Chapter 1

Poisson Brackets

1.1. Poisson brackets

Definition 1.1. A **Poisson manifold** is a manifold M endowed with a **Poisson bracket** on the space $C^\infty(M)$ of smooth functions, i.e., a Lie bracket

$$\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M),$$

satisfying the Leibniz identity:

$$\{f, g \cdot h\} = g \cdot \{f, h\} + \{f, g\} \cdot h, \quad \forall\, f, g, h \in C^\infty(M).$$

A **Poisson map** between Poisson manifolds $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$ is a smooth map $\Phi : M_1 \to M_2$ which induces a Lie algebra homomorphism:

$$\{f \circ \Phi, g \circ \Phi\}_1 = \{f, g\}_2 \circ \Phi, \quad \forall\, f, g \in C^\infty(M_2).$$

Recall that $(C^\infty(M), \{\cdot,\cdot\})$ being a **Lie algebra** means that the Poisson bracket is $\mathbb{R}$-bilinear and skew-symmetric and satisfies the **Jacobi identity**:

$$\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0, \quad \forall\, f, g, h \in C^\infty(M).$$

The **Leibniz identity** says that, for any $H \in C^\infty(M)$, the operation $\{H, \cdot\}$ is a derivation of the algebra $C^\infty(M)$; therefore it defines a vector field X_H on M via the relation

$$\{H, f\} = \mathscr{L}_{X_H}(f), \quad \forall\, f \in C^\infty(M). \tag{1.1}$$

This is called the **Hamiltonian vector field** of $H \in C^\infty(M)$.

A consequence of the Leibniz rule is that Poisson brackets are local in the sense that they can be restricted to open sets.

Proposition 1.2. *Any open subset U of a Poisson manifold $(M, \{\cdot,\cdot\})$ has an induced Poisson bracket $\{\cdot,\cdot\}_U$ for which the inclusion $U \hookrightarrow M$ is a Poisson map:*

$$\{f|_U, g|_U\}_U = \{f, g\}|_U, \quad \forall\, f, g \in C^\infty(M).$$

Proof. For $f, g \in C^\infty(U)$ and $p \in U$ we define

$$\{f, g\}_U(p) := \{\widetilde{f}, \widetilde{g}\}(p),$$

where $\widetilde{f}$, $\widetilde{g} \in C^\infty(M)$ are any smooth functions that coincide with f and g, respectively, on a neighborhood $O \subset U$ of p. To show that this is well-defined it suffices to show that, for any open set $O \subset M$ and any $f, g \in C^\infty(M)$, the bracket $\{f, g\}|_O$ depends only on $f|_O$ and $g|_O$. By skew-symmetry and bilinearity, it suffices to show that $g|_O = 0$ implies $\{f, g\}|_O = 0$. This holds because

$$\{f, g\}|_O = \mathscr{L}_{X_f}(g)|_O = \mathscr{L}_{X_f|_O}(g|_O).$$

Everything else (e.g., Jacobi) follows because it involves identities that can be checked on small enough neighborhoods of points where the functions involved have extensions to the entire M. □

In a local chart $(U, x^1, \ldots, x^m)$ a Poisson bracket $\{\cdot,\cdot\}$ takes the form

$$\{f, g\}|_U = \sum_{i,j=1}^{m} \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \tag{1.2}$$

for some smooth functions $\pi^{ij} \in C^\infty(U)$. To see this, decompose the Hamiltonian vector field of $f \in C^\infty(U)$ as

$$X_f = \sum_{j=1}^{m} X_f^j \frac{\partial}{\partial x^j}.$$

The Leibniz rule gives that the components satisfy

$$X_{f\cdot g}^j = f X_g^j + g X_f^j.$$

Hence, for each j, the map $f \mapsto X_f^j$ is a derivation of $C^\infty(U)$, and we conclude that the Hamiltonian vector field X_f can be written as

$$X_f = \sum_{i,j=1}^{m} \pi^{ij} \frac{\partial f}{\partial x^i} \frac{\partial}{\partial x^j},$$

which implies (1.2).

The functions π^{ij} are just the Poisson brackets of the local coordinates and are called the **structure functions** of the Poisson bracket with respect to the chart $(U, x^1, \ldots, x^m)$; i.e.,

$$\pi^{ij} = \{x^i, x^j\}_U \in C^\infty(U). \tag{1.3}$$

They form a skew-symmetric matrix of functions which by (1.2) determine the bracket locally. These functions are not arbitrary.

Exercise 1.3. Consider a skew-symmetric matrix of smooth functions $\pi^{ij} \in C^\infty(U)$, $1 \leq i, j \leq m$, and define an operation $\{\cdot,\cdot\}_U$ on $C^\infty(U)$ by (1.2). Show that the Jacobi identity for $\{\cdot,\cdot\}_U$ is equivalent to the following system of PDEs:

$$\sum_{l=1}^{m}\left(\pi^{il}\frac{\partial \pi^{jk}}{\partial x^l} + \pi^{jl}\frac{\partial \pi^{ki}}{\partial x^l} + \pi^{kl}\frac{\partial \pi^{ij}}{\partial x^l}\right) = 0 \quad (1 \leq i < j < k \leq m). \tag{1.4}$$

Remark 1.4. The system (1.4) is an *overdetermined nonlinear system of first-order PDEs*: there are $\binom{m}{3}$-equations on $\binom{m}{2}$ unknown functions π^{ij}. The space of local solutions of this system is poorly understood. This is the first indication that, in contrast with symplectic geometry, Poisson geometry is interesting even locally.

1.2. Orbits

The Hamiltonian vector fields defined by (1.1) give rise to a Lie subalgebra

$$\mathfrak{X}_{\mathrm{Ham}}(M,\{\cdot,\cdot\}) \subset \mathfrak{X}(M).$$

The fact that $\mathfrak{X}_{\mathrm{Ham}}(M,\{\cdot,\cdot\})$ is closed under the Lie bracket of vector fields follows from the Jacobi identity for the Poisson bracket.

Exercise 1.5. Show that the Jacobi identity is equivalent to the assignment $f \mapsto X_f$ being bracket preserving:

$$X_{\{f,g\}} = [X_f, X_g], \quad \forall\, f, g \in C^\infty(M). \tag{1.5}$$

So for a Poisson bracket this assignment is a Lie algebra homomorphism.

By moving along flows of Hamiltonian vector fields one generates an equivalence relation $\sim$ on M:

$$x \sim y \quad \Longleftrightarrow \quad \exists\ f_1, \ldots, f_k \in C^\infty(M) \ \text{ such that } \ \phi^1_{X_{f_1}} \circ \cdots \circ \phi^1_{X_{f_k}}(x) = y.$$

The previous exercise shows that the map

$$C^\infty(M) \to \mathfrak{X}(M), \quad f \mapsto X_f,$$

encodes an infinitesimal action of the Lie algebra $(C^\infty(M), \{\cdot,\cdot\})$ on M (see Section A.2). The definition of $\sim$ is inspired by the definition of the orbits of infinitesimal actions of finite-dimensional Lie algebras (see (A.13)). For this reason, the equivalence classes of $\sim$ will be called the **orbits** of the Poisson manifold $(M, \{\cdot,\cdot\})$.

Remark 1.6. For the time being we use the term **orbit**. Later on we will use the more standard term **symplectic leaf**. As we will see, each orbit is naturally an immersed submanifold of M and has an induced symplectic form.

In general it is not so easy to find the orbits of a given Poisson manifold by directly applying the definition. To get a preliminary idea of what the orbits look like it is very helpful to know the center of the Lie algebra $(C^\infty(M), \{\cdot,\cdot\})$. Its elements deserve a special name:

Definition 1.7. A function $C \in C^\infty(M)$ on a Poisson manifold $(M, \{\cdot,\cdot\})$ is called a **Casimir function** if

$$\{C, f\} = 0, \quad \forall\, f \in C^\infty(M).$$

Clearly, a Casimir function must be constant on each orbit. So to determine the orbits one can first try to find the Casimir functions, and then their common level sets would form a coarser partition than the orbit partition.

Next, note that the orbit directions are given by the Hamiltonian vector fields:

$$\{X_{H,x} : H \in C^\infty(M)\} \subset T_xM, \tag{1.6}$$

where $X_{H,x} = X_H(x)$ denotes the value of X_H at x. In fact, we shall see that these are indeed the tangent spaces to the orbits. Moreover, for the actual computation of the orbits, we state here the following very useful criterion.

Proposition 1.8. *Assume that $\mathcal{S}$ is a partition of a Poisson manifold $(M, \{\cdot,\cdot\})$ by connected immersed submanifolds such that, for each $S \in \mathcal{S}$,*

$$T_xS = \{X_{H,x} : H \in C^\infty(M)\}, \quad \forall\, x \in S.$$

Then the members of $\mathcal{S}$ are precisely the orbits of the Poisson manifold.

The proof will be given in Chapter 4.

1.3. Poisson and Hamiltonian diffeomorphisms

A **Poisson diffeomorphism** is a diffeomorphism which is also a Poisson map. The collection of all Poisson diffeomorphisms of $(M, \{\cdot,\cdot\})$ forms a subgroup

$$\mathrm{Diff}(M, \{\cdot,\cdot\}) \subset \mathrm{Diff}(M).$$

Infinitesimal Poisson diffeomorphisms are characterized as follows:

Exercise 1.9. Check that the flow ϕ_V^t of a vector field $V \in \mathfrak{X}(M)$ consists of Poisson diffeomorphisms if and only if

$$\mathscr{L}_V(\{f, g\}) = \{\mathscr{L}_V(f), g\} + \{f, \mathscr{L}_V(g)\}, \quad \forall\, f, g \in C^\infty(M). \tag{1.7}$$

By a **Poisson vector field** we mean a vector field $V \in \mathfrak{X}(M)$ satisfying (1.7). The following exercise shows that the collection of all Poisson vector fields forms a Lie subalgebra; we denote it by

$$\mathfrak{X}(M, \{\cdot,\cdot\}) \subset \mathfrak{X}(M).$$

Exercise 1.10. Prove the following:

(a) Every Hamiltonian vector field X_H is a Poisson vector field.

(b) A vector field V is a Poisson vector field if and only if

$$[V, X_H] = X_{\mathscr{L}_V(H)}, \quad \forall\, H \in C^\infty(M).$$

(c) Poisson vector fields form a Lie subalgebra of the Lie algebra of all vector fields.

