. Let $P^n_{\alpha_i}$, $i = 1, \dots, s$, be such polyhedra. We say that $P^n_{\alpha_i}$ are *incident* on P^{n-1}_β, and we say that $P^n_{\alpha_i}, P^n_{\alpha_j}$ are *adjacent*. Let $L_i : P^{n-1}_\beta \to \partial P^n_{\alpha_i}$ with $f_{\alpha_i} \circ L_i = f_\beta$. Pick an interior point $x \in P^{n-1}_\beta$, and let $x_i = L_i(x)$. We can assume, without loss of generality, that all maps L_i are isometric. Take $\epsilon > 0$ small, and let $U = \bigsqcup (P^n_{\alpha_i} \cap B_\epsilon(x_i))$. This is a saturated open set, so that $U^\star = q(U)$ is an open neighbourhood of $p = q(x_i) \in M$.

The set $U^\star$ is homeomorphic to $R_\epsilon = (\bigsqcup_{1 \le i \le s} B^+_\epsilon(0)_i)/\sim$, of a collection of s semiballs with the relation $(x,0)_i \sim (x,0)_j$, $i \neq j$, where $(x,y)_i$ denotes the point in the ith copy $B^+_\epsilon(0)_i \subset \mathbb{R}^n_+$. As M is a manifold around p, there is neighbourhood $V^\star \subset U^\star$ homeomorphic to either $B_\eta(0)$ if p is interior point or to $B^+_\eta(0)$ if p is a boundary point.

Suppose first that p is an interior point. Take the preimage of $B_\eta(0)$ on R_ϵ. This contains some R_δ, with $\delta < \epsilon$. So the preimage of $B_\eta(0) - \{0\}$, which is homotopy equivalent to S^{n-1}, retracts to $\partial \bar{R}_\delta$, which is the union of s discs glued along their boundary. Using homology (section 2.3), there should be an epimorphism

$$H_{n-1}(S^{n-1}) = \mathbb{Z} \to H_{n-1}(\partial \bar{R}_\delta) = \mathbb{Z}^{s-1}$$

(for $n = 2$, we can use the fundamental group, $\pi_1(S^1) = \mathbb{Z} \to \pi_1(\partial \bar{R}_\delta) = F_{s-1}$, the free group on $s-1$ generators). We conclude that $s = 2$, that is, only two n-cells are glued along P^{n-1}_β.

Now suppose that p is a boundary point. We do a similar argument with $B^+_\eta(0)$ instead of $B_\eta(0)$. Then we have $B^+_\eta(0) - \{0\}$ is contractible, so there is an epimorphism $\{0\} \to \mathbb{Z}^{s-1}$ and $s = 1$. Hence only one n-cell appears at a boundary point.

1.2.4. Existence of triangulations for surfaces. Now we shall prove that any compact topological surface without boundary admits a triangulation. This is a result proven by Radó [Rad]. We leave the case of surfaces with boundary as an exercise (Exercise 1.17). The case of open surfaces can be treated by an argument of exhaustion via compact embedded subsurfaces with boundary.

We shall use Theorem 1.36. This is a fundamental result in 2-dimensional topology, that has no analogue in higher dimensions $n \geq 3$ (as shown by the horned Alexander sphere [Ale]). Recall that a *Jordan curve* is a set $C \subset \mathbb{R}^2$ homeomorphic to S^1, and we call a *Jordan arc* to a set $C \subset \mathbb{R}^2$ homeomorphic to $[0,1]$. Theorem 1.36 implies that if $C \subset \mathbb{R}^2$ is a Jordan curve, then $\mathbb{R}^2 - C$ has two connected components, one bounded and the other unbounded. The bounded component has closure homeomorphic to a disc D^2.

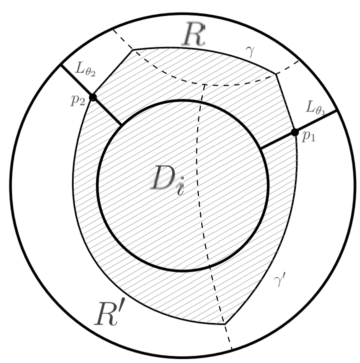
Theorem 1.55. *Let S be a compact topological surface. Then S admits a triangulation.*

Proof.

Step 1. We want to find a finite collection of closed subsets $D_i \subset S$, $1 \leq i \leq m$, which are homeomorphic to closed discs such that $\bigcup D_i = S$ and

$$\Gamma_i = \partial D_i \text{ is a collection of Jordan curves such that } \Gamma_i \cap \Gamma_j, i < j, \text{ is a finite collection of points or Jordan arcs.} \tag{1.4}$$

Let us construct such collection. We take for each point $p \in S$ an open neighbourhood U^p with a homeomorphism $\varphi : U \to B_1(0) \subset \mathbb{R}^2$, $\varphi(p) = 0$. We take the sets $U' = \varphi^{-1}(B_{1/2}(0))$, which cover S. By compactness, we have a finite collection of such sets $U'_1, \ldots, U'_m$ covering S, and denote $\varphi_i : U_i \to B_1(0)$, $i = 1, \ldots, m$. The sets D_i will be constructed as $\varphi_i^{-1}(D'_i)$, where D'_i are closed discs (i.e., subsets homeomorphic to D^2) such that $B_{1/2}(0) \subset D'_i \subset B_1(0)$, and $\Gamma_i = \partial D_i$. We do this by induction on i. For $i = 1$, we take $D'_1 = \bar{B}_{1/2}(0)$, $D_1 = \varphi_1^{-1}(D'_1)$, $\Gamma_1 = \partial D_1$. Now suppose that $D_1, \ldots, D_{i-1}$ have been constructed satisfying the property (1.4). Take the images $A_j = \varphi_i(\Gamma_j \cap U_i) \subset B_1(0)$, $j < i$. An obvious consequence of Theorem 1.36 is that any Jordan curve has empty interior. So $A = \bigcup_{j<i} A_j$ has empty interior. This implies that we can find two different segments $L_{\theta_1}, L_{\theta_2}$, $\theta_1 < \theta_2$, where $L_\theta = \{re^{\mathrm{i}\theta} \mid \frac{1}{2} \leq r \leq 1\}$, for $\theta \in [0, 2\pi]$, and two points $p_1 \in L_{\theta_1}$, $p_2 \in L_{\theta_2}$, which are not in A. Now the regions $R = \{re^{\mathrm{i}\theta} \mid \frac{1}{2} \leq r \leq 1, \theta_1 \leq \theta \leq \theta_2\}$ and $R' = \{re^{\mathrm{i}\theta} \mid \frac{1}{2} \leq r \leq 1, \theta_2 \leq \theta \leq \theta_1 + 2\pi\}$ are homeomorphic to discs. We shall construct an arc γ in R joining p_1 to p_2, and another arc γ' in R' joining p_1 to p_2, satisfying that both γ, γ' intersect A in a finite collection of points and arcs, and $\gamma \cap \partial R = \{p, q\} = \partial\gamma$, $\gamma' \cap \partial R' = \{p, q\} = \partial\gamma'$.



In that case, $C = \gamma \cup \gamma'$ is a Jordan curve, so by Theorem 1.36, C is the boundary of an embedded disc D'_i. Note that $B_{1/2}(0) \subset D'_i \subset B_1(0)$. Also $\Gamma_i = \partial D'_i = C$ satisfies (1.4) as well. To prove the above assertion, let

$$E = \{p \in R \mid \text{there exists an arc } \gamma \text{ from } p_1 \text{ to } p \text{ in } R \text{ such that } \gamma \cap A \text{ is a finite union of points and Jordan arcs}, \gamma \cap \partial R \subset \partial\gamma\}.$$

Let us see that $E = R$, by checking that E is open and closed. If $p \in E$, let γ be an arc from p_1 to p such that $\gamma \cap A$ is as required. If $p \notin A$, then there is a ball B around p with $B \cap A = \emptyset$. Take the first point where γ intersects ∂B, cut γ at that point, and join it with a segment to a point $q \in B$. This proves that $q \in E$, and so $B \subset E$. If $p \in A$, then take a ball B around p which does not intersect the components of A that do not pass through p. Let $q \in B$, join it to p with a path δ. This path intersects A at a point q' for the first time. Cut δ there, now take a path inside A from q' to p, and join the three paths, to get a path from p to q satisfying the required conditions. Therefore $q \in E$, and thus, $B \subset E$ again. Hence E is open.

Moreover, if $p \notin E$, the same argument shows that a small ball B around E satisfies that $B \subset R - E$. Hence $R - E$ is open, and so E is closed. By connectedness, $E = R$, and so we are done.

Step 2. Now that we have a finite collection of closed subsets $D_i \subset S$ homeomorphic to closed discs which cover S and such that (1.4) is satisfied. Then we can construct a triangulation as follows.

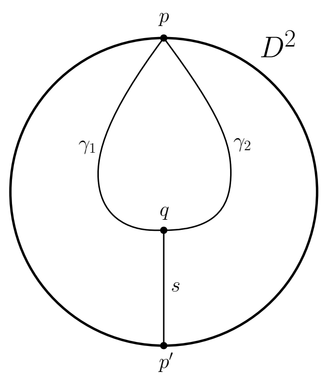
- The vertices are the isolated points of intersection of $\Gamma_i \cap \Gamma_j$, and the extremes of the arcs which are components of $\Gamma_i \cap \Gamma_j$. We add a vertex to any isolated Jordan curve.
- The edges form the finite graph $A = \bigcup \Gamma_i$. If some connected component is homeomorphic to a circle, then we insert a vertex on it.
- Let U be a connected component of $S - A$. If $p \in U$, then take the disc $D_i \cong D^2$ containing p in its interior. Then $U \subset \operatorname{Int} D_i$ and U is a connected component of $D_i - (A \cap D_i)$. Note that $A \cap D_i$ is the union of $\Gamma_j \cap D_i$, and hence it is composed by arcs inside Γ_j each of them bounded by some points in the intersection $\Gamma_i \cap \Gamma_j$. Let us see that D_i with the graph $A \cap D_i \subset D_i$ is triangulated (maybe adding new vertices and new edges).

  First, note the following facts. Consider a disc D^2 and add an arc $\gamma \subset D^2$ such that $\gamma \cap \partial D^2 = \partial\gamma$, so ∂D^2 is split into two arcs δ_1 and δ_2. Both $\delta_1 \cup \gamma$ and $\delta_2 \cup \gamma$ are Jordan curves, and hence they bound sets homeomorphic to discs. On the other hand, if we add an edge that is a Jordan curve γ with $\gamma \cap \partial D^2 = \{p\}$, then there is one region inside γ which is homeomorphic to a disc, and another region. Take the antipodal point $p' \in \partial D^2$ to p. Take $q \in \gamma$ the closest point to p', and add the segment s from p' to q. Split $\gamma = \gamma_1 \cup \gamma_2$ at q. Then the arc $\gamma_1 \cup s$ splits D^2 into two regions (homeomorphic to discs) and then γ_2 splits one of these regions into another two regions (discs).

