Preface

This book presents the fundamentals of shock wave theory. Shock waves are present in many natural situations as a consequence of *nonlinear* constitutive relations. Mathematical analysis of shock waves is based mostly on *conservation laws*. Consider the case of gas dynamics. There is the conservation of mass. The conservation of momentum follows from Newtonian physics. During the nineteenth century, the conservation of energy and the second law of thermodynamics were formulated. This was the scientific backdrop when Stokes [**119**] and Riemann [**112**] did their pioneering work on shock waves in the mid-nineteenth century. More conservation laws were subsequently formulated in the study of electro-magnetism, nonlinear elasticity, high-temperature gas dynamics, and other physical phenomena. There has been important, continuing progress in the development of shock wave theory since the time of Stokes and Riemann.

Mathematical study of shock waves requires thinking beyond the standard theory for partial differential equations. Shock wave theory is one of basic mathematical theories that had impacts on other fields in the mathematical sciences. The study of shock waves has helped to initiate new mathematical theory. The theory can involve sophisticated mathematical techniques.

Around the mid-twentieth century, it was recognized that the basic notions of shock wave theory can be understood with relative ease when scalar conservation laws are considered. The first part of the book, Chapter 2 through Chapter 5, covers the basic elements of shock wave theory by analyzing scalar conservation laws. Shock wave theory originates from consideration of natural phenomena, and so efforts are made to carry out exact mathematical analysis of the solution behavior by starting from intuitive geometric considerations. Solutions for hyperbolic conservation laws exhibit several striking kinds of solution behavior. In many physical situations viscous effects are important. To gain a basic understanding of these effects, in Chapter 4 we consider the Burgers equation, which is the simplest viscous conservation law. The main focus of the analysis is also on the explicit solution behavior. This first part of the book requires only the prerequisite of multi-variable calculus, and is suitable for an undergraduate course.

For the study of most natural phenomena, it is necessary to consider systems of conservation laws. The study of shock waves after the mid-nineteenth century focused mainly on gas dynamics. The key *nonlinearity* of the Euler equations in gas dynamics causing shocks to occur was identified, and elementary waves were constructed. Incorporation of the second law of thermodynamics through the *admissibility conditions (entropy conditions)* was carried out for the Euler equations in gas dynamics. A general mathematical theory for systems of conservation laws was subsequently developed, applicable to systems beyond the equations in gas dynamics.

The second part of the book presents this general theory for systems of hyperbolic conservation laws in Chapter 6 through Chapter 9.

Chapter 6 discusses the general theory of hyperbolic partial differential equations. Basic notions of hyperbolicity and finite speed of propagation of waves are explained. The equivalency of symmetric hyperbolicity and the existence of convex entropy is established. Progress toward shock wave theory based on the local existence theory of smooth solutions for general systems is also illustrated.

The shock wave theory for systems of hyperbolic conservation laws in one spatial dimension constitutes an important chapter in nonlinear partial differential equations and in nonlinear analysis. The next three chapters deal with this theory. Chapter 7 studies the fundamental *Riemann problem*, motivating the search for admissibility conditions and the construction of elementary waves. Chapter 8 studies the rich subject of *nonlinear interaction* of elementary waves. These topics are extensions of those for the scalar laws studied in the first part of the book. However, the basic states are now vectors and it is essential to keep in mind both the state space and the physical space. The analysis of wave interaction suggests various interaction measures to capture the nonlinear effects under different situations. With the basic understanding of wave behavior thus obtained, the *well-posedness* theory for the general initial value problem is then presented in Chapter 9. The existence theory, established via the Glimm scheme [**54**], and the subsequent analysis of continuous dependence on initial data and the solution behavior require explicit, constructive solution algorithms. This leads to a most significant well-posedness theory for *weak solutions* of quasilinear evolutionary partial differential equations.

Shock waves occur in various continuum media [**53**]. In a gas flow shock waves result from the *compression* of acoustic waves. There is a bow shock in front of a supersonic moving object. As the Earth moves in the solar wind, its magnetic field causes a large magnetohydrodynamics shock. An earthquake starts with longitudinal pressure waves, which form shock waves and in turn induce surface shear waves. In shallow water, surface shock waves are produced near the shore, leading to tsunamis.