The exercise shows that the Lie algebra of Hamiltonian vector fields is a Lie ideal of the Lie algebra of Poisson vector fields. Hence, as in symplectic geometry (see Section B.1), we have

$$\mathfrak{X}_{\mathrm{Ham}}(M, \{\cdot,\cdot\}) \subset \mathfrak{X}(M, \{\cdot,\cdot\}) \subset \mathfrak{X}(M).$$

While the Lie algebras $\mathfrak{X}(M)$ and $\mathfrak{X}(M, \{\cdot,\cdot\})$ correspond to the groups $\mathrm{Diff}(M)$ and $\mathrm{Diff}(M, \{\cdot,\cdot\})$, respectively, it is a bit less obvious which group gives rise to the Lie algebra $\mathfrak{X}_{\mathrm{Ham}}(M, \{\cdot,\cdot\})$. In principle this group should arise from flows of Hamiltonian vector fields. However, these are not enough and one needs to consider time-dependent functions.

Definition 1.11. Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. A **Hamiltonian diffeomorphism** is a diffeomorphism $\phi : M \to M$ with the following property: there exists a smooth family of diffeomorphisms $\phi^t : M \to M$, $t \in [0,1]$, with

$$\phi^0 = \mathrm{id}_M, \quad \phi^1 = \phi,$$

and such that the family is Hamiltonian; i.e., there is a smooth family of functions $\{H_t\}_{t\in[0,1]}$ on M such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi^t(x) = X_{H_t}(\phi^t(x)), \quad \forall\,(x,t) \in M \times [0,1].$$

The collection of all Hamiltonian diffeomorphisms is denoted $\mathrm{Ham}(M, \{\cdot,\cdot\})$ and is called the **Hamiltonian group** of $(M, \{\cdot,\cdot\})$.

A family $\{\phi^t\}_{t\in[0,1]}$ as in the definition is called a **Hamiltonian isotopy**. Note that ϕ^t is the flow $\Phi^{t,0}_{X_H}$ of the time-dependent Hamiltonian vector field X_{H_t} (see Section A.3). The fact that $\mathrm{Ham}(M, \{\cdot,\cdot\})$ is indeed a group follows from the exercise below.

Exercise 1.12. Let $\{\phi^t\}_{t\in[0,1]}$ and $\{\psi^t\}_{t\in[0,1]}$ be Hamiltonian isotopies. Show that $\{(\phi^t)^{-1}\}_{t\in[0,1]}$ and $\{\phi^t \circ \psi^t\}_{t\in[0,1]}$ are also Hamiltonian isotopies.

Exercise 1.13. Prove that $\mathrm{Ham}(M, \{\cdot,\cdot\})$ is a normal subgroup of the group of all Poisson diffeomorphisms of $(M, \{\cdot,\cdot\})$.

The proof of the following result is deferred until Chapter 4.

Proposition 1.14. *The orbits of the action of* $\mathrm{Ham}(M, \{\cdot,\cdot\})$ *on* M *are precisely the orbits of* $(M, \{\cdot,\cdot\})$.

1.4. Examples

Let us start with some concrete examples. For instance we have a Poisson bracket on $\mathbb{R}^2$ defined by

$$\{x, y\} = 1. \tag{1.8}$$

The only Casimir functions are the constant ones and there is only one orbit, the entire space $\mathbb{R}^2$.

Exercise 1.15. Consider the Poisson bracket on $\mathbb{R}^2$ defined by

$$\{x, y\} = x^2 + y^2.$$

Is it Poisson diffeomorphic to (1.8)?

Adding the variable z to (1.8) and declaring it to be a Casimir function, we obtain a Poisson bracket on $\mathbb{R}^3$:

$$\{x, y\} = 1, \quad \{x, z\} = 0, \quad \{y, z\} = 0. \tag{1.9}$$

Since a Casimir function is constant along leaves, the orbits are contained in the horizontal planes $z = c$. You should deduce using Proposition 1.8 that they actually coincide with these planes. Notice also that if we had considered instead

$$\{x, y\} = 1, \quad \{x, z\} = 0, \quad \{y, z\} = y,$$

then the Jacobi identity would not hold and so this would not be a Poisson bracket.

Next, consider the following *linear* bracket:

$$\{x, y\} = z, \quad \{x, z\} = y, \quad \{y, z\} = x. \tag{1.10}$$

You should check that the Jacobi identity holds. This Poisson bracket has the following Casimir function:

$$C(x, y, z) = x^2 - y^2 + z^2.$$

The origin being a zero of all structure functions, it is fixed by all Hamiltonian flows; therefore it is an orbit. The remaining orbits are the connected components of the level sets of C in $\mathbb{R}^3\backslash\{0\}$: the two components of the cone $C = 0$ and the sheets of the hyperboloids $C = r$, for $r \in \mathbb{R}^*$.

Exercise 1.16. Show the following:

(a) The Poisson brackets (1.9) and (1.10) are not Poisson diffeomorphic.

(b) Any Poisson automorphism of (1.10) must fix the origin.

Now we consider a *quadratic* Poisson bracket:

$$\{x, y\} = xy, \quad \{y, z\} = yz, \quad \{z, x\} = 0. \tag{1.11}$$

Again, you should verify that the Jacobi identity holds. There is a Casimir function

$$C(x, y, z) = xz,$$

which allows one to determine the orbits. Notice that the structure functions vanish along the plane $y = 0$, so points in this plane are orbits. In particular, this Poisson bracket cannot be Poisson diffeomorphic to the previous Poisson brackets on $\mathbb{R}^3$.

We now turn to general classes of examples of Poisson brackets of which the previous brackets will turn out to be special cases.

Example 1.17 (Symplectic structures)**.** Every symplectic manifold (M, ω) has an associated Poisson bracket $\{\cdot, \cdot\}$. It is defined such that the Hamiltonian vector field of $H \in C^\infty(M)$ satisfies

$$i_{X_H}\omega = \mathrm{d}H.$$

It follows that the notions of Hamiltonian and symplectic vector field from symplectic geometry (Section B.1) are consistent with the notions of Hamiltonian and Poisson vector field introduced above. As we will see later, besides providing basic examples, symplectic structures are also the building blocks of all Poisson manifolds.

Proposition 1.8 implies that the orbits of a symplectic manifold (M, ω) are the connected components of M.

In particular, we have the **canonical Poisson bracket** on $\mathbb{R}^{2s}$ which is given in linear coordinates $(q^1, \ldots, q^s, p_1, \ldots, p_s)$ by

$$\{f, g\} := \sum_{i=1}^{s} \left(\frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q^i} - \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} \right). \tag{1.12}$$

The structure functions of the canonical Poisson bracket are

$$\{q^i, q^j\} = \{p_i, p_j\} = 0, \quad \{p_i, q^j\} = \delta_i^j.$$

By Darboux's Theorem (Section B.1), any symplectic manifold can be covered by charts in which the Poisson bracket takes this canonical form.

Exercise 1.18. Consider the Poisson bracket on the symplectic manifold $(T^*N, \omega_{\text{can}})$. The evaluation on $X \in \mathfrak{X}(N)$ defines a smooth function

$$\text{ev}_X : T^*N \to \mathbb{R}, \quad \alpha_x \mapsto \alpha_x(X_x).$$

Show that

$$\text{ev}_{[X,Y]} = \{\text{ev}_X, \text{ev}_Y\}_{\text{can}}, \quad \forall X, Y \in \mathfrak{X}(N).$$

In other words, $\text{ev} : \mathfrak{X}(N) \to C^\infty(T^*N)$ embeds the Lie algebra of vector fields into the Lie algebra $(C^\infty(T^*N), \{\cdot,\cdot\}_{\text{can}})$.

Exercise 1.19 (Poisson maps versus symplectic maps)**.** Consider two symplectic manifolds (M_1, ω_1) and (M_2, ω_2) and a smooth map $\varphi : M_1 \to M_2$. Show the following:

(a) If Φ is a symplectic map, then it must be an immersion. Give examples of symplectic maps between symplectic manifolds of different dimensions.
(Hint: Inclusions.)

(b) If Φ is a Poisson map, then it must be a submersion. Give examples of Poisson maps between symplectic manifolds of different dimensions.
(Hint: Projections.)

(c) If Φ is a local diffeomorphism, then φ is a Poisson map if and only if it is a symplectic map.

Note: This exercise will become easier at the end of the next chapter.

Example 1.20 (The zero Poisson bracket)**.** Any manifold M carries the zero Poisson bracket $\{\cdot,\cdot\} \equiv 0$. Notice that its orbits are the points of M; hence we find ourselves at the opposite spectrum when compared to brackets coming from symplectic manifolds. As we will see later this example often turns out to be more interesting than one might expect at first. For now observe that a Poisson map

$$\mu = (\mu_1, \ldots, \mu_n) : (M, \{\cdot,\cdot\}) \to (\mathbb{R}^n, \{\cdot,\cdot\} \equiv 0) \tag{1.13}$$

is the same thing as a collection of n functions $\mu_i \in C^\infty(M)$ that pairwise Poisson commute:

$$\{\mu_i, \mu_j\} = 0.$$

In particular, when (M, ω) is symplectic and $\{\cdot,\cdot\}$ is the associated Poisson bracket, we recover two classical notions (discussed also in Section B.2):

(i) A Poisson map μ as in (1.13) is the same thing as the moment map of an **infinitesimal $\mathbb{R}^n$-Hamiltonian space**.

(ii) If in addition μ is a submersion almost everywhere and $\dim M = 2n$, then (M, ω, μ) is called a **completely integrable system**.

Example 1.21 (Constant Poisson brackets)**.** The simplest solutions to the Poisson equation (1.4) are the constant ones: $\pi^{ij}(x) = c^{ij}$. We then talk about a **constant Poisson bracket** on $\mathbb{R}^m$:

$$\{f, g\} = \sum_{i,j=1}^{m} c^{ij} \frac{\partial f}{\partial x^i} \frac{\partial g}{\partial x^j}, \quad \text{with} \quad c^{ij} \in \mathbb{R}. \tag{1.14}$$

In this case the orbit equivalence relation $\sim$ is stable under translation. Hence the orbits are all translates of the orbit W through the origin. This orbit is the vector subspace $W \subset \mathbb{R}^m$ spanned by the vectors

$$v^i = (c^{i1}, \ldots, c^{im}) \in \mathbb{R}^m \quad (1 \leq i \leq m). \tag{1.15}$$

The condition that $\{\cdot, \cdot\}$ is a constant bracket does not depend on the choice of linear coordinates. It makes sense on any finite-dimensional vector space V and can be characterized more intrinsically as the condition that the bracket of any two linear functions on V is a constant function.