  With this said, if $A \cap D_i$ is connected, then D_i with the graph $A \cap D_i$ is constructed by adding in finitely many steps, arcs of the two types described above. So the regions obtained are all homeomorphic to discs, hence providing a triangulation. If A is not connected, then we can make it connected as follows: If $A = A_1 \sqcup A_2$, then take $p_1 \in A_1$, $p_2 \in A_2$ at minimum distance. Add them as vertices, and the segment from one to the other as new edge. □

1.3. Planar representations of surfaces

Now we move to the specific study of surfaces. With the aid of triangulations, we are going to define a representation of a connected topological surface by means of a word



that allows us to recover the surface. This will be the key to proving the classification theorem for compact topological surfaces.

Definition 1.56. Let S be a compact connected topological surface. A *planar representation* of S is a planar (possibly curvilinear) polygon P_S together with a label on each edge consisting of a letter and an arrow indicating a direction from one of its vertices to the other. Associated to the label there is an equivalence relation $\sim$ on $\partial \mathrm{P}_S$ given as follows: if two edges α, α' have the same letter, parametrize them as $f_\alpha : [0,1] \to \alpha$, $f_{\alpha'} : [0,1] \to \alpha'$ following the prescribed direction, and declare $f_\alpha(t) \sim f_{\alpha'}(t)$, for all $t \in [0,1]$. We require that $S \cong \mathrm{P}_S/\sim$.

The above definition means that the edges of the polygon must be glued according to the letters, in the direction indicated by its corresponding arrows. Therefore a planar representation is a psuedo-triangulation consisting of just one polygon.

1.3.1. Local structure of a triangulation of a surface. Let us review the local structure of a triangulation. Let S be a connected compact surface (without or with boundary). Let $\tau = \{(P_\alpha^k, f_\alpha) | \alpha \in \Lambda, 0 \leq k \leq 2\}$ be a triangulation of S, consisting of a collection of polygons P_α^2. The quotient map $q : X = \bigsqcup P_\alpha^2 \to S$ identifies some edges and also some vertices. We have the following:

(1) Pick a 1-polyhedron $C_\beta^1 \subset S$. According to Remark 1.54, we have two possibilities:
  - If the points of $\mathrm{Int}(C_\beta^1)$ are in the interior of S, then there are two faces $P_\alpha^2, P_{\alpha'}^2$ incident with P_β^1. Note that the 2-polyhedra may be the same one, with two different edges with the same label.
  - If the points of $\mathrm{Int}(C_\beta^1)$ are in ∂S, then there is just one face P_α^2 incident with P_β^1.

(2) Let $v = P_\gamma^0$ be a vertex of the triangulation. Consider pairs (L, L') such that $L' : P_\gamma^0 \to P_\beta^1$ and $L : P_\beta^1 \to P_\alpha^2$ are the inclusions of the faces. For each L' there is one or two L depending on whether the points in the interior of C_β^1 are boundary or interior. If there are two, say L_1, L_2, we declare $(L_1, L') \approx (L_2, L')$. For each vertex in a polygon, there are two edges incident on it, that is two

pairs $(L_1, L_1'), (L_2, L_2')$ such that $L_1 \circ L_1' = L_2 \circ L_2'$. In that case, we declare $(L_1, L_1') \approx (L_2, L_2')$. This allows us to partition the set of (L, L') into chains:

- $(L_1, L_1'), \dots, (L_t, L_t')$, where we have $(L_i, L_i') \approx (L_{i+1}, L_{i+1}')$, $i = 1, \dots, t-1$, and $(L_t, L_t') \approx (L_1, L_1')$. Take the vertices $x_i = L_i(L_i'(v)) \in P^2_{\alpha_i}$ of the polygons and the set $U = \bigsqcup (B_\epsilon(x_i) \cap P^2_{\alpha_i})$. The quotient $U/\sim$ is homeomorphic to a ball $B_\epsilon(0)$.
- $(L_1, L_1'), \dots, (L_t, L_t')$, where we have $(L_i, L_i') \approx (L_{i+1}, L_{i+1}')$, $i = 1, \dots, t-1$, and $(L_t, L_t') \not\approx (L_1, L_1')$, and not related to any other. Take the vertices $x_i = L_i(L_i'(v)) \in P^2_{\alpha_i}$ of the polygons and the set $U = \bigsqcup (B_\epsilon(x_i) \cap P^2_{\alpha_i})$. The quotient $U/\sim$ is homeomorphic to a semiball $B^+_\epsilon(0)$.

Suppose that there are m chains of the first type and k chains of the second type. Then the union of the sets U for all the $m + k$ chains is a saturated open set of $\bigsqcup P^2_\alpha$. The image is an open set $V \subset S$ homeomorphic to

$$R = \Big(\Big(\bigsqcup_{i=1}^{m} B_\epsilon(0)_i \Big) \sqcup \Big(\bigsqcup_{j=1}^{k} B^+_\epsilon(0)_j \Big) \Big) / \sim .$$

This is a connected open neighbourhood of the point $p = q(v)$. As S is a 2-manifold around p, there is a small ball $B_\delta(p)$ included in R. As $B_\delta(p) - \{p\}$ is connected, so is $R - \{0\}$. But $R - \{0\}$ has $m + k$ connected components.

The conclusion is that either $m = 1$ and $k = 0$, in which case p is an interior point of S; or $m = 0$ and $k = 1$, in which case p is a boundary point of S.

Remark 1.57. By the above, once we know how the edges of the planar representation are identified, the vertices are identified in a precise way, namely, when they are in one of the chains constructed above.

Proposition 1.58. *Every compact connected topological surface S admits a planar representation.*

Proof. Let $\tau = \{(P^k_\alpha, f_\alpha) | \, \alpha \in \Lambda\}$ be a triangulation of S given by Theorem 1.55. Since $|\Lambda| < \infty$, the polyhedra P^2_α are compact and S is Hausdorff, the map

$$q : X = \bigsqcup_{\alpha \in \Lambda} P^2_\alpha \longrightarrow S$$

is a quotient map. If the set X is connected, i.e., if $|\Lambda| = 1$, then we are done. If not, take a connected component C_1 of X. If no edge of C_1 glues with an edge of $X - C_1$, then by the discussion on the local structure of the triangulation, no vertex of C_1 glues to a vertex of $X - C_1$. Therefore $S = q(C_1) \bigsqcup q(X - C_1)$ is disjoint union of non-empty closed subsets, which contradicts the fact that S is connected.

Therefore there must exist an edge l of C_1 which glues by q to an edge l' of $X - C_1$. Let C_2 be the connected component of $X - C_1$ which contains l'. Note that C_1 and C_2 are disjoint polygons in X, but if we glue the edges l and l' according to the map q, we then get from C_1 and C_2 just one polygon, and therefore we get a new space X' from X with an strictly lower number of connected components. Call $\pi : X \to X'$ the quotient map we have just described. Then the space X' is also a union of polygons, and it has a strictly fewer number of disjoint polygons. Moreover we have a quotient map $q' : X' \to S$ with $q' \circ \pi = q$. We repeat this process until we get just one polygon T. □

Let P_S be a planar representation. First the homeomorphism type of the quotient $P_S/\sim$ only depends on the isotopy classes of the homeomorphisms that identify the edges (Exercise 1.29). By Lemma 1.34, there are only two isotopy classes of homeomorphisms between intervals, determined by how the homeomorphism permutes the vertices. Therefore, to determine the quotient $P_S/\sim$ up to homeomorphism it is only necessary to name with the same letter any two edges which are to be identified and to put an arrow on both of them indicating in what direction they are to be glued together. Second, by the discussion on the local structure of a triangulation, any planar representation P in which the edges are either identified in pairs or they are alone, determines a well defined surface $S = P/\sim$. It has boundary if and only if there are letters which are not identified (they appear just once). The boundary ∂S is the image of the edges which appear just once.

Taking this into account, a planar representation can be given by a sequence of letters (corresponding to the edges) with directions. For this, fix a vertex of the polygon and a direction for running along the boundary ∂P_S (say clockwise or anticlockwise). We obtain a sequence of letters with exponents $+1$ or -1, depending on whether the arrow on the edge points in the same direction or the opposite direction to the running direction along the boundary. This produces a word $\mathbf{p}_S$ which determines P_S, and hence S, univocally.

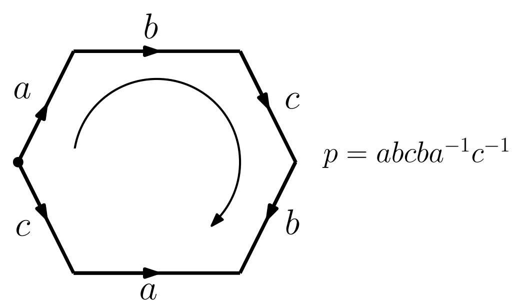


Figure 1.8. Word given by starting in the marked vertex (left-most) and in the given direction (clockwise).

We can also understand this as a map $\mathbf{p} \mapsto S_{\mathbf{p}}$ which assigns to each word the corresponding surface. We say that two words are equivalent, and write $\mathbf{p}_1 \approx \mathbf{p}_2$, if the corresponding surfaces are homeomorphic, $S_{\mathbf{p}_1} \cong S_{\mathbf{p}_2}$. Some easy instances of equivalent words are given by the choice of starting point and direction when reading the boundary of the polygon. These give the following equivalent words:

- A string obtained by rotation of the letters. If we denote $\mathbf{p} = \mathbf{p}_1\mathbf{p}_2$, where $\mathbf{p}_1, \mathbf{p}_2$ are words (subwords of $\mathbf{p}$ in this case), and the concatenation indicates putting a word after the other. Then we have $\mathbf{p}_1\mathbf{p}_2 \approx \mathbf{p}_2\mathbf{p}_1$. In Figure 1.8, we have the word $\mathbf{p} = abcba^{-1}c^{-1}$ starting clockwise in the left-most vertex. Then $\mathbf{p} \approx ba^{-1}c^{-1}abc$, obtained by starting clockwise in the right-most vertex.

- The inverse string obtained by formally inverting (using the group law) the word, $\mathbf{p} \approx \mathbf{p}^{-1}$. For the previous word $\mathbf{p} = abcba^{-1}c^{-1}$, the inverse string is $\mathbf{p}^{-1} = cab^{-1}c^{-1}b^{-1}a^{-1}$ (correspond to starting in the same vertex but in the other direction).
- The string substituting a letter a by its inverse a^{-1}, simultaneously in all occurrences, with the usual convention that $(a^{-1})^{-1} = a$. In our example, substituting the letter a by its inverse gives $\mathbf{p} \approx a^{-1}bcbac^{-1}$.

Example 1.59. Let us give the words of the surfaces of Example 1.48.

(1) The word aa^{-1} corresponds to the sphere. We can formally associate the empty word $\emptyset$ (no letters), and declare $aa^{-1} \approx \emptyset$. We shall see that this is coherent with the equivalence of words.
(2) For the projective plane we have the word aa.
(3) For the torus we have the word $aba^{-1}b^{-1}$.
(4) For the Klein bottle we have the word $abab^{-1}$.

The next result tells us how to obtain the word associated to the connected sum of two surfaces.

