Preface

I introduce in this book an extensive survey of many important topics in the theory of Hamilton–Jacobi equations with particular emphasis on modern approaches and viewpoints.

Firstly, I cover the basic well-posedness theory of viscosity solutions for first-order Hamilton–Jacobi equations. This is, by now, quite standard and there have been some great books in the literature on this subject since the 1980s. Nevertheless, it is important to have some key topics covered here in a self-contained way for use throughout the book. It is not our intention here to cover extensively the well-posedness of viscosity solutions for various different kinds of partial differential equations (PDEs).

Then, I aim at going beyond the well-posedness theory and studying further properties of viscosity solutions to Hamilton–Jacobi equations. Along these lines, I first discuss in depth the homogenization theory for Hamilton–Jacobi equations. Although this has always been a very active research topic since the late 1980s until the present (2021), there has not been any standard textbook covering it. I am hopeful that this book will serve as a gentle introductory reference on this subject. Various connections between homogenization and other research subjects are discussed as well. I focus on the periodic and almost periodic settings in the book and choose not to cover a more general and more complicated framework, which is the stationary ergodic setting.

Afterwards, dynamical properties, the Aubry–Mather theory, and weak Kolmogorov–Arnold–Moser (KAM) theory are studied. These appear naturally in the study of first-order Hamilton–Jacobi equations when the Hamiltonian is convex in the momentum variable. I will introduce both dynamical and PDE approaches to studying these theories. Then, I will discuss connections between homogenization and dynamical system, as well as the optimal rate of convergence in homogenization theory.

Let me emphasize that this is a textbook, not a research monograph. My hope is that it can be used by advanced undergraduate students, first- and second-year graduate students, and new researchers entering the fields of Hamilton–Jacobi equations and viscosity solutions as a learning tool. In this case, the readers can follow the flow of the book from the beginning (Chapters 1 and 2), then jump to the topics that the readers aim at. Besides, I intend to keep the contents of various topics covered here as independent as possible so that other interested readers are able to jump directly to a subject of interest in the book.

My intention when writing this book was to present the essential ideas in the clearest possible way, and thus, in various places, the assumptions/conditions imposed are not sharp. In many cases, the readers can improve the assumptions/conditions imposed right away. I will refer to a list of research articles and monographs at the end of each chapter that provide more general pictures of the situations.

Here is a quick outline of the contents of the book. Chapter 1 contains the basic theory of viscosity solutions for Hamilton–Jacobi equations. This includes the well-posedness theory of viscosity solutions, the classical Bernstein method to obtain gradient bounds, Perron's method to prove existence of viscosity solutions, finite speed of propagation for Cauchy problems, and the rate of convergence of the vanishing viscosity process via both the doubling variables method and the nonlinear adjoint method. Chapter 2 is about Hamilton–Jacobi equations with convex Hamiltonians. We discuss optimal control theory, the Dynamic Programming Principle, Legendre's transform, the Lagrangian viewpoint, and the Hopf–Lax formula. We then study some further hidden convex structures and also the maximal subsolutions with their representation formulas.

Chapter 3 is concerned with Hamilton–Jacobi equations with possibly nonconvex Hamiltonians. We discuss two-player zero-sum differential games, the upper and lower values of the games, and the corresponding equations. We then give representation formulas including the Hopf formula of the solutions to these equations. Finally, we give a brief introduction to finite difference approximations to first-order Hamilton–Jacobi equations.

In Chapter 4, I cover the periodic homogenization theory for Hamilton–Jacobi equations. Homogenization results, cell problems, properties of the effective Hamiltonian in the convex and nonconvex settings, and some rates of convergence are studied. In a similar way, the almost periodic homogenization theory is discussed in Chapter 5 although much less is well understood here.

Chapter 6 is devoted to the analysis of convex Hamilton–Jacobi equations in a flat torus. We introduce new representation formulas for solutions to the discount problems and give some applications. The discount problems already appear in Chapter 2. Then, backward characteristics corresponding to the cell problems and optimal rate of convergence in periodic homogenization are studied. This is related to the last part of Chapter 4. Besides, the backward characteristics provide a natural link between viscosity solutions and dynamical aspects of the corresponding Hamiltonian ODEs.

A gentle introduction to weak KAM theory is given in Chapter 7. Both Lagrangian methods and nonlinear PDE methods are presented. In particular, Mather measures, Mather set, and projected Aubry set are defined and analyzed. In Chapter 8, we study further properties of the effective Hamiltonians in the convex setting, which include strict convexity in certain directions, and the method of asymptotic expansion at infinity. Afterwards, the classical Hedlund example and its generalization are discussed.

The homework problems given in this book are of various levels of difficulty. Most of the time, the exercises in corresponding sections are helpful for further understanding of relevant methods, ideas, and techniques. Some of the problems are open-ended and are related to some active research areas.

I would like to thank my Ph.D. student Son Tu, who provided me the first draft of some of these notes based on a graduate topic course (Math 821) that I taught in the fall of 2016 at UW Madison. Solutions to some problems were drafted by him as well. I had been sitting on the notes for a long time before putting some real effort into writing this book.

Besides, I have also used some parts of my lecture notes taught at a topic course at the University of Tokyo, Tokyo, Japan (September 2014) and two topic courses at the University of Science, Ho Chi Minh City, Vietnam (July 2015 and July 2017) to form parts of this book. I would like to thank Professors Yoshikazu Giga, Hiroyoshi Mitake (University of Tokyo), and Huynh Quang Vu (University of Science, Ho Chi Minh City) for their hospitality.

I would like to thank my wife, Van Hai Van, and my daughter, An My Ngoc Tran, for their constant wonderful support during the writing of this book. Besides, I am extremely grateful for the friendship and the support from Wenjia Jing, Hiroyoshi Mitake, and Yifeng Yu.

In Appendix C, I include a characterization of the Legendre transform following Nam Quang Le's very useful suggestion. I very much thank Nam for this.

I appreciate Nattakorn Kittisut, Yeon-Eung Kim, Nam Quang Le, Yuchen Mao, Hiroyoshi Mitake, Loc Hoang Nguyen, Son Tu, Son Van, Lizhe Wan, and Yifeng Yu for pointing out various typos and giving some great suggestions to improve the presentation of this book. I thank the AMS for their tremendous support, especially Sergei Gelfand, Arlene O'Sean, and Christine M. Thivierge.

I was supported in part by NSF grant DMS-1664424 and NSF CAREER grant DMS-1843320 during the writing of this book.

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Summer 2021