Example 1.22 (Linear Poisson brackets)**.** A Poisson bracket $\{\cdot, \cdot\}$ on a vector space V with the property that the bracket of linear functions is again linear is called a **linear Poisson bracket**. The importance of these structures is highlighted by the following:

Proposition 1.23. *There is a canonical* 1*-to-*1 *correspondence*

$$\left\{ \begin{array}{c} \textit{linear Poisson brackets } \{\cdot,\cdot\} \\ \textit{on a vector space } V = \mathfrak{g}^* \end{array} \right\} \stackrel{\sim}{\longleftrightarrow} \left\{ \begin{array}{c} \textit{Lie algebra structures } [\cdot,\cdot] \\ \textit{on the dual vector space } \mathfrak{g} = V^* \end{array} \right\}$$

determined by the condition

$$\{\mathrm{ev}_u, \mathrm{ev}_v\} = \mathrm{ev}_{[u,v]}, \quad \forall\, u, v \in \mathfrak{g}, \tag{1.16}$$

where $\mathrm{ev} : \mathfrak{g} \to C^\infty(\mathfrak{g}^*)$ *is the evaluation map that identifies elements of* $\mathfrak{g} = V^*$ *with linear functions on* $\mathfrak{g}^* = V$*.*

The relation (1.16) says that, given a linear Poisson bracket $\{\cdot, \cdot\}$ on $V = \mathfrak{g}^*$, the corresponding Lie algebra structure $[\cdot, \cdot]$ on $\mathfrak{g}$ is obtained by restricting the Poisson bracket to linear functions. Conversely, given a Lie algebra structure $[\cdot, \cdot]$ on $\mathfrak{g}$, there is a unique linear Poisson bracket $\{\cdot, \cdot\}$ on $V = \mathfrak{g}^*$ satisfying (1.16). It is given on arbitrary functions $f, g \in C^\infty(\mathfrak{g}^*)$ by

$$\{f, g\}(\xi) := \langle [\mathrm{d}_\xi f, \mathrm{d}_\xi g], \xi \rangle, \quad \forall\, \xi \in \mathfrak{g}^*, \tag{1.17}$$

where the differential $\mathrm{d}_\xi f : T_\xi \mathfrak{g}^* \to \mathbb{R}$ is viewed as an element of $\mathfrak{g}$:

$$\langle \mathrm{d}_\xi f, \nu \rangle = \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0} f(\xi + t\nu), \quad \forall\, \nu \in \mathfrak{g}^*.$$

Exercise 1.24. Prove that (1.17) is indeed a linear Poisson bracket on $V = \mathfrak{g}^*$.

To write the correspondence in the proposition in coordinates, let $\{e^i\}$ be a basis of $\mathfrak{g}$ and denote by (x^i) the induced linear coordinates on $V = \mathfrak{g}^*$. Given a linear Poisson bracket, the resulting structure functions are the linear functions

$$\{x^i, x^j\} = \pi^{ij}(x) = \sum_k c_k^{ij} x^k,$$

where the c_k^{ij} are the structure constants of the Lie algebra $\mathfrak{g}$ w.r.t. the fixed basis:

$$[e^i, e^j] = \sum_k c_k^{ij} e^k.$$

Note that a Poisson bracket on a vector space is linear if and only if its structure functions are linear relative to any linear coordinate system.

Starting with the embedding of Lie algebras from (1.16)

$$\mathrm{ev} : (\mathfrak{g}, [\cdot,\cdot]) \hookrightarrow (C^\infty(\mathfrak{g}^*), \{\cdot,\cdot\})$$

and composing with the map $f \mapsto X_f$, one obtains an infinitesimal $\mathfrak{g}$-action on the manifold $\mathfrak{g}^*$:

$$a : (\mathfrak{g}, [\cdot,\cdot]) \to (\mathfrak{X}(\mathfrak{g}^*), [\cdot,\cdot]), \quad v \mapsto X_{\mathrm{ev}_v}.$$

This is precisely the coadjoint $\mathfrak{g}$-action ad^* recalled in Section A.2:

Exercise 1.25. Check that for any $v \in \mathfrak{g}$, one has

$$X_{\mathrm{ev}_v} = \mathrm{ad}_v^*.$$

We now explain that the orbits of the linear Poisson bracket coincide with the coadjoint orbits. These can be described using any connected Lie group G with Lie algebra $\mathfrak{g}$. Then the infinitesimal $\mathfrak{g}$-action comes from the coadjoint G-action

$$\mathrm{Ad}^* : G \to \mathrm{Diff}(\mathfrak{g}^*),$$

whose orbits are the coadjoint orbits. The coadjoint orbit

$$\mathcal{O}_\xi := G \cdot \xi \quad (\xi \in \mathfrak{g}^*)$$

is an immersed submanifold of $\mathfrak{g}^*$ with

$$T_\xi \mathcal{O}_\xi = \big\{(\mathrm{ad}_v^*)_\xi \,:\, v \in \mathfrak{g}\big\}.$$

For such general facts about smooth actions, see Section A.2. Therefore Exercise 1.25 and Proposition 1.8 imply:

Proposition 1.26. *The orbits of the linear Poisson bracket on $\mathfrak{g}^*$ coincide with the coadjoint orbits.*

Exercise 1.27. Let $\mathfrak{g} = \mathfrak{so}(3,\mathbb{R})$ be the Lie algebra of 3×3 skew-symmetric matrices with bracket the commutator of matrices. Show that if one identifies $\mathfrak{so}(3,\mathbb{R})$ with $\mathbb{R}^3$ so that the Lie bracket is identified with the vector product $\times$, then the linear Poisson bracket on $\mathfrak{so}(3,\mathbb{R})^*$ becomes the Poisson bracket on $\mathbb{R}^3$ given by the triple product:

$$\{f,g\}(\mathbf{x}) = (\nabla f(\mathbf{x}) \times \nabla g(\mathbf{x}))\cdot \mathbf{x} = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ x & y & z \end{vmatrix}. \tag{1.18}$$

Moreover, show the following:

(a) The equations for the orbits of the Hamiltonian vector field corresponding to the function $H(x,y,z) = \frac{x^2}{2I_x} + \frac{y^2}{2I_y} + \frac{z^2}{2I_z}$ are

$$\begin{cases} \dot{x} = \{H,x\} = \frac{I_y - I_z}{I_y I_z} yz, \\ \dot{y} = \{H,y\} = \frac{I_z - I_x}{I_z I_x} zx, \\ \dot{z} = \{H,z\} = \frac{I_x - I_y}{I_x I_y} yx. \end{cases} \tag{1.19}$$

These are the Euler equations describing the motion of a top in the absence of gravity, moving around its center of mass, with moments of inertia I_x, I_y, and I_z (see, e.g., [**13**] for such examples).

(b) The orbits of this Poisson bracket are the spheres centered at the origin, and the origin.

(c) The Poisson bracket is not Poisson diffeomorphic to (1.10).

Example 1.28 (Quadratic Poisson brackets)**.** Moving one degree higher, a **quadratic Poisson bracket** on a vector space is one for which the Poisson bracket of any two linear functions is a homogeneous polynomial of degree 2.

A relatively simple family of such brackets on $\mathbb{R}^m$ can be constructed as follows. Fix an $m \times m$ skew-symmetric matrix $A = (a^{ij})$ and define

$$\{f,g\}_A := \sum_{i,j=1}^{m} a^{ij} x^i x^j \frac{\partial f}{\partial x^i}\frac{\partial g}{\partial x^j}. \tag{1.20}$$

You should convince yourself that the Jacobi identity holds.

Exercise 1.29. There are even more general quadratic Poisson brackets than the Poisson brackets (1.20). Give such examples in $\mathbb{R}^2$ and $\mathbb{R}^3$.

Let us restrict the Poisson bracket $\{\cdot,\cdot\}_A$ to the open subset

$$\mathbb{R}^m_{>0} := \{(x^1,\dots,x^m) : x^i > 0,\ i = 1,\dots,m\}.$$

Fix $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ and define the following function on $\mathbb{R}^m_{>0}$:

$$H := \sum_{i=1}^{m} \left(\lambda_i \log x^i - x^i\right).$$

The flow of the Hamiltonian vector field X_H is the solution to the system of ODEs

$$\dot{x}^i = \{H, x^i\} = \varepsilon_i x^i + \sum_{j=1}^{m} a^{ij} x^i x^j \qquad (1 \le i \le m), \tag{1.21}$$

where we have introduced the constants $\varepsilon_i := \sum_{j=1}^{m} a^{ji}\lambda_j$. Equations (1.21) are the famous Lotka-Volterra equations which model the dynamics of the populations of n biological species interacting in an ecosystem [**91**]. For this reason we shall call (1.20) the **LV-type Poisson bracket** associated with the skew-symmetric matrix A.

Exercise 1.30. Find the orbits of the LV-type Poisson bracket on $\mathbb{R}^3$ with structure functions:

$$\{x, y\} = xy, \quad \{y, z\} = yz, \quad \{z, x\} = zx. \tag{1.22}$$

(Hint: Find a Casimir function.)

Exercise 1.31. Consider the map $\Phi : \mathbb{R}^{2m} \to \mathbb{R}^m$, $(q^i, p_i) \mapsto x^i$, defined by

$$x^i = e^{p_i - \frac{1}{2}\sum_{j=1}^{m} a^{ij} q^j}. \tag{1.23}$$

Show that Φ is a Poisson map when we equip $\mathbb{R}^{2m}$ with the canonical Poisson bracket (1.12) and $\mathbb{R}^m$ with the LV-type Poisson bracket (1.20).

1.5. Poisson actions and quotients

There are natural ways of producing new Poisson manifolds out of known Poisson manifolds. For example, one can form products of Poisson manifolds (see Problem 1.2). Another way is by forming quotients as we now discuss.