Proposition 1.60. *Let S_1 and S_2 be two compact connected surfaces without boundary with words $\mathbf{p}_1$ and $\mathbf{p}_2$, where we use disjoint sets of letters for S_1 and S_2. The connected sum $S_1 \# S_2$ admits either the word $\mathbf{p}_1\mathbf{p}_2$ or the word $\mathbf{p}_1(\mathbf{p}_2)^{-1}$ depending on the two different ways of identifying the circle to form the connected sum.*

Proof. First construct the polygons P_{S_1} and P_{S_2} associated to the words $\mathbf{p}_1$ and $\mathbf{p}_2$. Now we subtract small balls B_1 and B_2 from the polygons such that ∂B_1 and ∂B_2 pass through a vertex of each of the polygons. Now, if we make the gluing with the arrows pointing as indicated in Figure 1.9 below, then the string of the connected sum is $\mathbf{p}_1\mathbf{p}_2$. On the other hand, if we make the gluing by changing the direction of one arrow, then we get the string $\mathbf{p}_1(\mathbf{p}_2)^{-1}$. □

Definition 1.61. We introduce the following surfaces.

- The *connected orientable surface of genus g* is defined as the connected sum $\Sigma_g = T^2 \# \overset{(g)}{\cdots} \# T^2$ of g copies of the torus $T^2 = S^1 \times S^1$. By consistency, we denote $\Sigma_0 = S^2$. Note that $\Sigma_g \# \Sigma_h = \Sigma_{g+h}$, for all g, $h \geq 0$.
- The *connected non-orientable surface of genus* $k/2$ is defined as the connected sum $X_k = \mathbb{R}P^2 \# \overset{(k)}{\cdots} \# \mathbb{R}P^2$ of k copies of the real projective plane $\mathbb{R}P^2$, for $k \geq 1$. One has $X_k \# X_t = X_{k+t}$, for $k, t \geq 1$.

Example 1.62. By Proposition 1.60 and the words given in Example 1.59, we have

(1) Σ_g, $g \geq 1$, has its *canonical word* given by $a_1b_1a_1^{-1}b_1^{-1} \cdots a_gb_ga_g^{-1}b_g^{-1}$.
(2) X_k, $k \geq 1$, has its *canonical word* given by $a_1a_1 \cdots a_ka_k$.

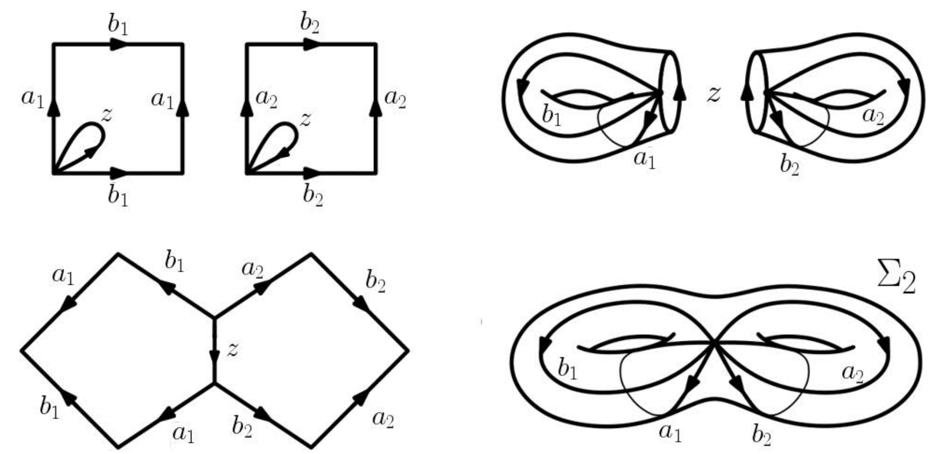


Figure 1.9. The surface Σ_2.

1.3.2. Euler-Poincaré characteristic.

Definition 1.63. Let (X, τ) be a compact triangulated space of dimension N. Let n_k be the number of k-polyhedra. We define the *Euler-Poincaré characteristic* of X as

$$\chi(X) = \sum_{k=0}^{N} (-1)^k n_k = n_0 - n_1 + n_2 - \cdots + (-1)^N n_N .$$

Remark 1.64.

(1) As X is compact, the number of k-polyhedra is finite, and conversely (Exercise 1.16). Therefore the sum of $\chi(X)$ makes sense.

(2) The Euler-Poincaré characteristic is invariant under PL isomorphism. This follows by proving that $\chi(X, \tau) = \chi(X, \tau')$, for τ, τ' equivalent triangulations (Exercise 1.18). Therefore

$$\chi : \mathbb{L}_{\mathbf{PLMan}_c^n} \to \mathbb{Z} \tag{1.5}$$

is a well defined map.

(3) Even though the definition of Euler-Poincaré characteristic depends, at first glance, on the triangulation τ chosen, we will eventually see that this number is independent of the chosen triangulation (Remark 2.114). Therefore it will be an invariant of the homeomorphism type of X (and actually an invariant of the homotopy type).

(4) For a triangulated surface (S, τ), $\chi(S) = v - e + f$, where v is the number of vertices of the triangulation, e the number of edges, and f the number of faces.

(5) In particular, if S is connected and P_S is a planar representation, we have $f = 1$ so $\chi(S) = v - e + 1$.

(6) We obtain the following corollary. For two connected surfaces S_1, S_2, we have

$$\chi(S_1 \# S_2) = \chi(S_1) + \chi(S_2) - 2. \tag{1.6}$$

To see it, just note that the word associated to $S_1 \# S_2$ is the concatenation of the strings of S_1 and S_2. On the other hand, when we subtract a ball of both surfaces and identify the boundaries, we identify two vertices of the triangulation to get just one vertex in $S_1 \# S_2$. Therefore, if v_1, e_1 and v_2, e_2 are the number of vertices and edges in the polygons P_{S_1} and P_{S_2}, we see that $\mathrm{P}_{S_1 \# S_2}$ has $v = v_1 + v_2 - 1$ vertices and $e = e_1 + e_2$ edges. Therefore $\chi(S_1 \# S_2) = v - e + 1 = (v_1 - e_1 + 1) + (v_2 - e_2 + 1) - 2 = \chi(S_1) + \chi(S_2) - 2$.

Example 1.65. Let us list the Euler-Poincaré characteristics of surfaces.

(1) The sphere has word aa^{-1}, whose fundamental polygon has two vertices, one edge, and one face. Thus $\chi(S^2) = 2 - 1 + 1 = 2$.

(2) Any platonic solid is a triangulation of S^2. The tetrahedron has $(v, e, f) = (4, 6, 4)$, the cube has $(v, e, f) = (8, 12, 6)$, the octahedron has $(v, e, f) = (6, 12, 8)$, the dodecahedron has $(v, e, f) = (12, 34, 20)$, and the icosahedron has $(v, e, f) = (20, 34, 12)$. All of them satisfy $v - e + f = 2$, a fact that seems to have been only noticed by Euler for the first time in 1750.

(3) Any planar graph also determines a triangulation of S^2. This justifies the formula $2 = v - e + f$, where v is the number of vertices of the graph, e the number of edges, and f the number of areas separated by the graph (including the unbounded one).

(4) The torus T^2 has word $aba^{-1}b^{-1}$, which has $e = 2$, $v = 1$, $f = 1$. Hence $\chi(T^2) = 1 - 2 + 1 = 0$.

(5) The projective plane $\mathbb{R}P^2$ has word aa, with $v = 1, e = 1, f = 1$. Thus $\chi(\mathbb{R}P^2) = 1 - 1 + 1 = 1$.

(6) Using (1.6), we have that $\chi(S \# T^2) = \chi(S) - 2$, and hence $\chi(\Sigma_g) = 2 - 2g$, for all $g \geq 0$. Alternatively, using the word in Example 1.62(1), the planar representation of Σ_g is a $4g$-gon, with one vertex, $2g$ edges (with letters $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$) and one face. Thus $\chi\left(\Sigma_g\right) = 1 - 2g + 1 = 2 - 2g$.

(7) Using (1.6), we have that $\chi(S \# \mathbb{R}P^2) = \chi(S) - 1$, and hence $\chi(X_k) = 2 - k$, for all $k \geq 1$. Alternatively, using the word 1.62(2), the planar representation of X_k is a $2k$-gon with one vertex, k edges (with letters $a_1, \ldots, a_k$) and one face, so $\chi(X_k) = 1 - k + 1 = 2 - k$.

1.4. Orientability

Now we are going to define the concept of orientability of a manifold. We shall do it in the smooth, PL, and topological categories.

1.4.1. Orientation of smooth manifolds.

Definition 1.66. Let V be a finite dimensional real vector space of dimension $n \geq 1$. We define an equivalence relation $\sim$ on the set of bases of V as follows. For two bases $B_1 = (v_1, \dots, v_n)$ and $B_2 = (w_1, \dots, w_n)$, we say that $B_1 \sim B_2$ if the matrix of change of coordinates between them has positive determinant. We denote $\mathrm{Or}(V) = \{\textit{basis of } V\}/\sim$ for the quotient set, which has two equivalence classes. We call *orientation* of V to an element $o = [B] \in \mathrm{Or}(V)$.

Recall that given a vector space V, the group $\mathrm{GL}(V)$ is the group of linear automorphisms of V, called the *general linear group*. We denote $\mathrm{GL}(n, \mathbb{R}) = \mathrm{GL}(\mathbb{R}^n)$. The matrices with positive determinant are $\mathrm{GL}^+(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) | \det(A) > 0\}$. Definition 1.66 means that two bases $(v_1, \dots, v_n), (w_1, \dots, w_n)$ define the same orientation if $(v_1, \dots, v_n) = (w_1, \dots, w_n)A$, with $A \in \mathrm{GL}^+(n, \mathbb{R})$. We also have the *special linear group* $\mathrm{SL}(n, \mathbb{R}) = \{A \in \mathrm{GL}(n, \mathbb{R}) | \det(A) = 1\}$.

Remark 1.67.

(1) If $V = \{0\}$, we define formally $\mathrm{Or}(V) = \{+, -\}$, so choosing an orientation is merely choosing a sign.
(2) If $\dim(V) = 1$, choosing an orientation is equivalent to choosing a positive direction. So $\mathrm{Or}(V) = \{[+v], [-v]\}$, with $v \in V$ any non-zero vector.
(3) If $\dim(V) = 2$, choosing an orientation is equivalent to choosing a positive direction of rotation for S^1. So $\mathrm{Or}(V) = \{[e_1, e_2], [e_2, e_1]\}$, with (e_1, e_2) any basis of V. The first orientation declares that $\gamma_1(t) = \cos(t)e_1 + \sin(t)e_2$ is the positively oriented circle, whereas the second orientation declares that $\gamma_2(t) = \cos(t)e_2 + \sin(t)e_1$ is the positively oriented circle.

In general, if $\mathrm{Or}(V) = \{o_1, o_2\}$, we denote $-o_1 = o_2$, that is, the "minus" consists of changing the orientation to its opposite. Note that if $o = [(v_1, v_2, \dots, v_n)]$, then $-o = [(-v_1, v_2, \dots, v_n)]$.

Let M be now a smooth manifold. Recall that we can associate to each point $p \in M$ a vector space T_pM called the *tangent space* (see section 3.1.1). In a chart (U, φ) around p, given as $\varphi = (x_1, \dots, x_n)$, the tangent space is generated by the coordinate vectors $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. If we have another chart (V, ψ), with $\psi = (y_1, \dots, y_n)$, then the tangent space is also generated by $\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$, and the change of basis is given by

$$\frac{\partial}{\partial x_j} = \sum_j \frac{\partial y_i}{\partial x_j} \frac{\partial}{\partial y_i} .$$

That is, the matrix of the change of basis is given by the differential of the change of charts, $d(\psi \circ \varphi^{-1}) = \left(\frac{\partial y_i}{\partial x_j}\right)$.