Due to the drastic occurring across them, shock waves constitute an essential signature of many natural phenomena and reflect the basic properties of the media carrying them. Because of their unique features, shock waves have been used as tools in the physical and medical sciences. With their steep gradients, shock waves are easily observed and often used to test material properties. Shock waves are used to compress steel by a definite ratio to test its strain response under great stress. For a high-temperature gas with several modes of internal energies, a shock tube is used to measure various physical parameters [**122**, **127**]. Shock wave lithotripsy is a standard medical practice used to break up kidney stones; see [**23**]. Industrial diamond can be produced by the compression from an imploding shock sphere.

The third and final part of the book, Chapter 10 to Chapter 14, goes back to the original motivation of shock wave theory by focusing on specific physical models. There has been very substantial progress in the general theory for hyperbolic conservation laws since 1950, as presented, for instance, in the first two parts of the book. The general theory provides useful concepts and techniques for exploring specific physical effects. This third part of the book can serve as a reference for readers interested in exploring new frontiers of shock wave theory. Besides exact analysis, formal calculations and intuitive arguments are presented. Potentially interesting research directions are also suggested in these chapters.

Chapter 10 discusses the effect of dissipation, such as viscosity and heat conductivity in gas dynamics. The simplest model is the Burgers equation studied in Chapter 4. A main concern is the coupling of nonlinear flux and the dissipation. Dissipation represents a form of *non-local* constitutive response, in the sense that the flux depends not only on the basic dependent variables, but also on their differentials. The study of viscous conservation laws enables the study of nonlinear waves for dissipative systems such as the Navier-Stokes equations in gas dynamics and the Boltzmann equation in kinetic theory. A more pronounced non-local response, the mechanism of *relaxation*, is presented in Chapter 11. The relaxation phenomenon is ubiquitous in science as a consequence of memory and other long-term effects. Chapter 11 also includes an introduction to the Boltzmann equation in kinetic theory.

Chapter 12 studies several topics of physical interest in the presence of *resonance.* The nonlinear nature of shock waves gives rise to rich nonlinear resonance phenomena. Basic elements of multi-dimensional Euler equations in gas dynamics are presented in Chapter 13. A new perspective on the difficult subject of multi-dimensional gas flows with shocks is provided. The final chapter, Chapter 14, concludes with some of the author's personal perspectives, including suggestions of open research directions.

Historical perspectives are indicated in the *Notes* toward the end of each chapter. This is to help readers gain further understanding of the material covered in the chapter. There is no attempt to provide a complete list of references. Rather, the references either are directly related to the topics covered in the book or provide further reading for possible future research in shock wave theory. The presentation of the book is self-contained. For the most part, multi-variable calculus is the only prerequisite. The book covers major topics of modern shock wave theory and is a suitable reference book for researchers in the mathematical sciences.

The book can be used as an introductory text for advanced undergraduate and graduate students in mathematics, engineering, and the physical sciences. Each chapter ends with *Exercises* for students.

Basic Notation

$A \equiv B$ means that the quantity A is defined to be B. Without ambiguity, $f(x) \equiv 0$ also means that the function $f(x)$ is zero for all values of x. The limiting values of a function $u(x)$ from the left and from the right are written as

$$u(x-0) \equiv \lim_{y\to 0, y>0} u(x-y), \quad u(x+0) \equiv \lim_{y\to 0, y>0} u(x+y).$$

$\mathbb{R}^m$ denotes m-dimensional real space. Bold letters $\boldsymbol{u}$, $\boldsymbol{x}$ etc. denote vectors in $\mathbb{R}^m$ for some $m > 1$. The inner product of two vectors $\boldsymbol{u} = (u_1, u_2, \ldots, u_m)$ and $\boldsymbol{v} = (v_1, v_2, \ldots, v_m)$ in $\mathbb{R}^m$ is given by

$$(\boldsymbol{u}, \boldsymbol{v}) \equiv u_1 v_1 + u_2 v_2 + \cdots + u_m v_m.$$

Plain letters $x, y, f, g, \ldots$ denote scalar quantities, in $\mathbb{R}^1$.

$\nabla_{\boldsymbol{x}} \equiv (\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_m})$ denotes the partial derivative with respect to the variables $\boldsymbol{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. The divergence of the function $f(\boldsymbol{x})$ is denoted by

$$\nabla_{\boldsymbol{x}} \cdot f(\boldsymbol{x}) \equiv \sum_{j=1}^{m} \frac{\partial f(\boldsymbol{x})}{\partial x_j} = \sum_{j=1}^{m} f_{x_j}(\boldsymbol{x}).$$

$\mathbb{M}$, $\mathbb{B}$ etc. denote matrixes, with $\mathbb{I}$ denoting the identity matrix.

Shock waves are highlighted in figures by thicker lines.

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