Given a proper and free **symplectic action** of a Lie group G on a symplectic manifold (M, ω), the quotient M/G has an induced Poisson bracket, which is uniquely determined by the property that the quotient map

$$p : M \to M/G$$

is a Poisson map. This follows because $C^\infty(M/G) = C^\infty(M)^G$ is closed under the Poisson bracket on M (see Exercise B.11). Even though we start with a symplectic manifold, the resulting Poisson bracket can have intricate geometry. Here is an explicit example.

Example 1.32. Start with the symplectic manifold:

$$M = \mathbb{C}^2 \backslash \{0\}, \quad \omega = \frac{1}{2}(\mathrm{d}z \wedge \mathrm{d}\bar{w} + \mathrm{d}\bar{z} \wedge \mathrm{d}w),$$

which admits the free and proper symplectic $G = \mathbb{S}^1$-action given by

$$\theta \cdot (z, w) = (e^{i\theta} z, e^{i\theta} w).$$

Consider the $\mathbb{S}^1$-invariant functions on M:

$$\sigma_1 = \frac{1}{2}\left(|z|^2 + |w|^2\right), \quad \sigma_2 = \frac{1}{2}\left(|z|^2 - |w|^2\right), \quad \sigma_3 = z\bar{w} + \bar{z}w.$$

The Poisson brackets of these functions are given by

$$\{\sigma_1, \sigma_2\} = \sigma_3, \quad \{\sigma_2, \sigma_3\} = -\sigma_1, \quad \{\sigma_1, \sigma_3\} = \sigma_2.$$

They induce a smooth map:

$$\sigma : M/\mathbb{S}^1 \to \mathbb{R}^3, \quad [x] \mapsto (\sigma_1(x), \sigma_2(x), \sigma_3(x)).$$

The restriction of σ to an open dense set $U \subset M/\mathbb{S}^1$ is an embedding. So (U, σ) is a chart on the quotient in which the Poisson bracket is linear.

In general, the orbits of the Poisson manifold M/G may be hard to determine. However, in the case of Hamiltonian G-spaces the situation improves. Recall — see Section B.2 — that such a Hamiltonian G-space consists of a symplectic G-space (M, ω) together with a G-equivariant map $\mu : M \to \mathfrak{g}^*$ satisfying the moment map condition:

$$i_{\mathcal{a}(v)}\omega = \mathrm{d}\mu_v, \quad \forall v \in \mathfrak{g},$$

where $\mathcal{a} : \mathfrak{g} \to \mathfrak{X}(M)$ denotes the infinitesimal $\mathfrak{g}$-action. We then have the following result relating the orbits and the symplectic quotients:

Proposition 1.33. *Let (M, ω) be a Hamiltonian G-space with moment map $\mu : M \to \mathfrak{g}^*$, and assume that the action is free and proper. Then the orbits of the Poisson manifold M/G are the connected components of the symplectic quotients $M/\!\!/_{\mathcal{O}} G := \mu^{-1}(\mathcal{O})/G \subset M/G$, where $\mathcal{O}$ ranges through the coadjoint orbits of $\mathfrak{g}^*$.*

Proof. The connected components of the symplectic quotients $M/\!\!/_{\mathcal{O}} G$, when $\mathcal{O}$ ranges through the coadjoint orbits of $\mathfrak{g}^*$, give a partition of M/G by connected immersed submanifolds. According to Proposition 1.8, all we have to check is that

$$T_y\big(M/\!\!/_{\mathcal{O}} G\big) = \{X_{H,y} : H \in C^\infty(M/G)\}, \quad \forall\, y \in M/\!\!/_{\mathcal{O}} G.$$

To see this we first observe that, since $p : M \to M/G$ is a Poisson map, if $H \in C^\infty(M/G)$, then the vector fields $X_H \in \mathfrak{X}(M/G)$ and $X_{H\circ p} \in \mathfrak{X}(M)$ satisfy

$$X_{H\circ p}(f \circ p) = X_H(f) \circ p, \quad \forall\, f \in C^\infty(M/G).$$

In other words,

$$\mathrm{d}p(X_{H\circ p,x}) = X_{H,p(x)}, \quad \forall\, x \in M.$$

Note that for each $\xi \in \mathcal{O}$ the projection restricts to a submersion $p : \mu^{-1}(\xi) \to M/\!\!/_{\mathcal{O}}G$, which induces the isomorphism:

$$M/\!\!/_{\xi}G := \mu^{-1}(\xi)/G_{\xi} \simeq M/\!\!/_{\mathcal{O}}G.$$

We claim that

$$T_x\mu^{-1}(\xi) = \{X_{H\circ p,x} : H \in C^{\infty}(M/G)\}, \tag{1.24}$$

so the result will follow.

We now observe that any Hamiltonian vector field $X_{H\circ p}$ is symplectic orthogonal to the G-orbits. Indeed, since the image of the infinitesimal action $a : \mathfrak{g} \to \mathfrak{X}(M)$ at x — see Section B.2 — coincides with the tangent space to the orbit through x, it is enough to observe that

$$\omega(a(v), X_{H\circ p}) = -a(v)(H \circ p) = 0, \quad \forall\, v \in \mathfrak{g}.$$

Now, (1.24) will follow from the following lemma:

Lemma 1.34. *For any Hamiltonian G-space, the orbits of the action are symplectic orthogonal to the fibers of the moment map. More precisely,*

$$T(G\cdot x)^{\perp_\omega} = \operatorname{Ker}\mathrm{d}\mu, \quad \forall\, x \in M.$$

Proof of the lemma. The image of a coincides with the tangent space to the orbits. The lemma follows by observing that the moment map condition

$$i_{a(v)}\omega = \mathrm{d}\mu_v$$

implies that $\omega(a(v), w) = 0$, whenever $v \in \mathfrak{g}$ and $w \in \operatorname{Ker}\mathrm{d}_y\mu$, and that $T_y(G\cdot x)$ and $\operatorname{Ker}\mathrm{d}_y\mu$ have complementary dimension. □

The inclusion $\supset$ in our claim (1.24) is now obvious. For the other inclusion, we need to prove that any tangent vector $X \in T_x\mu^{-1}(\xi)$ can be written as

$$X = X_{H\circ p,x},$$

for some $H \in C^{\infty}(M/G)$. Setting $\alpha = i_X\omega$, the lemma implies that α annihilates the tangent space to the orbits and hence is the pullback of a covector in M/G. Since any covector can be realized as the differential of a function, we can write

$$\alpha = p^*\mathrm{d}_xH,$$

for some function $H \in C^{\infty}(M/G)$. Then $X = X_{H\circ p,x}$, as required. □

Example 1.35 (Moment maps as Poisson maps)**.** Symplectic manifolds and linear Poisson brackets on duals of Lie algebras interact nicely within the Hamiltonian framework. We need here the infinitesimal version of this framework. A **$\mathfrak{g}$-Hamiltonian space** consists of a symplectic manifold (M, ω), an infinitesimal Lie algebra action $a : \mathfrak{g} \to \mathfrak{X}(M)$, and a smooth map $\mu : M \to \mathfrak{g}^*$ satisfying the following:

(i) ω is $\mathfrak{g}$-invariant.

(ii) μ is $\mathfrak{g}$-equivariant.

(iii) The moment map condition: $i_{a(v)}\omega = \mathrm{d}\mu_v$, $\forall v \in \mathfrak{g}$.

Note that the moment map condition (iii) implies that the map $a : \mathfrak{g} \to \mathfrak{X}(M)$ can be recovered from μ and ω. It is remarkable that all the other conditions can be packed into one single property, namely that the map

$$\mathfrak{g} \to C^\infty(M), \quad v \mapsto \mu_v$$

is a Lie algebra homomorphism — as discussed in Section B.2 — or, equivalently, that μ is a Poisson map! In other words, given (M, ω) and $\mathfrak{g}$ there is a 1-to-1 correspondence

$$\left\{ \begin{matrix} \text{Poisson maps} \\ \mu : (M, \omega) \to (\mathfrak{g}^*, \pi_{\mathfrak{g}}) \end{matrix} \right\} \overset{\sim}{\longleftrightarrow} \left\{ \begin{matrix} \mathfrak{g}\text{-Hamiltonian} \\ \text{spaces } (M, \omega) \end{matrix} \right\}.$$

Most of this discussion generalizes to Poisson manifolds. A **Poisson action** of a Lie group G on a Poisson manifold $(M, \{\cdot,\cdot\})$ is an action $\mathscr{A} : G \to \mathrm{Diff}(M)$ with the property that the translation by any $g \in G$ is a Poisson map:

$$\mathscr{A}_g : (M, \{\cdot,\cdot\}) \to (M, \{\cdot,\cdot\}), \quad x \mapsto g \cdot x.$$

The quotient construction immediately extends to the Poisson setting:

Proposition 1.36. *Given a free and proper Poisson action of a Lie group G on a Poisson manifold $(M, \{\cdot,\cdot\})$, the orbit space M/G has a unique Poisson bracket for which the projection $p : M \to M/G$ is a Poisson map.*

Example 1.37. There are many possible concrete illustrations of this construction. For example, one can construct interesting Poisson brackets on the real projective space $\mathbb{RP}^{n-1}$ by starting with an LV-type quadratic Poisson bracket $\{\cdot,\cdot\}_A$ on $\mathbb{R}^n \backslash \{0\}$ from Example 1.28 and the $\mathbb{R}^*$-action $(\lambda, x) \mapsto \lambda x$. The outcome can be quite interesting:

Exercise 1.38. Consider the quotient Poisson bracket on the projective plane $\mathbb{RP}^2$ induced by the LV-type Poisson bracket on $\mathbb{R}^3 \backslash \{0\}$ from Exercise

1.30. Show that the 0-dimensional orbits are the points on the three circles:

$$Z = \{[x : y : 0]\} \cup \{[x : 0 : z]\} \cup \{[0 : y : z]\},$$

and that 2-dimensional orbits are the four components of $\mathbb{RP}^2 \backslash Z$.