Definition 1.68. An *orientation* on a differentiable manifold M consists of a choice of a orientation $o(x)$ of T_xM for every $x \in M$, such that $x \mapsto o(x) \in \mathrm{Or}(T_xM)$ is smooth. This means that for every $x \in M$ there exists a neighbourhood $U^x \subset M$ and linearly independent smooth vector fields $V_1, \dots, V_n$ on U^x such that $o(y) = [(V_1(y), \dots, V_n(y))]$, for all $y \in U^x$.

A manifold which can be given an orientation is called *orientable*. We denote by $\mathrm{Or}(M)$ the set of orientations of M. In particular $\mathrm{Or}(M) = \emptyset$ if M is not orientable. If we make explicit the choice of the orientation $o \in \mathrm{Or}(M)$, then we say that (M, o) is an *oriented manifold*.

There is a useful characterization of orientable manifold. Suppose that M comes with an orientation o. Let $\varphi : U \to \mathbb{R}^n$ be a chart with U connected, with vector fields $V_1, \dots, V_n$ on U such that $o(x) = [(V_1(x), \dots, V_n(x)]$ for $x \in U$. Then we write $V_j(x) = \sum f_{ij}(x)\frac{\partial}{\partial x_i}$, $j = 1, \dots, n$. The function $\det(f_{ij})$ is smooth and non-zero on U. Therefore by connectedness, either it is everywhere positive or everywhere negative. If $\det(f_{ij}) > 0$, then $o(x) = \left[\left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}\right)\right]$, and we say that (U, φ) is an oriented (or positive) chart. If $\det(f_{ij}) < 0$, then $o(x) = -\left[\left(\frac{\partial}{\partial x_1}, \dots \frac{\partial}{\partial x_n}\right)\right]$, and we say that (U, φ) is a negative chart. Taking the reflection $r(t_1, t_2, \dots, t_n) = (-t_1, t_2, \dots, t_n)$, the chart $\phi = r \circ \varphi$ is positive. This means that

$$\mathcal{A}_+ = \{(U, \varphi) \text{ positive chart}\} \subset \mathcal{A}$$

is an atlas of X. It is called a *positive atlas*.

Proposition 1.69. *Let M be a smooth manifold. Then M is orientable if and only if we can find a smooth atlas $\mathcal{A}_+ = \{(U_\alpha, \varphi_\alpha)\}$ such that all the changes of coordinates $\varphi_\alpha \circ \varphi_\beta^{-1}$ have positive Jacobian, i.e.,* $\det(d(\varphi_\alpha \circ \varphi_\beta^{-1})) > 0$ *wherever it is defined.*

Proof. Suppose M is orientable, and take its positive atlas $\mathcal{A}_+$ as constructed above. Then for each non-empty intersection $U_\alpha \cap U_\beta$ take the corresponding smooth charts $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$ and $\varphi_\beta : U_\beta \to \mathbb{R}^n$ such that

$$o(x) = \left[\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right] = \left[\left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right)\right]$$

for all $x \in U_\alpha \cap U_\beta$. Therefore the determinant of the change of basis, the Jacobian, satisfies

$$J(\varphi_\beta \circ \varphi_\alpha^{-1}) = \det\left(\frac{\partial y_i}{\partial x_j}\right) > 0.$$

Conversely, suppose that there exists an atlas $\mathcal{A}_+$ whose change of coordinates all have positive Jacobian. Then we define the orientation on $x \in M$ by $o(x) = \left[\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right)\right]$ for a chart (U, φ) with $x \in U$. This definition is independent of the chart around x chosen, precisely because the change of charts in $\mathcal{A}_+$ have positive Jacobian. We conclude that this is a well defined orientation on M. □

For a smooth manifold with boundary M, we say that M is *orientable* if $\mathrm{Int}\, M$ is orientable as a manifold without boundary. If M is orientable, then the $(n-1)$-manifold ∂M inherits an orientation form that of M. This is constructed as follows. Take a chart $\varphi : U \to B_\varepsilon^+(0) \subset \mathbb{R}^n_+ = \mathbb{R}^n \cap \{x_1 \geq 0\}$ around $p \in \partial M$. Any vector $v = \sum v_i \frac{\partial}{\partial x_i}$ at p whose first coordinate satisfies $v_1 < 0$ is called *outward pointing*. This notion is well defined, i.e., it is independent of charts. If $\varphi = (x_1, \dots, x_n)$, $\psi = (y_1, \dots, y_n)$ are two charts as above, we note that $y_1 = y_1(x_1, \dots, x_n)$ satisfies that $y_1(0, x_2, \dots, x_n) = 0$ and $y_1(x_1, x_2, \dots, x_n) > 0$ for $x_1 > 0$. So $\frac{\partial y_1}{\partial x_1} > 0$ and $\frac{\partial y_1}{\partial x_i} = 0$ for $i > 1$, at p. Then if

$\nu = \sum \nu'_j \frac{\partial}{\partial y_j}$ in the chart ψ, we have

$$\nu = \nu_1 \frac{\partial}{\partial x_1} + \sum_{j>1} \nu_j \frac{\partial}{\partial x_j} = \nu_1 \frac{\partial y_1}{\partial x_1} \frac{\partial}{\partial y_1} + \sum_{j>1} \nu'_j \frac{\partial}{\partial y_j} ,$$

so $\nu'_1 = \nu_1 \frac{\partial y_1}{\partial x_1} > 0$, as well.

Proposition 1.70. *Let (M, o) be a oriented manifold with boundary. Then ∂M inherits a canonical orientation, denoted $o|_{\partial M}$.*

Proof. The case of dimension $n = 1$ has to be treated separately. In this case ∂M is 0-dimensional. We orient p by assigning the sign $+$ if ν gives the orientation of M, and the sign $-$ if $-\nu$ gives the orientation of M, where ν is an outward pointing vector at p (see Figure 1.10).

Suppose now that $n \geq 2$. Take a positive chart $\varphi : U \to B^+_\varepsilon(0)$ near a point $p \in \partial M$, and let ν be an outward pointing vector. We can arrange that the chart is positive by changing the sign of the last coordinate if necessary (possible since $n \geq 2$). Note that $\partial M \cap U$ is mapped by φ to $B^+_\varepsilon(0) \cap \{x_1 = 0\}$. We establish a positive basis for the orientation of ∂M at p as any basis $(v_2, \ldots, v_n)$ of $T_p(\partial M)$ such that $(\nu, v_2, \ldots, v_n)$ is a positive basis for T_pM. The outward pointing vector is $\nu = -\frac{\partial}{\partial x_1}$, so this means that the chart $(U \cap \partial M, \varphi|_{\partial M})$ is negative for the orientation of ∂M. This gives a well defined orientation on ∂M that varies smoothly (see Figure 1.10 for the case $n = 2$). □

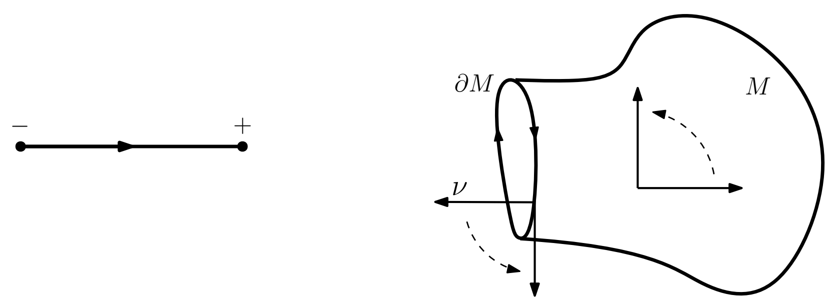


Figure 1.10. Induced orientations.

1.4.2. Orientation of PL manifolds. The concept of orientation can also be defined in the PL category by means of triangulations. First, an *orientation o of an n-polyhedron* P^n is simply an orientation of its interior $\operatorname{Int} P^n$, i.e., an orientation of $\mathbb{R}^n$. This induces an orientation $o|_{P^{n-1}}$ for any $(n-1)$-face $P^{n-1} \subset \partial P^n$. More explicitly, note that $P^n - \bigcup P^{n-2}_\gamma$, where P^{n-2}_γ are all the $(n-2)$-faces of P^n, is a smooth n-manifold with boundary, and the boundary consists of the interiors of the $(n-1)$-faces, which are given the induced orientation as described above. More concretely,

$$o\,|_{P^{n-1}} = [(w_2, \ldots, w_n)] \iff o = [(\nu, w_2, \ldots, w_n)],$$

with ν a tangent vector of P^n which is perpendicular to $P^{n-1} \subset \partial P^n$ and points outwards P^n.

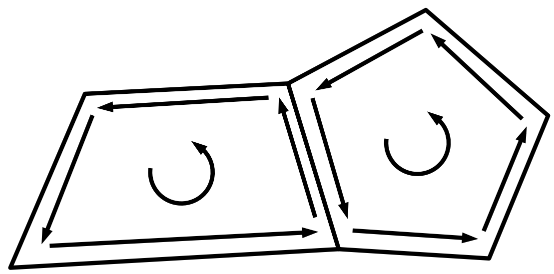


Figure 1.11. Compatible orientations.

Recall that for a PL manifold, a $(n-1)$-polyhedron P_γ^{n-1} is the boundary of at most two n-polyhedra (Remark 1.54). When P_γ^{n-1} consists of interior points, there are two (adjacent) n-polyhedra P_α^n, P_β^n, so that P_γ^{n-1} is a face of them. If o_α, o_β are the orientations of P_α^n, P_β^n, respectively, then we say that they are *compatible* along P_γ^{n-1} if they coincide at the points of the $(n-1)$-face. To be precise, let ν_a, ν_b be the outward normal to P_γ^{n-1} for P_α^n and P_β^n, respectively, so $\nu_\alpha = -\nu_\beta$. Let $(w_2, \dots, w_n)$ be a basis of P_γ^{n-1} such that $o_\alpha = [(\nu_\alpha, w_2, \dots, w_n)]$. Then $o_\beta = [(-\nu_b, w_2, \dots, w_n)] = -[(\nu_b, w_2, \dots, w_n)]$, and thus

$$o_\alpha|_{P_\gamma^{n-1}} = [w_2, \dots, w_n], \quad o_\beta|_{P_\gamma^{n-1}} = -[w_2, \dots, w_n].$$

So the orientations are compatible along P_γ^{n-1} if the induced orientations of P_α^n and P_β^n on P_γ^{n-1} are opposite (see Figure 1.11).

Definition 1.71. Let (M, τ) be a PL manifold of dimension n. An *orientation* consists of giving an orientation o_α to each n-polyhedron P_α^n such that for each $(n-1)$-polyhedron P_γ^{n-1} with two incident n-polyhedra P_α^n, P_β^n, the orientations induced by o_α, o_β on P_γ^{n-1} are opposite.

Remark 1.72. If M is an orientable smooth manifold, then M is also orientable with the induced PL structure (Exercise 1.23). So the smooth and PL notions of orientability agree.

If we have a planar representation P_S of a surface S, we can easily determine whether S is orientable or not.

