Preface

Why I wrote this book

This book is an introduction to hyperbolic geometry in three dimensions, with motivations and examples coming from the field of knots. It is also an introduction to knot theory with tools, techniques, and topics coming from geometry. As I write, I believe it is the only book that attempts to be both.

To be clear, there are dozens of excellent books on knot theory available from undergraduate to graduate levels, many of them classics that I learned from and continue to learn from. There are also several excellent books on hyperbolic geometry, particularly from the three-dimensional viewpoint. The aim of this book is to fill in a gap between them: to feature the contributions of hyperbolic geometry to knot theory, and the contributions of knot theory to hyperbolic geometry. It also aims to put techniques and tools from both fields into one place.

In recent years, the field of hyperbolic 3-manifolds has matured, with many open conjectures resolved in the early 2000s. The result is that we now have better insight than ever into the structure of hyperbolic manifolds. This insight can be applied to broad classes of 3-manifolds, including many knot and link complements. On the other hand, the area of knot theory has also ballooned in recent years with new tools arising from algebra, homology theory, quantum topology, representation theory, as well as geometry. As new knot and link invariants arise and new applications of knot theory to other fields develop, it is natural to ask how such invariants interact. In particular, how do these invariants interact with hyperbolic geometry, which contains some of the strongest information on knots and links? There are many open questions and conjectures about the interaction of hyperbolic geometry with other knot invariants, and many mathematicians are interested in learning hyperbolic geometry specifically as it applies to knot theory. This book is a more direct introduction to the hyperbolic geometry of knots.

Hyperbolic geometry was first applied to the study of knots and their complements in the 1970s. Since then, hyperbolic geometry has played an important role in the classification of knots, with invariants such as volume and canonical decomposition developing directly from geometry.

However, the contribution of knot theory to hyperbolic geometry should not be understated. Complements of knots and links have been the playground of the 3-dimensional hyperbolic geometer for decades, aided by diagrams and topology and by computational software, such as SnapPea by Weeks, to find hyperbolic structures on knots. Many conjectures in hyperbolic geometry are based upon geometric properties that were first observed in knots. Many results in hyperbolic geometry have been proved first by restricting to families of knots, especially twist knots, two-bridge knots, and alternating knots, all of which feature prominently in this text.

This book is a hands-on introduction to this mixing of fields, geometry and knots.

How I structured the book

The book starts with an introductory chapter giving basic definitions required from knot theory, and motivating some of the problems discussed in this book.

The first example of a hyperbolic knot, identified by Riley, is the unique prime knot with crossing number four, known as the figure-8 knot. In Chapter 1, we give an introduction to the complement of the figure-8 knot and describe how to decompose it into two polyhedra. The exercises outline a generalization of this decomposition to all knots and lead the reader through complications that arise when generalizing. This decomposition, particularly for the figure-8 knot, will then serve as a running example for later chapters.

In Chapters 2–6, we develop the basics of geometric structures on manifolds, particularly in dimensions two and three. Much of this material overlaps with other texts on hyperbolic geometry. Here, we try to keep our presentation heavily illustrated by examples, especially examples from knot theory. More specifically, Chapter 2 gives an introduction to the hyperbolic plane and hyperbolic 3-space, and gives properties and examples of calculations that we will need in the text. It is purposely brief, as it is not meant to be a comprehensive introduction to these spaces, but instead just enough to equip the reader with the tools required to calculate and compute in hyperbolic geometry. Chapter 3 introduces geometric structures on manifolds, and it works through examples in two dimensions, including careful examples of the torus and the 3-punctured sphere. Chapter 4 returns to 3-manifolds and knots, building the first examples of hyperbolic structures on knot complements by way of triangulations. This chapter covers Thurston's gluing and completeness equations, again using the figure-8 knot as a running example. Chapter 5 delves a little more deeply into properties of hyperbolic isometries, with a main goal of proving the thick-thin decomposition of hyperbolic 3-manifolds. This decomposition implies that thin parts of hyperbolic 3-manifolds can always be identified with knots or links in some 3-dimensional space. Finally, in Chapter 6, incomplete structures on hyperbolic 3-manifolds are carefully investigated. The main result is that such structures can often be viewed as Dehn fillings of hyperbolic manifolds.