One can also generalize Hamiltonian actions to Poisson manifolds. We say that a Poisson action of a Lie group G on $(M, \{\cdot,\cdot\})$ is a **Hamiltonian action** if there is a G-equivariant map $\mu : M \to \mathfrak{g}^*$ such that the infinitesimal action $\mathscr{a} : \mathfrak{g} \to \mathfrak{X}(M)$ satisfies the **moment map condition**:

$$\mathscr{a}(v) = X_{\mu_v}, \quad \forall\, v \in \mathfrak{g}. \tag{1.25}$$

Exercise 1.39. For a connected Lie group G, show that the G-orbits of a Hamiltonian G-space $(M, \{\cdot,\cdot\}, \mu)$ are always contained in the orbits of the Poisson manifold M. Is this still true for a Poisson action?
(Hint: Look at the Poisson action of Example 1.37.)

As for the analogue of Proposition 1.33, consider a proper and free Hamiltonian G-space $(M, \{\cdot,\cdot\})$ with moment map $\mu : M \to \mathfrak{g}^*$ and fix a coadjoint orbit $\mathcal{O} \subset \mathfrak{g}^*$. In this case, the statement becomes that the quotient

$$M /\!\!/_{\mathcal{O}} G := \mu^{-1}(\mathcal{O})/G$$

carries a unique Poisson bracket such that the inclusion $M /\!\!/_{\mathcal{O}} G \hookrightarrow M/G$ is a Poisson map. This will be discussed in detail in Section 8.1. For now we look at an example.

Example 1.40. Consider the linear Poisson bracket on $\mathfrak{so}^*(3, \mathbb{R}) \simeq \mathbb{R}^3$ as in Exercise 1.27. In coordinates (x, y, z) it is given by

$$\{x, y\} = z, \quad \{y, z\} = x, \quad \{z, x\} = y.$$

The action of $\mathbb{S}^1$ on $\mathbb{R}^3$ by rotations around the z-axis Oz is Hamiltonian with moment map

$$\mu : \mathbb{R}^3 \to \mathbb{R}, \quad \mu(x, y, z) = z.$$

The restriction of this action to the open set $M = \mathbb{R}^3 \backslash Oz$ is a proper and free Hamiltonian action. The induced Poisson bracket on the quotient $M/\mathbb{S}^1$ is zero: the $\mathbb{S}^1$-invariant functions $u = x^2 + y^2$ and $v = z$ give global coordinates on the quotient and we have

$$\{u, v\}_{M/\mathbb{S}^1} = \{x^2 + y^2, z\} = 0.$$

Now, recalling that $\mathfrak{so}(4, \mathbb{R}) \simeq \mathfrak{so}(3, \mathbb{R}) \oplus \mathfrak{so}(3, \mathbb{R})$, the linear Poisson bracket on $\mathfrak{so}(4, \mathbb{R})^* \simeq \mathbb{R}^3 \times \mathbb{R}^3$ is given in coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ by

$$\{x_i, y_i\} = z_i, \quad \{y_i, z_i\} = x_i, \quad \{z_i, x_i\} = y_i,$$

where the other structure functions are zero. We use the diagonal $\mathbb{S}^1$-action on $\mathbb{R}^3 \times \mathbb{R}^3$. This is still Hamiltonian with moment map:

$$\mu : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \quad \mu(x_1, y_1, z_1, x_2, y_2, z_2) = z_1 + z_2.$$

The action is proper and free on the open set $M = (\mathbb{R}^3 \backslash Oz) \times \mathbb{R}^3$. We leave as an exercise to determine the quotient Poisson bracket on $M/\mathbb{S}^1$ (it is nonzero!).

Problems

1.1. Recall that a first integral of a vector field $V \in \mathfrak{X}(M)$ is any function $f \in C^\infty(M)$ which is constant on the integral curves of V. Given a Poisson manifold $(M, \{\cdot,\cdot\})$ and a function $H \in C^\infty(M)$, show that if f and g are first integrals of X_H, then $\{f, g\}$ is a first integral of X_H.

1.2. Let $(M_1, \{\cdot,\cdot\}_1)$ and $(M_2, \{\cdot,\cdot\}_2)$ be Poisson manifolds. Show that on the product $M_1 \times M_2$ the following formula defines a Poisson bracket:

$$\{f, g\}(x_1, x_2) := \{f(\cdot, x_2), g(\cdot, x_2)\}_1(x_1) + \{f(x_1, \cdot), g(x_1, \cdot)\}_2(x_2).$$

Show that this is the unique Poisson bracket on the product for which the projections $p_i : M_1 \times M_2 \to M_i$ are Poisson maps and

$$\{p_1^*(f), p_2^*(g)\} = 0, \quad \forall\, f \in C^\infty(M_1), g \in C^\infty(M_2).$$

1.3. Consider on $G = \mathbb{R}_+ \times \mathbb{R}$ the Poisson bracket of LV-type:

$$\{x, y\} = xy.$$

Consider also the group operation $m : G \times G \to G$:

$$m((x_1, y_1), (x_2, y_2)) := (x_1 x_2, y_1 + x_1 y_2).$$

Show that $m : G \times G \to G$ is a Poisson map, where we use the product Poisson structure on the domain.
NOTE: A pair $(G, \{\cdot,\cdot\}_G)$ where G is a Lie group and $\{\cdot,\cdot\}_G$ is a Poisson bracket for which multiplication $m : (G \times G, \{\cdot,\cdot\}_{G\times G}) \to (G, \{\cdot,\cdot\}_G)$ is called a **Poisson-Lie group**.

1.4. In the standard coordinates $(z_0 = x_0 + iy_0, z_1 = x_1 + iy_1, \ldots, z_n = x_n + iy_n)$ on $\mathbb{C}^{n+1}$, consider the bracket defined by

$$\{f, g\} := i \sum_{j=0}^{n} z_j \overline{z}_j \left(\frac{\partial f}{\partial z_j} \frac{\partial g}{\partial \overline{z}_j} - \frac{\partial g}{\partial z_j} \frac{\partial f}{\partial \overline{z}_j} \right). \tag{1.26}$$

(a) Verify that this formula defines a (real) Poisson bracket on $\mathbb{C}^{n+1}$.

(b) Show that the action of $\mathbb{C}^*$ on $\mathbb{C}^{n+1}\backslash\{0\}$ is by Poisson diffeomorphism and hence there is a quotient Poisson bracket on $\mathbb{CP}^n$.

(c) Determine the Hamiltonian vector field X_{H_j} on $\mathbb{CP}^n$ of the function

$$H_j([z_0 : \cdots : z_n]) = \frac{|z_j|^2}{|z_0|^2 + \cdots + |z_n|^2}.$$

1.5. Show that for any quadratic Poisson bracket on $\mathbb{R}^n$ the Euler vector field

$$E = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$$

is a Poisson vector field which is not Hamiltonian.

1.6. Show that on a 2-dimensional manifold M any skew-symmetric, bilinear bracket $\{\cdot,\cdot\}$ on $C^\infty(M)$ satisfying the Leibniz identity also satisfies the Jacobi identity.

1.7. Let $\mathscr{A} : G \times M \to M$ be an action of a Lie group G on a manifold M and denote by $a : \mathfrak{g} \to \mathfrak{X}(M)$ the corresponding infinitesimal action. If $(M, \{\cdot,\cdot\})$ is a Poisson manifold and the G-action is by Poisson diffeomorphisms, verify that $a(v)$ is a Poisson vector field, for all $v \in \mathfrak{g}$. Show that the converse holds provided G is a connected Lie group.

1.8. Let $C : M \to \mathbb{R}$ be a Casimir function of a Poisson manifold $(M, \{\cdot,\cdot\})$. If 0 is a regular value of C, show that $C^{-1}(0)$ has a unique Poisson bracket for which the inclusion $i : C^{-1}(0) \hookrightarrow M$ is a Poisson map.

1.9. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Consider the linear Poisson bracket on $\mathfrak{g}^*$. Show that the coadjoint action $G \times \mathfrak{g}^* \to \mathfrak{g}^*$ is Hamiltonian with moment map $\mu = \mathrm{Id}_{\mathfrak{g}^*} : \mathfrak{g}^* \to \mathfrak{g}^*$.

1.10. Identify the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})$ with $(\mathbb{R}^3)^*$ by identifying a traceless 2×2 real matrix

$$\begin{pmatrix} a & b-c \\ b+c & -a \end{pmatrix}$$

with the linear functional $(x, y, z) \mapsto ax + by + cz$.

(a) Show that under this identification the Poisson bracket on $\mathfrak{sl}(2,\mathbb{R})^*$ becomes the following Poisson bracket on $\mathbb{R}^3$:

$$\{f, g\}(\mathbf{x}) = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \\ 2x & 2y & -2z \end{vmatrix}. \tag{1.27}$$

(b) Verify that $C(x, y, z) = x^2 + y^2 - z^2$ is a Casimir function for this Poisson bracket.

(c) Find the orbits of this Poisson bracket.

(d) Show that this Poisson structure is Poisson diffeomorphic to (1.10).

1.11. Let G be a Lie group and consider the symplectic manifold $(T^*G, \omega_{\text{can}})$. Let G act on itself (on the left) by right translations:

$$\mathscr{A} : G \times G \to G, \ \mathscr{A}_g(h) \mapsto hg^{-1},$$

and consider the lifted symplectic action $G \times T^*G \to T^*G$. Show that the resulting Poisson quotient T^*G/G is isomorphic to $\mathfrak{g}^*$ with the linear Poisson bracket.

Chapter 3

Local Structure of Poisson Manifolds

Poisson structures can exhibit a very rich and interesting geometry, even locally. In this chapter we will discuss some classical aspects of the local structure of Poisson manifolds. The main result of the chapter, the Weinstein Splitting Theorem, states that a Poisson manifold is locally the product of a symplectic manifold and of a Poisson manifold with a zero. This yields a very simple local structure for regular Poisson manifolds. It follows that, to understand a general Poisson manifold locally, it suffices to look around zeros. At such a point there is a canonical first-order approximation of the Poisson bivector — the linear Poisson structure corresponding to the *isotropy Lie algebra*. The linearization problem asks whether a Poisson structure is locally isomorphic around a zero to its first-order approximation. In the end of this chapter we will discuss Conn's Linearization Theorem, a deep, difficult, and beautiful result in Poisson geometry.

3.1. The Weinstein Splitting Theorem

Definition 3.1. For a bivector field $\pi \in \mathfrak{X}^2(M)$ the dimension of the image of $\pi_x^\sharp : T_x^*M \to T_xM$ is called the **rank** of π at $x \in M$.

