Preface

This book was built on material from the courses on ordinary differential equations we offered at IMPA (Instituto de Matemática Pura e Aplicada) in the years from 2011 to 2018. Differential equations is a main theme in IMPA's masters program, where it has been taught since the very inception of graduate studies in the 1960s.

Ever since the work of Henri Poincaré revolutionized this field, a little over a century ago, the theory of differential equations has not stopped growing, branching, and acquiring new tools and innumerable applications. It seemed to us that the time was ripe to rethink the discipline as a whole, reflect on its indispensable core ideas and on the additional topics more suited for the training of a young mathematician of our times, "pure" and "applied" alike.

The countless applications of differential equations rely both on the mathematical theory and on the numerical calculation of solutions. A key principle in our train of thought was that both aspects—theoretical and numerical—must be contemplated in our presentation, in an organic and integrated fashion. The goal is not to write a textbook on the numerical analysis of differential equations, a rich and very active research field which already has some excellent bibliographic references, but rather to provide enough tools for the reader to explore numerically the various models presented within the text, and to present interesting opportunities for using such tools.

Structure of this text. Curiously, but perhaps not surprisingly, this line of reasoning led us back to the original vision of Poincaré who, more than hundred years ago, advocated that differential equations should be approached through a combination of *qualitative analysis* and *numerical calculation* of the solutions. This vision is explained and illustrated in Chapter 1, where we also introduce some basic notions of the theory, starting from the very definition of a differential equation. From there on, the text is organized into six cycles of ideas.

The first cycle, comprising Chapters 2 and 3, deals with **foundations**, that is, the basic questions about existence and uniqueness of solutions, and their dependence on the questions: *Does every ordinary differential equation have a solution? If so, how many? How do the solutions change when we modify the differential equation itself? Are they defined on the whole real line? Otherwise, why not?*

The second cycle, consisting of Chapters 4 and 5, introduces several **basic tools**, both theoretical and practical, so that, already at an early stage of the text, the reader becomes capable of analyzing and solving concrete cases. These tools fall under two categories: numerical methods for solving differential equations, together with techniques for estimating the corresponding calculation errors; and the theoretical formalism of the qualitative theory, including concepts of flow, the Poincaré map, equivalence and conjugacy. Using this opportunity, we shall also prove the Poincaré recurrence theorem, a striking illustration of the power of the qualitative approach.

The third cycle, developed in Chapters 6 and 7, introduces the **linear theory** of differential equations, first in the autonomous case and then in the general linear setting. The notion of exponentiation of a matrix, the Liouville–Ostrogradskiĭ formula, and Floquet's theorem enter the picture at this stage. The study of linear equations provides important insight for the general case and lies at the heart of many more sophisticated developments.

The fourth cycle consists of Chapter 8 and is dedicated to Lyapunov **stability theory**. This is a classical subject, contemporary to Poincaré himself, and a beautiful illustration of qualitative analysis. It is also technically accessible, has many practical and theoretical applications, and historically paved the way for important recent progress, including the theory of Lyapunov exponents.

The fifth cycle, in Chapters 9 and 10, deals with **local theory**, that is, the study of the differential equation in the vicinity of certain special trajectories, such as the stationary or the periodic ones. We state and prove two major results, the Grobman–Hartman theorem and the stable manifold theorem, in which the notion of hyperbolicity plays a key role. The proofs use ideas that can be generalized to many other contexts in differential equations and dynamical systems.

The sixth and last cycle, formed by Chapters 11 and 12, introduces the reader to the **global theory** of differential equations, that is, to results about the behavior of the flow as a whole, in connection with the properties of the ambient space. At this point, in order to take full advantage of the theory, it is practically obligatory to expand its scope to differential equations *on manifolds*. We shall discuss some results which are specific for surfaces, such as the theorems of Poincaré–Bendixson and Mayer, and then finish with the beautiful Poincaré–Hopf theorem.