Chapters 7–12 focus on families of knots and links that have been particularly amenable to study through hyperbolic geometry, and to tools used to study these knots and links, including tools coming from more general 3-manifold topology. Chapter 7, just after the chapter on hyperbolic Dehn filling, discusses knots described by Dehn filling links in the 3-sphere; many of these links have very explicit hyperbolic geometry. This chapter explores consequences of Dehn fillings for these families. Chapter 8 then provides an interlude, with results from 3-manifold topology, defining essential surfaces, normal surfaces, and returning to hyperbolic geometry via angle structures. Chapter 9 develops the powerful tools of angle structures and volumes of 3-manifolds. The main result in the chapter is a proof of the theorem of Casson and Rivin relating volumes of angle structures to hyperbolic geometry. Angle structures have had great success as applied to the family of two-bridge knots, and this is the subject of Chapter 10. The chapter develops topological descriptions of the knots as gluings of tetrahedra and works through a proof that these tetrahedra are geometric using the theorems of Chapter 9. In Chapter 11, we study alternating links. This chapter gives a proof, using properties of these knots, of the theorem of Menasco that a prime alternating knot with more than one twist region is hyperbolic. Chapter 12 discusses the geometry of surfaces embedded in knot and link complements, including three and four punctured spheres, and checkerboard surfaces.

The final chapters, Chapters 13–15, explore some of the more important knot and link invariants arising from hyperbolic geometry. One of the most important geometric invariants of a hyperbolic knot is its volume, and Chapter 13 is devoted to volumes of knots and links. It contains methods to bound the volume of a knot. Chapter 14 discusses the Ford domain and canonical polyhedral decomposition, also called the Epstein–Penner decomposition of a manifold. This decomposition provides a tool that can be used to identify when two 3-manifolds are isometric; for example, it is used by the software SnapPea (and SnapPy). Chapter 15 gives a brief introduction to the overlap of hyperbolic geometry and algebraic geometry, introducing gluing and character varieties of knots, and the A-polynomial, which is a polynomial invariant directly related to the hyperbolic geometry of a knot.

Prerequisites and notes to students

I have tried to keep prerequisites to a minimum. A basic course in topology is required, as well as some knowledge of basic algebraic topology, particularly the fundamental group and covering spaces. Occasionally, experience with Riemannian geometry will be helpful, but it is not required, with one exception: we assume standard results from a first course in Riemannian geometry in parts of Chapter 13. We also occasionally assume basic results in differential topology, such as the fact that smooth manifolds admit tubular neighborhoods, and that submanifolds can be isotoped to meet transversely.

Also, this book is written to be interactive, with examples and exercises. I hope you work through the examples as they are presented and generalize them in exercises. Many important results are saved for exercises.

Acknowledgments

The first form of this book appeared as lecture notes for a unit at Brigham Young University (BYU). The subject was inspired by my participation in a workshop on interactions between hyperbolic geometry, quantum topology, and number theory held at Columbia University in 2009. I have also given related graduate student workshops at Iowa in 2014, at Melbourne in 2016, and at Luminy in 2018. I thank the organizers of these workshops for inviting me, and various agencies for supporting the workshops, and for supporting fundamental research in mathematics.

I learned much of the material in the first part of this book as a graduate student under the direction of Steve Kerckhoff, reading notes of William Thurston from the 1970s that were ghostwritten by Kerckhoff and Bill Floyd [**Thu79**]. Learning along with me were fellow graduate students David Futer and Henry Segerman. Many of their insights and elucidations helped me develop my own understanding; those insights are contained in this book, and I thank Steve, David, and Henry for them. I also owe thanks to Henry Segerman and Saul Schleimer for figures, particularly Figures 6.4, 6.5, 6.6, and 12.6. Thanks to Saul Schleimer for Figure 12.5, and to David Bachman, Saul Schleimer, and Henry Segerman for Figure 12.7. Discussions with David Futer about various parts of this book, especially two-bridge knots, have been invaluable. I also thank Jim Cannon, who attended my course on this material, and provided ideas for helping students get involved with exercises, and I thank Kenneth Perko, Abhijit Champanerkar, Ilya Kofman, Yi Wang, and Yair Minsky for feedback on drafts of the book.

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Remaining errors are, of course, my own. Please tell me about them.

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