For a Poisson bivector the rank is the dimension of the Hamiltonian directions (2.10). By skew-symmetry, it is an even integer. Moreover, the rank cannot drop locally: every $x \in M$ has a neighborhood U such that

$$\text{rank } \pi_x \leq \text{rank } \pi_y, \quad \forall\, y \in U.$$

Recall that π is called a nondegenerate bivector field if $\pi^\sharp : T^*M \to TM$ is a vector bundle isomorphism or, equivalently, if rank $\pi_x = \dim M$, for all $x \in M$. As discussed in the previous chapter, in this case $\omega = \pi^{-1}$ is a symplectic structure. So by Darboux's Theorem, Theorem B.7, π can be put locally in the canonical form

$$\pi = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} \quad (2s = \dim M).$$

The following important result generalizes Darboux's Theorem and is the Poisson geometric analogue of Frobenius's Theorem:

Theorem 3.2 (Weinstein Splitting Theorem)**.** *Let (M, π) be a Poisson manifold, and let $x \in M$. There exist coordinates $(U, p_1, \ldots, p_s, q^1, \ldots, q^s, y^1, \ldots, y^q)$ centered at x such that*

$$\pi|_U = \sum_{i=1}^{s} \frac{\partial}{\partial p_i} \wedge \frac{\partial}{\partial q^i} + \sum_{1\leq a<b\leq q} \theta^{ab}(y) \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b} \qquad (2s = \text{rank } \pi_x),$$

where the $\theta^{ab}(y)$ are smooth functions of $(y^1, \ldots, y^q)$ such that $\theta^{ab}(0) = 0$.

Note that, by shrinking U, the chart $\chi = (p_1, \ldots, p_s, q^1, \ldots, q^s, y^1, \ldots, y^q)$ can be chosen such that $\chi(U) = V \times W$, where $V \subset \mathbb{R}^{2s}$ and $W \subset \mathbb{R}^q$ are open neighborhoods of 0. These **Weinstein splitting charts** give local Poisson isomorphisms with a product

$$\chi : (U, \pi|_U) \xrightarrow{\sim} (V, \pi_{\text{can}}) \times (W, \theta), \tag{3.1}$$

where π_{can} is the canonical Poisson structure on $\mathbb{R}^{2s}$ and

$$\theta = \sum_{a<b} \theta^{ab}(y) \frac{\partial}{\partial y^a} \wedge \frac{\partial}{\partial y^b} \in \mathfrak{X}^2(W)$$

is a Poisson structure that vanishes at the point $y = 0$. Such charts should be seen as the Poisson analogue of the

- *Darboux charts* for symplectic structures,
- charts resulting from the Frobenius Theorem, which we call *foliated charts*.

See the discussion concerning Theorems B.7 and C.3. Accordingly, the Weinstein Splitting Theorem is the Poisson analogue of the theorems of Darboux and Frobenius.

In the next chapter we will use the splitting charts to define the smooth structure on the orbits of a Poisson manifold, in a similar way as one uses Frobenius's Theorem to describe the leaves of regular foliations. Namely, note that, for a splitting chart (3.1), the submanifold

$$V_0 := \chi^{-1}(V \times \{0\}) = \{y^1 = 0, \ldots, y^q = 0\}$$

plays the role of the *plaque* through $x \in M$ for Im $\pi^\sharp$. In other words, since $\theta(0) = 0$, the Hamiltonian directions are given by

$$\operatorname{Im}\, \pi^\sharp|_{V_0} = \operatorname{Span}\Big\{\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_s}, \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^s}\Big\}.$$

In general, for $y \neq 0$, the submanifold $V_y := \chi^{-1}(V \times \{y\})$ is not necessarily a plaque of Im $\pi^\sharp$, as only one inclusion holds:

$$\operatorname{Im}\, \pi^\sharp|_{V_y} \supset \operatorname{Span}\Big\{\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_s}, \frac{\partial}{\partial q^1}, \dots, \frac{\partial}{\partial q^s}\Big\}.$$

For the proof of the Weinstein Splitting Theorem, we need the following standard lemma, which is a consequence of the flow box theorem (the case $k = 1$) and the fact that flows of commuting vector fields also commute for small times.

Lemma 3.3. *Let $V_1, \dots, V_k$ be vector fields defined on a neighborhood of $x \in M$, which are linearly independent at x and pairwise commute:*

$$[V_i, V_j] = 0, \quad 1 \le i, j \le k.$$

Then there exists a chart centered at x, $(U, x^1, \dots, x^m)$, such that

$$V_i|_U = \frac{\partial}{\partial x^i}, \quad 1 \le i \le k. \tag{3.2}$$

Proof of Theorem 3.2. We will prove the statement by induction on the rank π_x. If rank $\pi_x = 0$, there is nothing to prove. Assume that rank $\pi_x > 0$ and that the result holds for any Poisson structure with rank smaller than rank π_x.

Since $\pi_x \neq 0$ there exists a function p defined around x such that $p(x) = 0$ and $X_p|_x = \pi_x^\sharp(\mathrm{d}_x p) \neq 0$. By Lemma 3.3 we can find a coordinate chart $(U, x^1, \dots, x^m)$ centered at x in which $X_p|_U = \frac{\partial}{\partial x^1}$. Set $q := x^1$. Observe that the relations

$$X_p(p) = 0, \quad X_q(q) = 0, \quad X_p(q) = \{p, q\} = 1, \quad X_q(p) = \{q, p\} = -1$$

imply that X_p and X_q are linearly independent. These vector fields commute:

$$[X_p, X_q] = X_{\{p,q\}} = X_1 = 0.$$

Lemma 3.3 gives a new chart centered at x, still denoted $(U, x^1, \dots, x^m)$, such that

$$X_q|_U = \frac{\partial}{\partial x^1}, \quad X_p|_U = \frac{\partial}{\partial x^2}. \tag{3.3}$$

After possibly shrinking U, $(U, q, p, x^3, \dots, x^m)$ is also a coordinate system centered at x. This follows because the differentials of these functions are independent:

$$\begin{aligned}\mathrm{d}q \wedge \mathrm{d}p \wedge \mathrm{d}x^3 \wedge \cdots \wedge \mathrm{d}x^m &= \Big(\frac{\partial q}{\partial x^1}\frac{\partial p}{\partial x^2} - \frac{\partial q}{\partial x^2}\frac{\partial p}{\partial x^1}\Big)\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m \\ &= \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^m,\end{aligned}$$

where in the last equality we have used the relations

$$\begin{aligned}\frac{\partial p}{\partial x^1} &= X_q(p) = -1, & \frac{\partial p}{\partial x^2} &= X_p(p) = 0, \\ \frac{\partial q}{\partial x^1} &= X_q(q) = 0, & \frac{\partial q}{\partial x^2} &= X_p(q) = 1.\end{aligned}$$

By (3.3), the new coordinates $(U, q, p, x^3, \dots, x^m)$ satisfy

$$\{p, q\} = 1, \quad \{p, x^i\} = 0, \quad \{q, x^i\} = 0 \quad (3 \leq i \leq m).$$

Therefore, π takes the form

$$\pi|_U = \frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} + \sum_{3 \leq a < b \leq m} \theta^{ab}(q, p, x^3, \dots, x^m)\frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}.$$

Hence, in these coordinates $X_p|_U = \frac{\partial}{\partial q}$ and $X_q|_U = -\frac{\partial}{\partial p}$. On the other hand, the Jacobi identity gives

$$\frac{\partial}{\partial q}(\theta^{ab}) = X_p(\theta^{ab}) = \{p, \{x^a, x^b\}\} = \{\{p, x^a\}, x^b\} + \{x^a, \{p, x^b\}\} = 0,$$

and similarly $\frac{\partial}{\partial p}(\theta^{ab}) = 0$. So, after possibly shrinking U again, we may assume that the functions θ^{ab} do not depend on the variables p and q. The Jacobi identity for the variables $x^3, \dots, x^m$ shows that

$$\theta = \sum_{3 \leq a < b \leq m} \theta^{ab}(x^3, \dots, x^m)\frac{\partial}{\partial x^a} \wedge \frac{\partial}{\partial x^b}$$

is a Poisson structure defined around 0 in $\mathbb{R}^{m-2}$. Note that

$$\operatorname{rank}\, \theta_0 = \operatorname{rank}\, \pi_x - 2.$$

So the theorem follows by applying the induction hypothesis to θ. □

Chapter 11

Contravariant Geometry and Connections

11.1. Contravariant connections on vector bundles

Following once again the point of view that in Poisson geometry the correct tangent bundle is the cotangent Lie algebroid, one is led to the following notion of connection:

Definition 11.1. Let (M, π) be a Poisson manifold, and let $E \to M$ be a vector bundle. A **contravariant connection** on E is an $\mathbb{R}$-bilinear operation

$$\Omega^1(M) \times \Gamma(E) \to \Gamma(E), \quad (\alpha, s) \mapsto \nabla_\alpha s,$$

satisfying the following properties:

$$\nabla_{f\alpha} s = f \nabla_\alpha s, \quad \nabla_\alpha(fs) = f\nabla_\alpha s + \mathscr{L}_{\pi^\sharp\alpha}(f)s.$$

We will call a pair (E, ∇) a **contravariant vector bundle**.

Given a local chart $(U, x^1, \dots, x^n)$ where E admits a basis of sections $\{e^1, \dots, e^r\}$, a contravariant connection is determined locally by some functions $\Gamma^{il}_k \in C^\infty(U)$, called the **Christoffel symbols**

$$\nabla_{\mathrm{d}x^i} e^l = \sum_{k=1}^{r} \Gamma^{il}_k e^k \qquad (1 \le i \le n,\ 1 \le l \le r). \tag{11.1}$$

Exercise 11.2. For a contravariant connection ∇ on E, given a section $s \in \Gamma(E)$, show that $\nabla_\alpha s|_x$ depends only on $\alpha|_x \in T^*_x M$. Deduce that any $\xi \in T^*M$ defines a map $\nabla_\xi : \Gamma(E) \to \mathbb{R}$.