Proposition 1.73. *A compact connected surface S is orientable if and only if for every letter in the word $\mathbf{p}_S$ of S that appears twice, it does so with different exponents $+1$ and -1.*

Proof. Set an orientation on the interior of the 2-polyhedron P_S. Then the induced orientation is given by going around the boundary in the chosen direction (clockwise or anticlockwise). If two edges with the same letter a appear with different exponents, then this means that they appear in the boundary with opposite directions under the identification. Hence the two induced orientations on a are opposite to one another, which means that they satisfy the compatibility criterium. □

Example 1.74. The previous discussion is illustrated in the following example, where the planar representation of the Möbius band makes it clear that it is not orientable. Note that its planar representation is $abac$, so there are two letters a with the same exponent $+1$, and therefore it is non-orientable (see Figure 1.12).

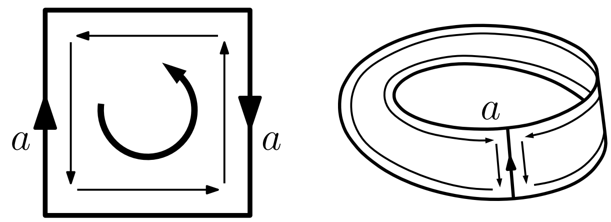


Figure 1.12. The Möbius band is not orientable.

Example 1.75. We can determine easily the orientability of the surfaces in examples 1.59 and 1.62 using Proposition 1.73, as we have words for them: S^2 and T^2 are orientable, $\mathbb{R}P^2$ and Kl are non-orientable. Also Σ_g, $g \geq 1$, are orientable and X_k, $k \geq 1$, are non-orientable (this justifies the words orientable and non-orientable in Definition 1.61).

Corollary 1.76. *A surface is non-orientable if and only if it contains an embedded Möbius band.*

Proof. If there exists an embedding of a Möbius band in S, then S cannot be orientable, because the Möbius band would inherit the orientation, which is impossible. Note that if S is orientable and $S' \subset S$ is a subsurface, then S' is also orientable.

Conversely, if the surface is non-orientable, then some of its connected components are non-orientable. So we can suppose that S is connected. Take a planar representation P_S of S. It must have two edges labelled with the letter to the same exponent. Consider a small closed interval in the interior of both edges, which match under the identification of the boundary, and take a band (i.e., topological rectangle) inside P_S that joins these two closed intervals. In the quotient, this rectangle defines a Möbius band inside S. □

Remark 1.77. For the projective plane, a Möbius band is easily drawn. It turns out to be a small neighbourhood along a projective line inside $\mathbb{R}P^2$ (in the left of Figure 1.13 it is a neighbourhood of a horizontal line). But clearly, it does not matter which line is used, so we can use the line at infinity, which is given as ∂D^2 in the planar model (see the right of Figure 1.13). A neighbourhood of the line at infinity is a Möbius band and its complement is a ball.

1.4.3. Orientation of topological manifolds. The concept of orientability can be defined for a topological n-manifold. For $n \geq 3$, it requires the use of homology groups, that will be introduced in section 2.3. So we will focus here in the case of surfaces ($n = 2$) using the fundamental group, and we leave the case of $n \geq 3$ to Exercise 2.22.

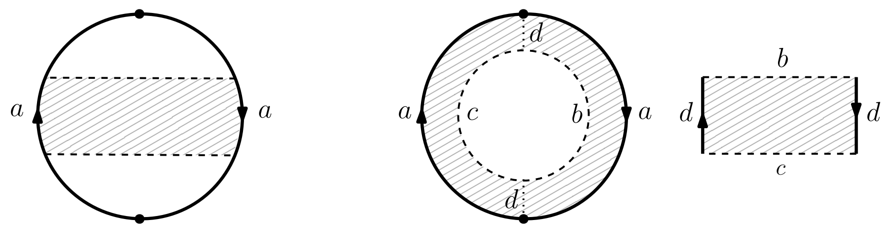


Figure 1.13. $\mathbb{R}P^2$ is the union of a disc and a Möbius band.

Let S be a topological surface. Let $p \in S$, and consider a chart $\varphi : U = U^p \to B_1(0)$, $\varphi(p) = 0$. Then $U - \{p\}$ has the homotopy type of the circle $S^1_\varepsilon(p) = \varphi^{-1}(S^1_\varepsilon(0))$, via a deformation retract $r(x) = \varepsilon \frac{x}{||x||}$, for $0 < \varepsilon < 1$. Thus there is an isomorphism $\pi_1(U - \{p\}) \cong \pi_1(S^1_\varepsilon(p)) \cong \mathbb{Z}$. The first isomorphism is canonical (it does not depend on the chart φ chosen), but the second one depends on a choice of a generator of $\pi_1(S^1_\varepsilon(p))$. Note that this is given by a direction (clockwise or anticlockwise) for going around $S^1_\varepsilon(p)$ (which agrees with an orientation of the image of the chart $\varphi(U) \subset \mathbb{R}^2$ as explained in Remark 1.67(3)).

First note that for the isomorphism $\pi_1(U - \{p\}) \cong \pi_1(S^1_\varepsilon(p))$, we need to choose a basepoint in $U - \{p\}$. However, for any $q_1, q_2 \in U - \{p\}$, the isomorphism $\pi_1(U - \{p\}, q_1) \cong \pi_1(U - \{p\}, q_2)$ is *canonical*. Secondly, we have to choose the radius of the circle, but clearly for $0 < \delta < \varepsilon$, the isomorphism $\pi_1(S^1_\varepsilon(p)) \cong \pi_1(S^1_\delta(p))$ is canonical. Finally, it is independent of charts: if we have two charts $(U, \varphi), (U', \psi)$, take a smaller open subset $V \subset U \cap U'$, and $\varepsilon > 0$ small enough so that $\varphi^{-1}(S^1_\varepsilon(0)) \subset V$ and $\psi^{-1}(S^1_\varepsilon(0)) \subset V$. Then we have a canonical isomorphism $\pi_1(\varphi^{-1}(S^1_\varepsilon(0))) \cong \pi_1(V - \{p\}) \cong \pi_1(\psi^{-1}(S^1_\varepsilon(0)))$.

We call *orientation at p* a choice of generator $o(p) = [\gamma] \in \pi_1(S^1_\varepsilon(p))$. To define the continuity, take a small circle $S^1_\varepsilon(p)$ as above and let $V = \varphi^{-1}(B_\varepsilon(0)) \subset U$. Then for any $q \in V$ we have a canonical isomorphism $\pi_1(S^1_\varepsilon(q)) \cong \pi_1(U - \{q\}) \cong \pi_1(U - V) \cong \pi_1(S^1_\varepsilon(p)) \cong \pi_1(U - \{p\})$. An orientation $o(p)$ at p and $o(q)$ at q are compatible if they coincide under this isomorphism. So $o(p)$ determines an orientation at all points $q \in V$.

Definition 1.78. Let M be a topological manifold. An *orientation* of M consists of a choice of orientations $\{o(p) | \, p \in M\}$ such that for each point $p \in M$ there exists a small neighbourhood U^p such that all the orientations $\{o(q) | \, q \in U^p\}$ are compatible with $o(p)$.

As expected, the notions of orientation for PL manifolds and for topological manifolds agree (Exercise 2.22).

Remark 1.79. Let M be a connected manifold (smooth, PL, or topological). If M is orientable, then it admits exactly two orientations. We see this as follows: let o be one orientation, so $-o$ is another orientation. If o' is an orientation, then the sets $V_1 = \{x \in M | \, o'(x) = o(x)\}$ and $V_2 = \{x \in M | \, o'(x) = -o(x)\}$ are open (by the continuity property of orientations). As they are disjoint and cover M, either $M = V_1$ or $M = V_2$ by

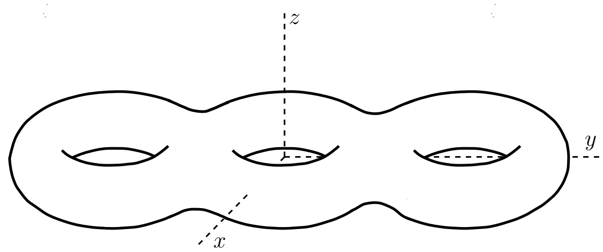


Figure 1.14. The surface Σ_g.

connectedness, so $o' = o$ or $o' = -o$. If M has r connected components, then a choice of orientation for M is to give an orientation for each of the connected components. Hence M has 2^r possible choices of orientations.

Remark 1.80. If (M, o) is a topological n-manifold with an orientation, a homeomorphism $f : M \to M$ *preserves orientation* if $f(o(x)) = o(f(x))$, for all $x \in M$ (cf. Exercise 1.24). Otherwise we say that f *reverses orientation*. When M is connected, these notions make sense for an orientable manifold, even if an orientation has not been fixed. Also for M connected, if f preserves the orientation at a point, then it preserves the orientation everywhere.

Remark 1.81. The orientable surfaces Σ_g admit an orientation reversing homeomorphism. Just locate Σ_g in $\mathbb{R}^3$ lying over the horizontal plane and define $f(x, y, z) = (x, y, -z)$ (see Figure 1.14). As we will shortly see, Σ_g are all the compact connected orientable surfaces (Theorem 1.82), and Remark 1.37 gives the uniqueness of the connected sum of surfaces.

1.5. Classification of compact surfaces

1.5.1. Classification theorem for compact surfaces. The aim of this section is to prove the classification theorem of compact surfaces. Recall that it is enough to restrict to connected surfaces. The following theorem gives the list $\mathbb{L}^{co}_{\mathbf{PLMan}^2_c}$.

Theorem 1.82. *Let S be a compact connected triangulated surface without boundary. Then S is one and only one of the following.*

- *If S is orientable, then S is homeomorphic to Σ_g, for some $g \geq 0$.*
- *If S is non-orientable, then S is homeomorphic to X_k, for some $k \geq 1$.*

In other words, $\mathbb{L}^{co}_{\mathbf{PLMan}^2_c} = \{\Sigma_g, X_k \,|\, g \geq 0, k \geq 1\}$.

Proof. Let S be a compact connected surface without boundary. By Proposition 1.58, S admits a planar representation P_S given by a word $\mathbf{p}_S$. We will show that the word

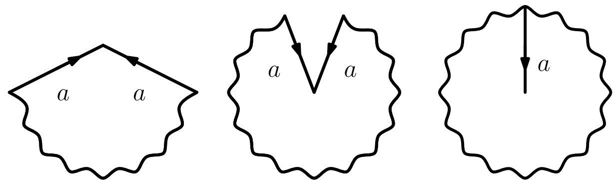
$\mathbf{p}_S$ is equivalent to one of the canonical words given in Example 1.62, that is

$$\begin{aligned} S^2 &: \quad aa^{-1} \approx \emptyset, \\ \Sigma_g &: \quad a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1}, \qquad g \geq 1, \\ X_k &: \quad a_1 a_1 a_2 a_2 \ldots a_k a_k, \qquad k \geq 1. \end{aligned}$$

If $\mathbf{p}_S \approx \mathbf{p}_c$, where $\mathbf{p}_c$ is one of the canonical words above, then $S \cong S_c$, where S_c is one of the surfaces S^2, Σ_g, X_k, respectively. This will complete the proof of the theorem.

Case I. S is orientable. Then all letters in $\mathbf{p}_S$ appear with different exponent. Define the distance between two edges with the same letter by counting the number of edges in between at either side and choosing the smaller one. Take an edge a such that this distance is minimal.