Computational applications, exercises and notes. A distinctive feature of this book is that every chapter proposes a problem for the reader to analyze by computational methods. For each of these proposed numerical experiments, after an explanation of the problem, its context, and related ideas, we list a few objectives that also hint at possible approaches to the problem.

Each chapter also contains exercises related to its respective contents, including several computational ones. But the numerical experiments are something very different: their objectives are stated in a loose, sometimes outright vague way ("find interesting solutions",...), all the more to stress the exploratory nature of the tasks we propose. In our experience, giving the students freedom in interpreting and performing those tasks often leads to surprisingly innovative approaches and solutions.

We end every chapter with a section of Notes, which has multiple purposes. To begin with, most of the bibliographical references have been pushed to those sections, in order to retain the fluidity of the text. Also, we often discuss in the Notes related topics that are left out of the preceding sections. Finally, the Notes contain brief biographical comments about the personalities behind the results, including chronological information which may help the reader in forming a clearer impression of the ways those ideas unfolded.

Prerequisites. The last two chapters require some familiarity with notions from the theory of differentiable manifolds, such as tangent spaces, differential forms, Riemannian metrics and curvature. For the reader's convenience, these and other related notions are recalled in the Appendix. We also cover in the Appendix some basic ideas from the theory of metric spaces, such as compactness, completeness, and the Ascoli–Arzelà theorem, which are used in the text. Additionally, knowledge of a few fundamental concepts of linear algebra (eigenvalue, eigenvector, determinant) and analysis (implicit function theorem, inverse function theorem) has been assumed.

How to use. Depending on the time available, it may be difficult to cover the entire book in a single course. The idea is that a suitable curriculum can be constructed out of the text by making some choices and possibly leaving others topics to be presented by the students during seminars.

For a 40h lecture time course, we recommend the following curriculum, which is consistent with our experience teaching these matters.[1]

Chapter 1: Introduction to the theory (Sections 1.1 to 1.3). [1.5h]

Chapter 2: Theorems of Picard and Peano (Sections 2.1, 2.2, and 2.5.1). Statement and discussion of the continuous and differentiable dependence theorems (Sections 2.3 and 2.4). [3.0h]

Chapter 3: Maximal solutions (Sections 3.1 and 3.2) and Gronwall lemma (Section 3.3). Statement and discussion of the global dependence theorems (Theorems 3.11, 3.13, and 3.14). [3.0h]

Chapter 4: One-step methods (Sections 4.1 and 4.2) and their error estimates (Section 4.3). [3.0h]

Chapter 5: Flows (Section 5.1), tubular flow theorem (Section 5.2) and Poincaré maps (Section 5.3). [3.0h]

Chapter 6: Exponential of a matrix (Sections 6.1 to 6.3). [3.0h]

Chapter 7: Linear homogeneous equations, fundamental solution (Sections 7.1 and 7.2). Liouville–Ostrogradskiĭ formula (Section 7.3). Nonhomogeneous linear equations (Section 7.4). [4.5h]

Chapter 8: Linear stability (Section 8.1). Lyapunov functions, Lyapunov theorem and invariant set theorem (Section 8.2). [3.0h]

Chapter 9: Hyperbolicity, Grobman–Hartman theorem (Sections 9.1 to 9.3), statement and sketch of proof. Statement of the theorem for diffeomorphisms (Theorem 9.16). [4.5h]

Chapter 10: Stable manifold theorem (Sections 10.1 to 10.3), statement and sketch of proof. Statement of the theorem for periodic orbits (Theorem 10.15). [4.5h]

Chapter 11: Limit sets, Poincaré–Bendixson theorem (Sections 11.1 and 11.2). [3.0h]

Chapter 12: Index and Euler characteristic (Sections 12.1 and 12.2). Flows on manifolds. Poincaré–Hopf theorem (Sections 12.3 and 12.4), statement and sketch of proof. [3.0h]

Among the themes best suited for students' presentations, we count the following.