Many of the usual constructions for ordinary connections extend to contravariant connections in a more or less straightforward way. For example, the **curvature** of a contravariant connection ∇ on E is the $\mathrm{End}(E)$-valued bivector field given by

$$R_\nabla \in \mathfrak{X}^2(M; \mathrm{End}(E)) := \Gamma\Big(\bigwedge^2 TM \otimes \mathrm{End}(E)\Big),$$

$$R_\nabla(\alpha, \beta)s := \nabla_\alpha(\nabla_\beta s) - \nabla_\beta(\nabla_\alpha s) - \nabla_{[\alpha,\beta]_\pi} s,$$

for all $\alpha, \beta \in \Omega^1(M)$ and $s \in \Gamma(E)$. The connection is said to be **flat** if its curvature vanishes identically.

Exercise 11.3. Show that the formula above for the curvature defines indeed a section of the vector bundle $\bigwedge^2 TM \otimes \mathrm{End}(E)$.

Example 11.4. Let $E \to M$ be a vector bundle with an ordinary connection $\overline{\nabla}$. If the base is a Poisson manifold (M, π), we can produce a contravariant connection ∇ on E by setting

$$\nabla_\alpha s = \overline{\nabla}_{\pi^\sharp(\alpha)} s.$$

The curvature tensors $R_{\overline{\nabla}}$ and R_∇ of $\overline{\nabla}$ and ∇ are related by

$$R_\nabla(\alpha, \beta) = R_{\overline{\nabla}}(\pi^\sharp(\alpha), \pi^\sharp(\beta)).$$

Although this gives a quick way of producing contravariant connections, these connections do not play a significant role in Poisson geometry.

Example 11.5 (Pullback connections)**.** Let $\Phi : T\Sigma \to T^*M$ be a cotangent map into a Poisson manifold (M, π), covering a map $\phi : \Sigma \to M$. Given a vector bundle $p : E \to M$ with a contravariant connection ∇, the pullback vector bundle

$$\phi^* E = \big\{(u, x) \,:\, p(u) = \phi(x)\big\}$$

has an induced ordinary connection $\overline{\nabla} = \Phi^* \nabla$, uniquely determined by the condition

$$\overline{\nabla}_v(\phi^* s) = \nabla_{\Phi(v)} s \quad (v \in T_x\Sigma,\ s \in \Gamma(E)).$$

That this is well-defined uses only the first property of contravariant maps: $\pi^\sharp \circ \Phi = \mathrm{d}\phi$, Lemma 10.6. The full condition implies that $R_{\overline{\nabla}} = \Phi^* R_\nabla$; i.e.,

$$R_{\overline{\nabla}}(v, w) = R_\nabla(\Phi(v), \Phi(w)) \quad (v, w \in T_x\Sigma).$$

In particular, one can pull back connections along cotangent paths.

Example 11.6. Let (M, π) be a regular Poisson structure and consider the conormal bundle to the symplectic foliation

$$\nu^*(\mathcal{F}_\pi) := (T\mathcal{F}_\pi)^\circ = \operatorname{Ker} \pi^\sharp.$$

We have a canonical contravariant connection

$$\nabla_\alpha \beta := [\alpha, \beta]_\pi, \quad \alpha \in \Omega^1(M),\ \beta \in \Gamma(\nu^*(\mathcal{F}_\pi)),$$

called the **contravariant Bott connection**. The Jacobi identity implies flatness: $R_\nabla \equiv 0$. Recall that the usual Bott connection $\overline{\nabla}$ on $\nu^*(\mathcal{F}_\pi)$ is the partial connection

$$\overline{\nabla}_X \beta = \mathscr{L}_X \beta, \quad X \in \mathfrak{X}(\mathcal{F}_\pi),\ \beta \in \Gamma(\nu^*(\mathcal{F}_\pi)).$$

Using the expression for $[\cdot, \cdot]_\pi$ one sees that the two are related by

$$\nabla_\alpha \beta = \overline{\nabla}_{\pi^\sharp(\alpha)} \beta.$$

Example 11.7. Let (M, π) be a Poisson manifold. The line bundle $\bigwedge^{\text{top}} T^*M$ carries a canonical contravariant connection ∇. This is defined on exact 1-forms by

$$\nabla_{\mathrm{d}f} \mu := \mathscr{L}_{X_f} \mu, \quad f \in C^\infty(M),\ \mu \in \Omega^{\text{top}}(M),$$

and it is extended to all 1-forms by requiring $C^\infty(M)$-linearity. Using that $[\mathrm{d}f_1, \mathrm{d}f_2]_\pi = \mathrm{d}\{f_1, f_2\}$, we see that this connection is flat:

$$R_\nabla(\mathrm{d}f_1, \mathrm{d}f_2)\mu = \mathscr{L}_{X_{f_1}}(\mathscr{L}_{X_{f_2}}\mu) - \mathscr{L}_{X_{f_2}}(\mathscr{L}_{X_{f_1}}\mu) - \mathscr{L}_{X_{\{f_1, f_2\}}}\mu = 0.$$

Let $L \to M$ be a line bundle over a Poisson manifold (M, π) equipped with a flat contravariant connection ∇. Assume first that L has a nowhere vanishing section μ, so L is actually trivial. Then

$$\nabla_\alpha \mu = c_\mu(\alpha)\mu, \tag{11.2}$$

for some C^∞-linear map $c_\mu : \Omega^1(M) \to C^\infty(M)$, i.e., a vector field $c_\mu \in \mathfrak{X}(M)$. Since we assume the connection to be flat, we find that

$$\begin{aligned}
0 &= \nabla_\alpha \nabla_\beta \mu - \nabla_\beta \nabla_\alpha \mu - \nabla_{[\alpha, \beta]} \mu \\
&= \nabla_\alpha (c_\mu(\beta)\, \mu) - \nabla_\beta (c_\mu(\alpha)\, \mu) - c_\mu([\alpha, \beta])\mu \\
&= (\mathscr{L}_{\pi^\sharp(\alpha)} c_\mu(\beta))\, \mu - (\mathscr{L}_{\pi^\sharp(\beta)} c_\mu(\alpha))\, \mu - c_\mu([\alpha, \beta])\mu \\
&= (\mathrm{d}_\pi c_\mu)(\alpha, \beta)\, \mu.
\end{aligned}$$

We conclude that c_μ is a Poisson vector field: $\mathrm{d}_\pi c_\mu = 0$.

Exercise 11.8. If $\mu' = \pm e^g \mu$ is another nowhere vanishing section, show that

$$c_{\mu'} = c_\mu - X_g;$$

i.e., $c_{\mu'}$ and c_μ differ by a Hamiltonian vector field.

It follows that the Poisson cohomology class

$$c(L, \nabla) := [c_\mu] \in H^1_\pi(M)$$

does not depend on the choice of μ.

When L is not trivializable, we can still define $c(L, \nabla)$ as follows. Form the tensor product $L^{\otimes 2} = L \otimes L$ and equip it with the flat connection

$$\widetilde{\nabla}_\alpha(\mu \otimes \xi) := \nabla_\alpha \mu \otimes \xi + \mu \otimes \nabla_\alpha \xi.$$

Since $L^{\otimes 2}$ has a nowhere vanishing section, we can define

$$c(L, \nabla) := \tfrac{1}{2}[c(L^{\otimes 2}, \widetilde{\nabla})] \in H^1_\pi(M).$$

Exercise 11.9. If L is trivializable, the two definitions of $c(L, \nabla)$ agree.

Definition 11.10. Let (M, π) be a Poisson manifold, and let $L \to M$ be a line bundle with a flat contravariant connection ∇. The Poisson cohomology class $c(L, \nabla) \in H^1_\pi(M)$ is called the **characteristic class** of (L, ∇).

Example 11.11 (Modular class)**.** We saw in Example 11.7 that, for any Poisson manifold (M, π), the line bundle $\bigwedge^{\text{top}} T^*M$ carries a canonical flat connection. Hence, we have a characteristic class

$$c\Big(\bigwedge^{\text{top}} T^*M, \nabla\Big) \in H^1_\pi(M).$$

Exercise 11.12. If M is orientable, show that this class coincides with the modular class $c(\bigwedge^{\text{top}} T^*M, \nabla) = \operatorname{mod}(M, \pi)$.

This exercise allows us to define the **modular class** for any Poisson manifold (M, π), orientable or not, as

$$\operatorname{mod}(M, \pi) := c\Big(\bigwedge^{\text{top}} T^*M, \nabla\Big).$$

11.2. Parallel transport along cotangent paths

Connections are useful for connecting and comparing fibers over different base points. This is done using parallel transport along paths. As one may expect, the generalization to contravariant connections requires the use of cotangent paths.

For this, fix a Poisson manifold (M, π) and a vector bundle $p : E \to M$ with a contravariant connection ∇. Consider a cotangent path $a : [0, 1] \to T^*M$ and a path $c : [0, 1] \to E$ "above" a, i.e., such that

$$p(c(t)) = \gamma_a(t), \quad \forall\, t \in [0, 1].$$

One can find a time-dependent section $s_t \in \Gamma(E)$ extending c, i.e., such that

$$s_t(\gamma_a(t)) = c(t), \quad \forall\, t \in [0,1].$$

This allows one to define

$$(D_a c)(t) = \nabla_{a(t)} s_t + \frac{\mathrm{d}}{\mathrm{d}t} s_t\Big|_{\gamma_a(t)}. \tag{11.3}$$

Exercise 11.13. Check that expression (11.3) is independent of the choice of time-dependent extension s_t. Would $D_a c$ still be well-defined if a was not a cotangent path?
Hint: No!

Definition 11.14. One calls $D_a c$ the **contravariant derivative** of c along the cotangent path a.

The local expression of the contravariant derivative is

$$D_a c(t) = \sum_{k=1}^{r} \Big(\frac{\mathrm{d}}{\mathrm{d}t} c_k(t) + \sum_{\substack{1 \le i \le n \\ 1 \le l \le r}} \Gamma_k^{il}(\gamma_a(t))\, a_i(t)\, c_l(t) \Big) e^k, \tag{11.4}$$

where we use the Christoffel symbols (11.1), and we write

$$a(t) = \sum_{i=1}^{n} a_i(t) \mathrm{d}x^i, \quad c(t) = \sum_{l=1}^{r} c_l(t) e^l.$$

This follows by using the time-dependent section $s_t(x) := c(t)$ in (11.3).