If the distance is zero, this means that the letters a, a^{-1} come together, so $\mathbf{p}_S = \mathbf{p}_1 aa^{-1} \mathbf{p}_2 \approx aa^{-1} \mathbf{p}_2 \mathbf{p}_1$ (maybe changing the direction of the arrows a). If there are no more letters than a, then $\mathbf{p}_S = aa^{-1}$, and we are finished. Otherwise, the following pasting procedure shows that



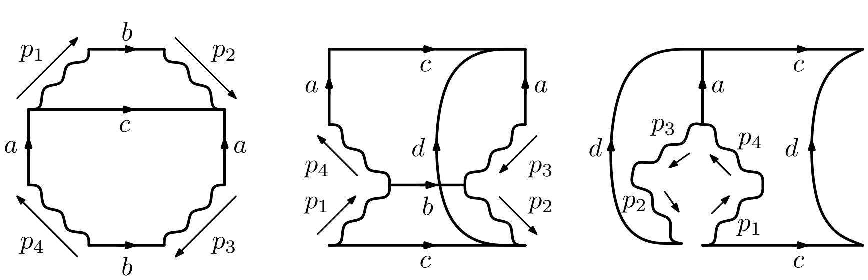
$$\mathbf{p}_S \approx aa^{-1} \mathbf{p}_2 \mathbf{p}_1 \approx \mathbf{p}_2 \mathbf{p}_1 \approx \mathbf{p}_1 \mathbf{p}_2, \tag{1.7}$$

reducing the number of letters.

Now suppose that the (minimal) distance between the two edges labelled a is not zero. Then there is an edge b in between. The other edge b cannot be at the same side, since the distance between the edges b would then be strictly smaller than the distance between the edges a, contrary to assumption. So, after rotation and possibly changing the direction of the arrows a, b, we can write

$$\mathbf{p}_S \approx a \mathbf{p}_1 b \, \mathbf{p}_2 a^{-1} \mathbf{p}_3 b^{-1} \mathbf{p}_4 \, .$$

Now we perform the following cutting and pasting procedure.



We cut the planar representation along c, and glue along b. In the resulting new planar representation, we cut along d and glue along a. All these new planar representations correspond to the same surface S (i.e., there is a quotient map to S). This means that

$$\mathbf{p}_S \approx a\mathbf{p}_1 b\,\mathbf{p}_2 a^{-1}\mathbf{p}_3 b^{-1}\mathbf{p}_4 \approx dcd^{-1}c^{-1}\mathbf{p}_1\mathbf{p}_4\mathbf{p}_3\mathbf{p}_2\,. \tag{1.8}$$

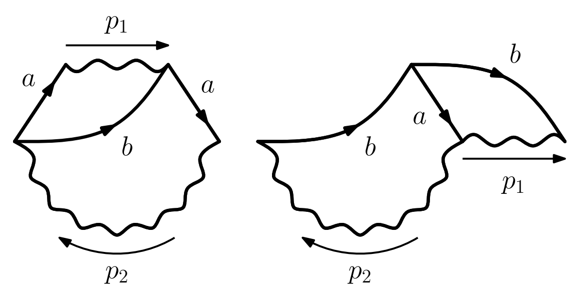
So $S = T^2\#S'$, where S' has word $\mathbf{p}_1\mathbf{p}_4\mathbf{p}_3\mathbf{p}_2$, which has fewer letters.

Proceeding inductively, we get that $S \cong T^2\#\cdots\#T^2 = \Sigma_g$, for some $g \geq 0$.

Case II. S is not orientable. In this case, there are letters with the same exponent. Let a be one such letter. After rotating and possibly changing the direction of the arrows, we have

$$\mathbf{p}_S \approx a\mathbf{p}_1 a\mathbf{p}_2\,.$$

We perform the following cutting and pasting procedure to get



$$\mathbf{p}_S \approx a\mathbf{p}_1 a\mathbf{p}_2 \approx bb(\mathbf{p}_1)^{-1}\mathbf{p}_2\,. \tag{1.9}$$

Therefore $S = \mathbb{R}P^2\#S_2$, for some surface S_2. Arguing by induction, we get

$$\mathbf{p}_S \approx b_1b_1\cdots b_kb_k\mathbf{p}',$$

where $\mathbf{p}'$ only has letters with different exponents. This means that $S = \mathbb{R}P^2\#\overset{(k)}{\cdots}\#\mathbb{R}P^2\#S'$, where S' is an orientable connected surface. If there is no such S', then $S = \mathbb{R}P^2\#\overset{(k)}{\cdots}\#\mathbb{R}P^2 = X_k$, $k \geq 1$, and we are finished.

Finally, we treat the case where $S' = \Sigma_g$, $g \geq 1$. The key point is that $\mathbb{R}P^2\#T^2 \cong X_3$, which is equivalent to $aabcb^{-1}c^{-1} \approx aabbcc$. This is proved by using (1.9), that is $a\mathbf{p}_1a\mathbf{p}_2 \approx aa(\mathbf{p}_1)^{-1}\mathbf{p}_2$, repeatedly. We get the following.

$$\begin{array}{lclclcl} aabcb^{-1}c^{-1} & \approx & ac^{-1}b^{-1}ab^{-1}c^{-1} & \approx & b^{-1}ab^{-1}c^{-1}ac^{-1} & \approx & babcac \\ & \approx & bba^{-1}cac & \approx & cbba^{-1}ca & & \\ & \approx & ccab^{-1}b^{-1}a & \approx & ab^{-1}b^{-1}acc & & \\ & \approx & aabbcc & & & & \end{array}$$

In the first line we put $\mathbf{p}_1 = c^{-1}b^{-1}, \mathbf{p}_2 = b^{-1}c^{-1}$. It is followed by a rotation and a change of direction of the arrow. In the second line we use $\mathbf{p}_1 = a, \mathbf{p}_2 = cac$, followed by a rotation. In the third line take $\mathbf{p}_1 = bba^{-1}, \mathbf{p}_2 = a$, and in the fourth line $\mathbf{p}_1 = b^{-1}b^{-1}, \mathbf{p}_2 = cc$.

Now we have (for $k \geq 1$ and $g \geq 1$)

$$S = X_k\#\Sigma_g = X_{k-1}\#X_1\#T^2\#\Sigma_{g-1} = X_{k-1}\#X_3\#\Sigma_{g-1} = X_{k+2}\#\Sigma_{g-1}\,.$$

Repeating this, we get finally $S = X_{k+2g}$, as required.

Uniqueness. First, X_k cannot be homeomorphic to any Σ_g since the later is orientable while the former is not. Second, the Euler-Poincaré characteristic gives a well defined map $\chi : \mathbb{L}_{\mathbf{PLMan}_c^2} \to \mathbb{Z}$. For $g \neq g'$, we have $\chi(\Sigma_g) = 2 - 2g \neq 2 - 2g' = \chi(\Sigma_{g'})$, hence $\Sigma_g \ncong \Sigma_{g'}$. Also for $k \neq k'$, we have $\chi(X_k) = 2 - k \neq 2 - k' = \chi(X_{k'})$, hence $X_k \ncong X_{k'}$. □

Remark 1.83. The list of topological surfaces $\mathbb{L}^{co}_{\mathbf{TMan}_c^2}$ is also given by

$$\mathbb{L}^{co}_{\mathbf{TMan}_c^2} = \{\Sigma_g, X_k | \, g \geq 0, k \geq 1\}.$$

As we have proven that any topological surface is triangulable (Theorem 1.55), we only need to see that this list has no repetitions, that is that the surfaces Σ_g, X_k, $g \geq 0$, $k \geq 1$, are topologically distinct. This will follow once we prove that the Euler-Poincaré characteristic $\chi(S)$ is a topological invariant (Remark 2.114). So a topological surface S is determined by a couple of topological invariants: $\chi(S)$ and the orientability.

The list of smooth surfaces is also $\mathbb{L}^{co}_{\mathbf{DMan}_c^2} = \{\Sigma_g, X_k | \, g \geq 0, k \geq 1\}$. This will be completed in Theorem 6.56.

Remark 1.84. The rules (1.7), (1.8), and (1.9) hold for any planar representation, so they hold also for surfaces with boundary. However, the rule $\mathbf{p}_S = \mathbf{p}_{S_1}\mathbf{p}_{S_2}$ for $S = S_1 \# S_2$ (Proposition 1.60) only holds for surfaces without boundary (see Exercise 1.30).

Remark 1.85.

(1) The Klein bottle is homeomorphic to X_2, since $abab^{-1} \approx ccb^{-1}b^{-1}$, by using (1.9).

(2) $\mathbb{R}P^2$ minus a ball is a Möbius band (cf. Remark 1.77). A Möbius band is given by $acad \approx bbc^{-1}d$. We can consider $c^{-1}d$ as a single edge, so that the Möbius band is given by the word bbe. Now we glue a disc with boundary e, and we get the surface with word bb, that is, a projective plane. We can also write $\mathbb{R}P^2 = Mob \cup_{S^1} D^2$.

(3) Also $Kl = Mob \cup_{S^1} Mob$, since $X_2 = \mathbb{R}P^2 \# \mathbb{R}P^2$ is obtained by removing a ball from the first $\mathbb{R}P^2$, another ball from the second $\mathbb{R}P^2$, and gluing along the two boundaries.

(4) Note also that $X_1 \# T^2 \cong X_3$ can be rewritten as $Kl \# \mathbb{R}P^2 \cong T^2 \# \mathbb{R}P^2$. A topological interpretation of this isomorphism appears in Exercise 1.37.

Corollary 1.86. *The classification of connected compact triangulated surfaces with boundary,* $\mathbf{PLMan}^2_{\partial,c}$, *is given by the following list.*

(1) *Orientable:* $\Sigma_g - \big(\bigsqcup_{i=1}^{b} B_i\big)$, $g, b \geq 0$, *where* B_i *are disjoint balls.*

(2) *Non-orientable:* $X_k - \big(\bigsqcup_{i=1}^{b} B_i\big)$, $k \geq 1, b \geq 0$, *where* B_i *are disjoint balls.*

Here $b = 0$ *means that the boundary is empty.*

Proof. Let S be a connected compact surface with boundary. The boundary ∂S is a compact 1-manifold, maybe disconnected, and without boundary. Let $b \geq 0$ be the number of connected components of ∂S. As $\mathbb{L}^{co}_{\mathbf{TMan}^1_c} = \{S^1\}$, we have $\partial S \cong \bigsqcup_{i=1}^b S^1_i$, where S^1_i is a copy of the circle. Consider discs D^2_i and homeomorphisms $f_i : S^1_i \to \partial D^2_i$. Take

$$\bar{S} = S \cup \big(\cup_{f_1} D^2_1 \cup_{f_2} D^2_2 \cdots \cup_{f_b} D^2_b \big).$$

This is a compact surface without boundary. Hence $\bar{S} = \Sigma_g$ or X_k, depending on whether it is orientable or not, and $S = \bar{S} - \big(\bigsqcup_{i=1}^b \overset{\circ}{D}{}^2_i \big)$.

Observe that the choice of the discs is irrelevant. By Exercise 1.14 it does not matter from where the ball is removed. By the argument of Theorem 1.35, it does not depend on the homeomorphisms φ_i. □

Remark 1.87. There is also a theorem of classification for open surfaces (an open manifold is a non-compact connected manifold without boundary). This appears in [Ric].

