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Preface

The genesis of this book is a pair of courses I taught, one at Indiana University and another at the University of Tennessee. Both of these followed a standard two semester course in functional analysis, though this book is written with only a one semester course in functional analysis as its prerequisite. The aim is to cover with varying depth a variety of subjects that are central to operator theory. Many of these topics have treatises devoted to their explication, and some care has been taken to alert the reader to additional sources for deeper study. So you can think of this book as a sequel to a basic functional analysis course or as a foundation for the study of operator theory.

The prerequisites for this book are a bit fuzzy. The reader is assumed to know fundamental functional analysis. Specifically it will be assumed that the reader knows the material in the first seven chapters of [ACFA]. (Repeated reference to [ACFA] is made in this text and it is referenced in this way rather than the form of other references.) The desire is to make the book as accessible as possible. It is incumbent on me to avoid the arrogance of assuming or requiring familiarity with one of my previous books. But I have to set the standard somewhere, so I will assume the reader knows what I believe to be the material common to all basic courses in functional analysis: the three basic principles of Banach spaces, the definition and elementary properties of locally convex spaces, the foundations of Banach algebras, including the Riesz functional calculus, and the rudiments of operator theory, including the spectral theory of compact operators. This constitutes the first seven chapters of [ACFA].

This book starts with an introduction to C^*-algebras followed by a chapter on normal operators that culminates with the Spectral Theorem and the functional calculus. There is considerable overlap between these first two chapters and Chapters VIII and IX in [ACFA]. If the reader really knows these first nine chapters of [ACFA], he/she will be able to fly through the first two chapters here. Such a reader could realistically begin reading Chapter 3 of the present book. There is, however, material in the first two chapters of this book, specifically §4, that does not appear in [ACFA].

Chapter 3 examines compact operators. Again there is some overlap with [ACFA], but most of this material does not appear there. Chapter 4 begins the study of non-normal operators, which has seen so much significant progress in recent times. In particular the reader is introduced to some rather deep connections between operator theory and analytic functions. This is a hallmark of much that has been done in recent years. The Fredholm index appears here. This is not part of the stated prerequisites and is the subject of Chapter XI in [ACFA]. On the other hand, the need for the index is not that substantial that the reader should be dismayed by encountering it. Indeed, Fredholm theory is discussed later in §37, where the reader will see statements of the pertinent results from this topic, some proofs, and references to [ACFA] for the omitted proofs. The reader who feels some insecurity on this point may examine §37 during the study of Chapter 4.

Chapter 5 returns to the theory of C^*-algebras and examines irreducible representations as well as completely positive maps. As an application, a proof is presented of the Sz.-Nagy Dilation Theorem, which is the basis for a large portion of modern operator theory. The chapter closes with the existence of a quasicentral approximate identity in a C^*-algebra. This is used in the following chapter.

Chapter 6 explores the general topic of compact perturbations. After an abbreviated treatment of Fredholm theory, the Weyl–von Neumann–Berg Theorem is proved as is Voiculescu's Theorem on the approximation of representations of a separable C^*-algebra. The chapter concludes with some applications of these ideas to single operators.

Chapter 7 is a rather extended introduction to von Neumann algebras. The classification scheme is obtained as is complete information on Type I algebras. This is used to recapture the multiplicity theory of normal operators. The chapter includes a proof that a von Neumann algebra is finite if and only if it has a faithful, centered-valued trace.

Chapter 8, the last chapter, gives an introduction to reflexive subspaces of operators. Here the word "reflexive" is used differently from its meaning in Banach space theory. Reflexive subspaces are spaces of operators that are determined by their invariant subspaces. An operator is reflexive if and only if the weakly closed algebra it generates is a reflexive subspace. This, together with the related notion of a hyperreflexive subspace, is a still developing area of research. In many ways this subject is one of the more successful episodes in the modern exploration of asymmetric algebras.

There are many topics in operator theory that are not included here. Given the vastness of the subject, that is no surprise. Many important topics have been omitted because good treatments already exist in the literature. For example, nothing is in this book on the Brown–Douglas–Fillmore theory, which would have been a natural sequel to the chapter on compact perturbations. But whatever approach I would have used would not have differed in any substantial way from that in Davidson [1996]. Also the theory of dual algebras is not presented, but this is due more to space and time limitations. So much has happened since Bercovici, Foias, and Pearcy [1985] that the area is ripe for a further exposition. Such an exposition would have almost doubled the size of the present book. This list of omissions could continue.

Like all books, especially more advanced ones, the material is not linearly dependent. The reader can skip around, especially after the first three chapters. For example, Chapter 8 on reflexivity does not depend on the preceding three chapters in any substantial way. Developing a dependency chart was a temptation, but instead I'll encourage readers to skip around, covering the topics that interest them and filling in the gaps as necessary.

One final caveat. Be aware that I am a bit schizophrenic about separability. On the one hand, I don't wish to present anything in that setting as though it depends on the assumption of separability. On the other hand, in Hilbert space this is where the interest lies. There are also parts of operator theory that really only hold in a reasonable way when the underlying Hilbert space is separable. There are others that are connected to measure theory and I did not want to get into discussing non-separable measure spaces. So I start out with no assumption of separability, but occasionally giving a result that does depend on this. Later, in §51, all Hilbert spaces are assumed to be separable for the remainder of the book.

Throughout the text I give references for further study. I frequently also will cite the source of some, but not all, of the results. I have confidence in the attributions I give, but I certainly am not infallible. I'll maintain a list of corrections and updates on this book linked to my web page (http://www.math.utk.edu/∼conway), and any corrections or changes in attribution will be found there as well in future printings should they come to be.

I have many people to thank for their assistance during the preparation of this book. My former student Nathan Feldman read various editions of the manuscript and made many helpful suggestions. Similar help came from my current students Gabriel Prajitura and Sherwin Kouchekian, who, with Nathan, were in one of the courses that eventually led to this book. In the final analysis, of course, I am the one who is responsible for any errors.