Partial differential equations (Section 2.5.2). [1.5h]

Proofs of the dependence theorems (Sections 2.3 and 2.4). [3.0h]

Proofs of the global dependence theorems (Sections 3.4 and 3.5). [3.0h]

Adams methods (Section 4.4) and their error estimates (Section 4.5). [3.0h]

Equivalence and conjugacy (Section 5.4). [1.5h]

Classification of linear hyperbolic flows (Sections 6.4 and 6.5). [1.5h]

Stiff problems (Section 4.6). [1.5h]

Floquet theorem (Section 7.5). [1.5h]

Lyapunov functions of nonautonomous differential equations (Section 8.3). [1.5h]

Lyapunov exponents (Section 8.4). [1.5h]

Differentiable conjugacy, statement of Sternberg's theorem (Section 9.5). [1.5h]

Remarks about limit sets of flows on surfaces (Section 11.3). [1.5h]

Euler characteristic (Section 12.2 and Section A.8), preceding Mayer and Poincaré–Hopf theorems. [1.5h]

Mayer's theorem (Section 11.4). [3.0h]

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[1]Graduate courses at IMPA comprise 32 lectures of 1.5h each, adding to 48h total lecture time. However, most institutions seem to run shorter academic terms.

Problem sessions and computational platform. Problem sessions dedicated to computational aspects of the course (1.5h per week) are recommended as a supplement to the theoretical classes; they can also complement the theoretical discussion about numerical integration and error estimates. Our suggestion is to dedicate one or two initial sessions to introducing the computational ambient to be used in the course, and then one session to each of the experiments presented in the book.

There are many choices for the computational platform, some more suited than others. For our courses at IMPA we opted for MATLAB, which is one of the most popular solutions worldwide, but there are many others which cater perfectly to the needs of the course. Various manuals, classes, tutorials and discussion forums on such computational platforms are freely accessible on the internet.

Additional references. Several excellent textbooks on differential equations are available that the reader can use to complement the material presented here, as well as to obtain alternative viewpoints on the topics we cover. The short list that follows is very significant, but far from complete.

Among the classic works whose influence on our own text is most evident, we include the books of V. Arnold [**10**], P. Hartman [**160**], M. Hirsch, S. Smale [**170**] (see also M. Hirsch, S. Smale, R. Devaney [**171**]) and J. Sotomayor [**375**]. Another classic reference still much used is Coddington, Levinson [**91**]. Among more recent publications, let us mention Barreira, Valls [**19**] and Teschl [**393**], whose approaches are substantially distinct from ours.

The numerical aspects of ordinary differential equations are covered by Burden, Faires [**59**], Butcher [**65**], Hairer, Nørsett, Wanner [**156**], LeVeque [**238**], and Morton [**287**], among others. Hubbard, West [**177**] is one of relatively few texts which, like ours, try to bridge the gap between theory and numerics. Trefethen, Birkisson, Driscoll [**397**] proposes an interesting approach, built on the exploration of specific examples, and is accompanied by the computational package Chebfun.

Other more specific references are given within the text, especially in the Notes section of each chapter. That includes several references to the original works that built this area of mathematics. Whenever possible, we consulted the primary sources to try and minimize the imprecisions one so often finds in historical surveys of this kind. Fortunately, nowadays there exist several *online* repositories that make access to the classic works much easier. Nevertheless, the task remains difficult and delicate: despite all our efforts, errors surely remain, for which we take full responsibility.

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We also benefited from constructive criticism from various colleagues, including several anonymous reviewers. Aparecido Jesuíno was the first to use the book to teach a course, at the Universidade Federal de Campina Grande, and provided us with a valuable list of comments. Paulo Ney de Souza read preliminary versions of the text and helped us with several corrections and tips, in addition to contributing a good number of exercises. Luiz Henrique de Figueiredo also read a preliminary version and contributed very useful observations. Advice from Alexis Blake helped shape Chapter 4 and is also gratefully acknowledged.

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