1.5.2. The semigroup ($\mathbb{L}^{co}_{\mathbf{TMan}^n_c}, \#$). It is interesting to note that the list of compact connected n-manifolds has the structure of a commutative semigroup ($\mathbb{L}^{co}_{\mathbf{TMan}^n_c}, \#$). The operation *connected sum* $\#$ is commutative and associative, i.e., $X\#Y \cong Y\#X$ and $(X\#Y)\#Z \cong \#(Y\#Z)$. The neutral element is the sphere S^n, since $S^n\#X \cong X$. However, in general we do not have inverses. Indeed, given a connected surface S different from the sphere, there does not exist another surface S' with $S\#S' \cong S^2$, since $\chi(S\#S') = \chi(S) + \chi(S') - 2 = 2$ implies that $\chi(S) = \chi(S') = 2$ and hence $S \cong S' \cong S^2$. Also the cancellation law is not satisfied, that is $X\#Y \cong X\#Z \not\Longrightarrow Y \cong Z$ (see Remark 1.85(4)).

For the case of surfaces, let us determine the semigroup $G = \left(\mathbb{L}^{co}_{\mathbf{TMan}^2_c}, \#\right)$. It is generated by the surfaces $u = X_1 = \mathbb{R}P^2$ and $v = \Sigma_1 = T^2$ with the relation $u + v = X_1\#\Sigma_1 \cong X_3 = 3u$. Therefore we have an isomorphism of semigroups

$$G = \left(\mathbb{L}^{co}_{\mathbf{TMan}^2_c}, \#\right) \cong \frac{\mathbb{N}\langle u, v\rangle}{\langle u + v = 3u\rangle}.$$

In general, semigroups are difficult to manipulate. So it is convenient to convert an Abelian semigroup into an Abelian group via the Grothendieck construction $G \mapsto K(G)$. To describe it, we return to some notions of categorical nature.

Definition 1.88.

(1) An *initial object* in a category $\mathcal{C}$ is $X_I \in \mathrm{Obj}(\mathcal{C})$ such that for every object $X \in \mathrm{Obj}(\mathcal{C})$ there exists a unique arrow $X_I \to X$.
(2) A *final object* in $\mathcal{C}$ is $X_F \in \mathrm{Obj}(\mathcal{C})$ such that for every object $X \in \mathrm{Obj}(\mathcal{C})$ there exists a unique arrow $X \to X_F$.

Initial and final objects may or may not exist, depending on the category. But if an initial (or final) object exists, then it is unique up to unique isomorphism. Imagine that X_I, X_I' are two initial objects. Then as X_I is initial, there is a unique arrow $f : X_I \to X_I'$. As X_I' is initial, then there is a unique arrow $f : X_I' \to X_I$. So $g \circ f : X_I \to X_I$ is an arrow from an initial object. But $1_{X_I} : X_I \to X_I$ is another one, so by uniqueness $g \circ f = 1_{X_I}$. Analogously $f \circ g = 1_{X_I'}$, so f, g are isomorphisms and $X_I \cong_{\mathcal{C}} X_I'$.

Example 1.89. We give some examples.

- In **Set** and **Top**, $X_I = \emptyset$ and $X_F = \star$, the singleton.
- In $\mathbf{Top}_*$, $X_I = \star$ and $X_F = \star$.
- In $\mathbf{Vect_k}$, $X_F = X_I = 0$.
- In the category of commutative rings **Ring**, $X_I = \mathbb{Z}$ but there is no X_F.
- Consider the groupoid category $\Pi_1(X)$ for a topological space X. Then if X is simply connected, all points are isomorphic, and they are initial and final objects. If X is path connected but not simply connected, then all points are isomorphic, but there are neither initial nor final objects. If X is not path connected, then not all points are isomorphic and there are neither initial nor final objects.

Definition 1.90. Let $(G, +)$ be an Abelian semigroup. We call the *K-theory group* of G or the *Grothendieck group* to an Abelian group $K(G)$ with a homomorphism of semigroups $\iota : G \to K(G)$ satisfying that, for any Abelian group A and any semigroup homomorphism $f : G \to A$, there exists a unique group homomorphism $\bar{f} : K(G) \to A$ such that $f = \bar{f} \circ \iota$.

$$\begin{array}{ccc} G & \xrightarrow{f} & A \\ {\scriptstyle\iota}\big\downarrow & \nearrow_{\bar{f}} & \\ K(G) & & \end{array}$$

The definition of $K(G)$ is given in terms of what is usually called a *universal property*. Let us see that this is an initial/final object property in a suitable category. Let $\mathcal{S}$ be the category defined as:

- its objects are semigroup homomorphisms $f : G \to A$, where A is an Abelian group;
- the homomorphisms between objects $f : G \to A$ and $g : G \to B$ are given by group homomorphisms $\phi : A \to B$ such that the following diagram commutes.

$$\begin{array}{ccc} G & \xrightarrow{f} & A \\ \big\| & & \big\downarrow{\scriptstyle\phi} \\ G & \xrightarrow{g} & B \end{array}$$

Such a category is usually called an *arrow category*, since its objects are arrows of other categories, and its morphisms form diagrams in two levels (the top and bottom levels are the arrows defining the objects "from" and "to" of the morphism). With the above description of $\mathcal{S}$, an initial object is $G \to K(G)$, so it must be unique.

The existence has to be proved by a constructive procedure. In this case, we explicitly construct $K(G)$ as the set of equivalence classes of pairs $(x, y) \in G \times G$ with the equivalence relation $\sim$ given by $(x_1, y_1) \sim (x_2, y_2)$ if and only if there exists $s \in G$ such that $x_1 + y_2 + s = x_2 + y_1 + s$. This is easily proved to be an equivalence relation (the insertion of the term $s \in G$ is necessary for the transitivity). The quotient

$$K(G) = (G \times G)/\sim$$

is an Abelian group with operation $[(x_1, y_1)] + [(x_2, y_2)] = [(x_1 + x_2, y_1 + y_2)]$, neutral element $0 = [(x, x)]$, and symmetric element $-[(x, y)] = [(y, x)]$. The homomorphism $\imath : G \to K(G)$ is defined by $\imath(x) = [(x + y, y)]$, for any $y \in G$.

It is natural to think of the equivalence classes $[(x, y)]$ formally as the difference $x - y$, since $[(x, y)] = \imath(y) - \imath(x)$. This construction is quite typical, and it appears in the axiomatic constructions of the integers $\mathbb{Z}$ from the natural numbers $\mathbb{N}$, i.e., $\mathbb{Z} = K(\mathbb{N}, +)$, and the non-zero rationals $\mathbb{Q}^*$ from $\mathbb{Z}$, $\mathbb{Q}^* = K(\mathbb{Z} - \{0\}, \cdot)$.

For the semigroup $G = (\mathbb{L}^{co}_{\mathbf{TMan}^2_c}, \#)$, the K-theory is given by

$$K(G) \cong K\left(\frac{\mathbb{N}\langle u, v\rangle}{\langle u + v = 3u\rangle}\right) \cong \frac{\mathbb{Z}\langle u, v\rangle}{\langle u + v = 3u\rangle} \cong \mathbb{Z}\langle u\rangle \cong \mathbb{Z}\,.$$

Actually, in $K(G)$, $\Sigma_g = \Sigma_g + \mathbb{RP}^2 - \mathbb{RP}^2 = X_{2g+1} - \mathbb{RP}^2 = X_{2g+1} - X_1 = X_{2g}$. The isomorphism $K(S) \cong \mathbb{Z}$ can be given using the Euler-Poincaré characteristic. For this, consider the map

$$F : G \to \mathbb{Z}, \qquad F(S) = 2 - \chi(S).$$

This is easily seen to be a homomorphism since

$$F(S\#S') = 2-(\chi(S\#S')) = 2-(\chi(S)+\chi(S')-2) = (2-\chi(S))+(2-\chi(S')) = F(S)+F(S').$$

It sends $F(u) = 1$, $F(v) = 2$, hence it is an isomorphism.

Definition 1.91. We define the genus of a compact connected surface without boundary X as $\frac{1}{2}F(X)$. Therefore Σ_g has genus g and X_k has genus $k/2$. It is clearly additive.

Problems

Exercise 1.1. Prove that, in a category $\mathcal{C}$, if a morphism has inverse, then this is unique. Prove that "being isomorphic" is an equivalence relation on $\mathrm{Obj}(\mathcal{C})$, and that the set of automorphisms of an object X (isomorphisms from X to X) is a group.

Exercise 1.2. Prove the theorem of invariance of dimension for $n = 1$, that is, if $U \subset \mathbb{R}$ and $V \subset \mathbb{R}^m$ are open subsets which are homeomorphic, then $m = 1$.

Exercise 1.3. Classify connected topological 1-manifolds (directly from the definition).

Exercise 1.4. Classify connected smooth 1-manifolds (directly from the definition).

Exercise 1.5. Classify connected PL 1-manifolds (directly from the definition).

Exercise 1.6. Prove that any topological 1-manifold admits a triangulation (this gives a solution to Exercise 1.3 from Exercise 1.5).

Exercise 1.7. Classify (connected) 1-manifolds with (non-empty) boundary (in the topological, PL, and smooth categories).

Exercise 1.8. Let Ω be the set of all *countable* ordinals. In $[0, 1] \times \Omega$, glue $(1, \omega) \sim (0, \omega + 1)$, for each $\omega \in \Omega$. Prove that the resulting space X is Hausdorff and locally Euclidean, but not second countable. This is called the *long line*. Does it admit a smooth atlas?

Exercise 1.9. Let X be a topological space. The *suspension of* X is the space $\mathrm{S}(X) = X \times [0, 1]/\sim$, where $(x, 0) \sim (x', 0)$, $(x, 1) \sim (x', 1)$, for all $x, x' \in X$. Show that $\mathrm{S}(S^n)$ is homeomorphic to S^{n+1} for all $n \geq 0$.

Exercise 1.10. Prove that the product of two topological manifolds with boundaries is a topological manifold with boundary. Determine its boundary. Show that $((0,1)\times(0,1))\cup(\{1\}\times[0,1])$ is not a manifold with boundary.

Exercise 1.11. Let $f : \partial D^n \to \partial D^n$ be a homeomorphism. Prove that $M = D^n \cup_f D^n$ is homeomorphic to S^n.

Show that if f is a diffeomorphism, then M is a smooth manifold. Moreover, if f is smoothly isotopic to the identity (that is, there is $H : \partial D^n \times [0,1] \to \partial D^n$ smooth, with $f_t = H(\cdot, t)$ diffeomorphisms, $f_0 = \mathrm{Id}$ and $f_1 = f$), then M is diffeomorphic to S^n.

Exercise 1.12. Let M be a manifold, and let $f : M \to M$ be a homeomorphism. Prove that the mapping torus T_f is a manifold. If M is a manifold with boundary, then T_f is a manifold with boundary, and the boundary is the mapping torus of ∂M.

Exercise 1.13. Prove that if M is a smooth manifold and $f : M \to M$ is a diffeomorphism, then T_f is a smooth manifold.

Exercise 1.14. Let M be a connected topological manifold, and let $p, q \in M$. Prove that there is a homeomorphism of M that sends p to q. If M is a smooth manifold, show that there is one such diffeomorphism.

Exercise 1.15. Prove that the (differentiable) connected sum of two smooth connected surfaces S_1, S_2 does not depend on the choice of points or charts.

Exercise 1.16. Let X be a triangulated space. Prove that it is compact if and only if the number of cells is finite.

