Preface

This book grew out of the lecture notes for a graduate course we taught during the summer semester of 2018 at the Max-Planck Institute (MPI) for Mathematics in the Sciences in Leipzig, Germany. This was part of the general lecture series (called *Ringvorlesung* in German) offered biannually by the International Max-Planck Research School (IMPRS). The aim of our course was to introduce the theme of *Nonlinear Algebra*, which is also the name of the research group that started at MPI Leipzig in early 2017.

Linear algebra is the foundation of much of mathematics, particularly applied mathematics. Numerical linear algebra is the basis of scientific computing, and its importance for the sciences and engineering can hardly be overestimated. The ubiquity of linear algebra has overshadowed the fairly recent growth in the use of nonlinear models across the mathematical sciences. There has been a proliferation of methods based on systems of multivariate polynomial equations and inequalities. This expansion is fueled by recent theoretical advances, development of efficient software, and an increased awareness of these tools. At the heart of this growing area lies algebraic geometry, but there are links to many other branches of mathematics, such as combinatorics, algebraic topology, commutative algebra, convex and discrete geometry, tensors and multilinear algebra, number theory, representation theory, and symbolic and numerical computation. Application areas include optimization, statistics, and complexity theory, among many others.

Nonlinear algebra is not simply a rebranding of algebraic geometry. It represents a recognition that a focus on computation and applications, and the theoretical underpinnings that this requires, results in a body of inquiry that is complementary to the existing curriculum. The term nonlinear algebra is intended to capture these trends, and to be more friendly to applied scientists. A special research semester with that title, held in the fall of 2018 at the Institute for Computational and Experimental Research in Mathematics (ICERM) in Providence, Rhode Island, explored the theoretical and computational challenges that have arisen and charted a course for the future. This book supports this effort by offering students and researchers a warm welcome to the theme of nonlinear algebra.

Our presentation is structured into 13 chapters, one for each week in a semester. Many of the chapters are rather ambitious in that they promise a first introduction to an area of mathematics that would normally be covered in a full-year course. But what we offer is really just an invitation. Readers are encouraged to go further in their studies by exploring other sources. We think that students will enjoy our presentation. We hope that nonlinear algebra will encourage them to think critically and deeply, and to question the historic boundaries between "pure" and "applied" mathematics.

Mateusz Michałek and Bernd Sturmfels

Acknowledgments

The term nonlinear algebra was introduced in the setting of theoretical physics two decades ago. The second author is grateful to Shamil Shakirov for introducing him to that literature. We wish to cite the book by Dolotin and Morozov [**18**], which presents a perspective that aligns well with ours.

A two-week intense course based on our own manuscript took place at MPI Leipzig in June 2019. We are grateful to the following participants for their lectures and comments: Zachary Adams, Yulia Alexandr, Tobias Boege, Marie-Charlotte Brandenburg, Madeline Brandt, Türkü Çelik, Rodica Dinu, Eliana Duarte, Yassine El Maazouz, Yuhan Jiang, Paul Görlach, Alex Heaton, Nidhi Kaihnsa, Max Kölbl, András Lörincz, Orlando Marigliano, Milo Orlich, Yue Ren, Jose Samper, Mahsa Sayyary, Emre Sertöz, Tim Seynnaeve, Isabelle Shankar, Stefana Sorea, Martin Vodicka, and Maddie Weinstein.

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