Exercise 1.17. Prove that any compact surface with boundary admits a triangulation.

Exercise 1.18. Let X be a compact triangulated space. Prove that the Euler-Poincaré characteristic is invariant by any subdivision of a triangulation.

Exercise 1.19. Give a formula for $\chi(M_1 \# M_2)$, where M_1, M_2 are compact, triangulated, n-dimensional connected manifolds.

Exercise 1.20. Prove that $\mathrm{GL}(n,\mathbb{R})$ has two path connected components, which are the sets $\mathrm{GL}^+(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) |\det(A) > 0\}$ and $\mathrm{GL}^-(n,\mathbb{R}) = \{A \in \mathrm{GL}(n,\mathbb{R}) |\det(A) < 0\}$ (cf. section 4.1.1). Conclude that for a finite dimensional real vector space V of dimension $n \geq 1$, the set $\mathcal{B}$ of all the bases of V has two path connected components, and that $\mathrm{Or}(V)$ is in bijection with the set of path connected components.

Exercise 1.21. Prove that $\mathbb{R}P^n$ is orientable if and only if n is odd.

Exercise 1.22. Let $H \subset \mathbb{R}^n$ be a smooth hypersurface. Prove that an orientation of H is equivalent to the choice of a (continuously varying) unitary normal vector $N_p \in (T_pH)^\perp$, $p \in H$.

Exercise 1.23. If M is an orientable smooth manifold, then M is also orientable with the induced PL structure.

Exercise 1.24. Let (M, o) be an oriented (topological, PL, or smooth) manifold, and let $f : M \to M$ be an isomorphism (homeomorphism, PL isomorphism, or diffeomorphism). Define properly the image of the orientation $f(o(x))$, for $x \in M$.

Exercise 1.25. We define the n-dimensional Möbius band as $Mob_n = Mob \times [0,1]^{n-2}$, for $n \geq 2$. Prove that, in the PL category, a manifold is non-orientable if and only if it contains an embedded Mob_n.

Exercise 1.26. Let M_1, M_2 be two PL connected n-manifolds. Prove that if M_1 is non-orientable, then the two connected sums $M_1 \#_{\mathrm{Id}} M_2$ and $M_1 \#_r M_2$, are homeomorphic (use Exercise 1.25).

Exercise 1.27. Let M_1, M_2 be two orientable connected n-manifolds. Show that both connected sums $M_1 \#_f M_2$ are orientable. Show that if one of M_1, M_2 is not orientable, then $M_1 \#_f M_2$ is not orientable.[3]

Exercise 1.28. Give a *planar* representation (that is, a triangulation with only one curvilinear 3-polyhedron) of $\mathbb{R}P^3$, and prove that it is orientable.

Prove that any (compact) triangulable connected n-manifold has a planar representation.

Exercise 1.29. Let $P \subset \mathbb{R}^2$ be a polygon with edges $l_1, \ldots, l_n$ parametrized by functions $\alpha_1(t), \ldots, \alpha_n(t)$, Let $\beta_1(t), \ldots, \beta_n(t)$ be another parametrization of the same edges with $\beta_i(0) = \alpha_i(0)$, $\beta_i(1) = \alpha_i(1)$. Identify the edges according to a partition $A_1, \ldots, A_r$ of the set $\{1, \ldots, n\}$, that is $\alpha_i(t) \sim \alpha_j(t)$ if $i, j \in A_s$, $t \in [0,1]$. Analogously, we define $\approx$ via $\beta_i(t) \approx \beta_j(t)$ if $i, j \in A_s$, $t \in [0,1]$. Show that $P/\sim$ and $P/\approx$ are homeomorphic.

Exercise 1.30. Give a formula for the word associated to the connected sum of two connected surfaces with boundary given by words $\mathbf{p}_X, \mathbf{p}_Y$, respectively.

Exercise 1.31. Prove that always $aa\mathbf{p}_1\mathbf{p}_2 \approx \mathbf{p}_1 aa\mathbf{p}_2$, where $\mathbf{p}_1, \mathbf{p}_2$ are subwords. However show that $\mathbf{p}_1 \approx \mathbf{p}_1'$ does not imply $\mathbf{p}_1\mathbf{p}_2 \approx \mathbf{p}_1'\mathbf{p}_2$, where $\mathbf{p}_2$ does not share letters with $\mathbf{p}_1, \mathbf{p}_1'$. Explain this in relation to Proposition 1.60.

Exercise 1.32. Prove the theorem of classification of connected surfaces with boundary by using only manipulation of words.

Exercise 1.33. Determine the surfaces with boundary given by the words,

$$\mathbf{p}_X = a\,b\,a\,f\,c\,d\,b^{-1}\,d^{-1}\,g\,e\,c\,e, \quad \text{and} \quad \mathbf{p}_Y = a\,b\,a\,f\,c\,d\,b^{-1}\,d^{-1}\,g\,e\,c^{-1}\,e.$$

Exercise 1.34. Let X be the space parametrizing (unordered) subsets of two points in S^1. Describe the surface X.

Exercise 1.35. Let $\mathcal{C}$ be a category. Prove that there is a category, called the *opposite category*, $\mathcal{C}^{op}$, with $\mathrm{Obj}(\mathcal{C}^{op}) = \mathrm{Obj}(\mathcal{C})$ and $\mathrm{Mor}_{\mathcal{C}^{op}}(X,Y) = \mathrm{Mor}_{\mathcal{C}}(Y,X)$, for objects X, Y. Prove that an initial (resp., final) object for $\mathcal{C}$ is a final (resp., initial) object for $\mathcal{C}^{op}$. Finally prove that a covariant (resp., contravariant) functor $F : \mathcal{C}_1 \to \mathcal{C}_2$ is a contravariant (resp., covariant) functor $F : \mathcal{C}_1 \to \mathcal{C}_2^{op}$.

Exercise 1.36. Determine the semigroup of the compact connected surfaces with boundary $\left(\mathbb{L}^{co}_{\mathbf{TMan}^2_{\partial,c}}, \#\right)$ and the corresponding Grothendieck ring.

Exercise 1.37. Let M be a connected n-manifold. Gluing a *handle* $H = S^{n-1} \times [0,1]$ consists of removing two small disjoint balls $B_1, B_2 \subset M$ and gluing $\partial B_1, \partial B_2$ with the two connected components of ∂H. Prove that there are two ways to glue a handle. Moreover, consider the manifolds $S^{n-1} \times S^1$ and the n-dimensional Klein bottle Kl_n, which is the mapping torus of the reflection on S^{n-1}. Then prove that the two manifolds that are the result of gluing a handle to M are homeomorphic to $M\#(S^{n-1} \times S^1)$ and $M\#Kl_n$.

References and extra material

Basic reading. The following references have material closely related to what is covered in this chapter. For a general introduction to manifolds, we recommend [Boo] and [Sp1]. The notions related to categories can be found in [McL] and sheaves in chapter II of [Wel]. Topological notions can be found in [Kos]. The important theorem of classification of triangulated surfaces appears in [Ma1], and the existence of triangulations can be read from [G-X].

[3] We suggest that the reader try exercises 1.25, 1.26, and 1.27 in the PL category, or in the topological category for $n = 2$. The results hold in the topological category for all n as well, and the exercises can be completed using the theory in section 2.2.2.

[Boo] W.M. Boothby, *An Introduction to Differentiable Manifolds and Riemannian Geometry*, Pure and Applied Mathematics, Vol. 120, 2nd Edition, Academic Press, 2002.

[G-X] J. Gallier, D. Xu, *A Guide to the Classification Theorem for Compact Surfaces*, Geometry and Computing, Vol. 9, Springer, 2013.

[McL] S. MacLane, *Categories for the Working Mathematician*, Graduate Texts in Mathematics, Springer, 1998.

[Ma1] W.S. Massey, *Algebraic Topology: An Introduction*, Graduate Texts in Mathematics, Vol. 56, Springer, 1977.

[Mun] J.R. Munkres, *Elements of Algebraic Topology*, Taylor Francis Inc, 1996.

[Sp1] M. Spivak, *A Comprehensive Introduction to Differential Geometry*, Vol. 1, 3rd Edition, Publish-or-Perish, 1999.

[Wel] R.O. Wells, *Differential Analysis on Complex Manifolds*, Graduate Texts in Mathematics, 3rd Edition, 2008.

Further reading. When teaching a course based on this book, the professor can propose some topics for a short dissertation or specific study to be done by the students. We recommend the following topics related to the content of this chapter.

- **Classification of non-compact surfaces**. This gives an example of a very rich classification. It can be found in:

  [Ric] I. Richards, *On the classification of noncompact surfaces*, Trans. Amer. Math. Soc. 106, 259-269, 1963.

- **Jordan-Schönflies theorem**. This completes the triangulation of surfaces. It can be found at:

  [Cai] S.S. Cairns, *An elementary proof of the Jordan-Schönflies theorem*, Proc. Amer. Math. Soc. 2, 860-867, 1951.

- **Structures C^r on manifolds**. We recommend the treatment in:

  [Hir] M.W. Hirsch, *Differential Topology*, Graduate Texts in Math., Springer, 1997.

- **Whitney embedding**. A smooth manifold can be embedded as a submanifold of $\mathbb{R}^N$.

  [Ada] M. Adachi, *Embeddings and Immersions*, Translations of Mathematical Monographs, Vol. 124, American Mathematical Society, 1993.

- **Exotic spheres**. A proof of the first exotic sphere found by Milnor appears in the original article:

  [Mi1] J.W. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Annals Math. 64, 399-405, 1956.

- **Alexander horned sphere**. This provides an example of a surface in $\mathbb{R}^3$ homeomorphic to S^2 but not bounding a ball. This shows that the Jordan-Schönflies theorem fails in high dimensions.

  [Ale] J.W. Alexander, *An example of a simply connected surface bounding a region which is not simply connected*, Proc. Nat. Acad. Sci. USA 10, 8-10, 1924.

- **Topological quantum field theory**. A TQFT is a functor from a category of manifolds and cobordisms to an algebraic category. In the case of surfaces, it uses the classification of surfaces with boundary.

  [Koc] J. Kock, *Frobenius Algebras and 2D Topological Quantum Field Theories*, London Mathematical Society student texts, Cambridge University Press, 2003.

References. We add some references that have been mentioned within the text. However, most of them are clearly a level beyond the scope of the book. We also include expository references to historical developments, which are more accessible.

[Do1] S. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Diff. Geom. 18, 279-315, 1983.

[Fra] G. Franzoni, *The Klein bottle: variations on a theme*, Notices Amer. Math. Soc. 59, 1076-1082, 2012.

[Fre] M. Freedman, *The topology of four-dimensional manifolds*, J. Diff. Geom. 17, 357-453, 1982.

[Har] J. Harris, *Algebraic Geometry: A First Course*, Graduate Texts in Mathematics, Vol. 133, Springer, 1995.

[Mi2] J.W. Milnor, *Differential topology forty-six years later*, Notices Amer. Math. Soc. 58, 804-809, 2011.

[Moi] E. Moise, *Affine structures in* 3*-manifolds. V. The triangulation theorem and Hauptvermutung*, Annals of Math. (2), 56, 96-114, 1952.

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[Wh2] H. Whitney, *Differentiable manifolds*. Annals of Math. (2), 37, 645-680, 1936.