One verifies immediately that the contravariant derivative D satisfies the following:

(i) Linearity: If $c_1, c_2 : [0,1] \to E$ are paths above a and $\lambda_1, \lambda_2 \in \mathbb{R}$, then

$$D_a(\lambda_1 c_1 + \lambda_2 c_2) = \lambda_1 D_a c_1 + \lambda_2 D_a c_2.$$

(ii) Leibniz: If $c : [0,1] \to E$ is a path above a and $f \in C^\infty(M)$, then

$$D_a((f \circ \gamma_a)\, c) = (f \circ \gamma_a)\, D_a c + \Big(\frac{\mathrm{d}}{\mathrm{d}t} f \circ \gamma_a \Big)\, c.$$

These properties reveal an alternative description of the contravariant derivative:

Exercise 11.15. Let $\overline{\nabla} = a^* \nabla$ be the pullback connection along the cotangent path $a : [0,1] \to T^*M$, as in Example 11.5. Show that

$$D_a c = \overline{\nabla}_{\frac{\mathrm{d}}{\mathrm{d}t}} c,$$

where we identify curves $c : [0,1] \to E$ above γ_a with sections $c \in \Gamma(\gamma_a^* E)$.

The contravariant derivative gives more geometric insight into the notion of curvature. For this, consider a cotangent map defined on the square:

$$\Phi : T([0,1]\times[0,1]) \to T^*M,$$
$$\Phi(t,\varepsilon) = \Phi_1(t,\varepsilon)\,\mathrm{d}t + \Phi_2(t,\varepsilon)\,\mathrm{d}\varepsilon,$$

with base map $\gamma : [0,1]\times[0,1] \to M$. Consider now any smooth map $c : [0,1]\times[0,1] \to E$ above γ. Then, as explained in Section 10.4:

- For fixed a $\varepsilon = \varepsilon_0$, the map $t \mapsto \Phi_1(t,\varepsilon_0)$ is a cotangent path covering the path $t \mapsto \gamma(t,\varepsilon_0)$. Therefore we have the contravariant derivative of c in the t-direction, resulting in a new map above γ:

$$D_{\Phi_1}c : [0,1]\times[0,1] \to E,$$
$$(D_{\Phi_1}c)(t,\varepsilon_0) := \big(D_{\Phi_1(\cdot,\varepsilon_0)}c(\cdot,\varepsilon_0)\big)(t).$$

- Similarly, by freezing t, we obtain the contravariant derivative of c in the ε-direction, which defines a new map above γ:

$$D_{\Phi_2}c : [0,1]\times[0,1] \to E,$$
$$(D_{\Phi_2}c)(t_0,\varepsilon) := \big(D_{\Phi_2(t_0,\cdot)}c(t_0,\cdot)\big)(\varepsilon).$$

The curvature has the following interpretation:

Proposition 11.16. *Given a contravariant vector bundle (E,∇) over a Poisson manifold (M,π), one has*

$$R_\nabla(\Phi_1,\Phi_2)c = D_{\Phi_1}D_{\Phi_2}c - D_{\Phi_2}D_{\Phi_1}c,$$

for any cotangent map $\Phi : T([0,1]\times[0,1]) \to T^*M$, covering a base map $\gamma : [0,1]\times[0,1] \to M$, and any map $c : [0,1]\times[0,1] \to E$ above γ.

Proof. This result can be proven by pulling back the connection via the map Φ to a classical connection on γ^*E and then using that the curvatures are related via Φ — see Example 11.5. The statement is then reduced to the similar statement for classical connections, which can be found for example in [**137**]. We also give a self-contained proof.

Choose a (t,ε)-dependent section $s_{t,\varepsilon} \in \Gamma(E)$ extending $c(t,\varepsilon)$,

$$s_{t,\varepsilon}(\gamma(t,\varepsilon)) = c(t,\varepsilon),$$

and choose (t,ε)-dependent 1-forms $\alpha_{t,\varepsilon}, \beta_{t,\varepsilon} \in \Omega^1(M)$ extending Φ_1 and Φ_2,

$$\alpha_{t,\varepsilon}(\gamma(t,\varepsilon)) = \Phi_1(t,\varepsilon), \quad \beta_{t,\varepsilon}(\gamma(t,\varepsilon)) = \Phi_2(t,\varepsilon).$$

According to the definition of contravariant derivative (11.3), we have

$$D_{\Phi_1}c(t,\varepsilon) = \Big(\nabla_{\alpha_{t,\varepsilon}} s_{t,\varepsilon} + \frac{\mathrm{d}}{\mathrm{d}t} s_{t,\varepsilon}\Big)\Big|_{\gamma(t,\varepsilon)},$$
$$D_{\Phi_2}c(t,\varepsilon) = \Big(\nabla_{\beta_{t,\varepsilon}} s_{t,\varepsilon} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon} s_{t,\varepsilon}\Big)\Big|_{\gamma(t,\varepsilon)}.$$

It follows that

$$D_{\Phi_1}D_{\Phi_2}c(t,\varepsilon) = \Big(\nabla_{\alpha_{t,\varepsilon}}\nabla_{\beta_{t,\varepsilon}} s_{t,\varepsilon} + \frac{\mathrm{d}}{\mathrm{d}t}\nabla_{\beta_{t,\varepsilon}} s_{t,\varepsilon} + \nabla_{\alpha_{t,\varepsilon}}\frac{\mathrm{d}s_{t,\varepsilon}}{\mathrm{d}\varepsilon} + \frac{\mathrm{d}^2 s_{t,\varepsilon}}{\mathrm{d}t\mathrm{d}\varepsilon}\Big)\Big|_{\gamma(t,\varepsilon)},$$
$$D_{\Phi_2}D_{\Phi_1}c(t,\varepsilon) = \Big(\nabla_{\beta_{t,\varepsilon}}\nabla_{\alpha_{t,\varepsilon}} s_{t,\varepsilon} + \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\nabla_{\alpha_{t,\varepsilon}} s_{t,\varepsilon} + \nabla_{\beta_{t,\varepsilon}}\frac{\mathrm{d}s_{t,\varepsilon}}{\mathrm{d}t} + \frac{\mathrm{d}^2 s_{t,\varepsilon}}{\mathrm{d}\varepsilon\mathrm{d}t}\Big)\Big|_{\gamma(t,\varepsilon)}.$$

Taking the difference of these two equations, we obtain

$$\begin{aligned} &D_{\Phi_1}D_{\Phi_2}c(t,\varepsilon) - D_{\Phi_2}D_{\Phi_1}c(t,\varepsilon) \\ &\qquad = \Big(\nabla_{\alpha_{t,\varepsilon}}\nabla_{\beta_{t,\varepsilon}} s_{t,\varepsilon} - \nabla_{\beta_{t,\varepsilon}}\nabla_{\alpha_{t,\varepsilon}} s_{t,\varepsilon} + \nabla_{\frac{\mathrm{d}}{\mathrm{d}t}\beta_{t,\varepsilon} - \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\alpha_{t,\varepsilon}} s_{t,\varepsilon}\Big)\Big|_{\gamma(t,\varepsilon)}. \end{aligned}$$

Using property (iii) from Proposition 10.17, we obtain the result

$$\begin{aligned} &D_{\Phi_1}D_{\Phi_2}c(t,\varepsilon) - D_{\Phi_2}D_{\Phi_1}c(t,\varepsilon) \\ &\qquad = (R_\nabla(\alpha_{t,\varepsilon},\beta_{t,\varepsilon})s_{t,\varepsilon})\circ\gamma(t,\varepsilon) = R_\nabla(\Phi_1,\Phi_2)c(t,\varepsilon). \qquad \square \end{aligned}$$

Let us now turn to parallelism and parallel transport.

Definition 11.17. Let (E,∇) be a contravariant vector bundle over (M,π). We say that $c:[0,1]\to E$ is a **parallel curve** along a cotangent path $a:[0,1]\to T^*M$ if c lies above a and

$$D_a c = 0.$$

Proposition 11.18. *Let (E,∇) be a contravariant vector bundle over (M,π). Given a cotangent path $a:[0,1]\to T^*M$ and a point $u\in E_{\gamma_a(0)}$ there is a unique parallel curve $c_u:[0,1]\to E$ along a, starting at u. Moreover, the end point $c_u(1)$ of this curve depends linearly on u.*

Proof. This result can be proven by pulling back ∇ via a to a classical connection on $\gamma_a^*E\to[0,1]$, as in Exercise 11.15. This reduces the result to the existence of parallel transport of a classical connection over the interval. Here we give a self-contained argument.

Assume first that the base path γ_a belongs to the domain of a coordinate chart (U,x^i) where E admits a basis of sections $\{e_l\}$. By (11.4), a parallel curve $c(t)$ along $a(t)$ with initial condition u is a solution of the system of

ODEs

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} c_k(t) = -\sum_{\substack{1\le i\le n \\ 1\le l\le r}} \Gamma_k^{il}(\gamma_a(t))\, a_i(t)\, c_l(t) \quad (k = 1, \dots r), \\ c(0) = u. \end{cases}$$

Since $a(t)$ and $\gamma_a(t)$ are given, this is a linear system of ODEs with time-dependent coefficients. So, for any $u \in E_{\gamma_a(0)}$ a unique solution exists, which is defined as long as the coefficients are defined, i.e., on $[0, 1]$, and the solution depends linearly on u.

In general, consider a partition $0 = t_0 < t_1$ such that each segment $\gamma_a([t_p, t_{p+1}])$ is covered by a chart as above. By the first part, we find inductively parallel paths $c_p : [t_p, t_{p+1}] \to E$ over $a|_{[t_p,t_{p+1}]}$ satisfying the initial conditions

$$c_0(0) = u \quad \text{and} \quad c_p(t_p) = c_{p-1}(t_p) \quad (p \ge 1).$$

The path $c : [0, 1] \to E$ obtained by gluing the paths c_p is smooth at the points $t_1, \dots, t_{q-1}$. This holds by local uniqueness around these points. □

In conclusion, any cotangent path $a : [0, 1] \to T^*M$ yields a linear map

$$\tau_a : E_{\gamma_a(0)} \to E_{\gamma_a(1)}, \quad u \mapsto c_u(1).$$

The map τ_a is called the **parallel transport** of the contravariant connection ∇ along the cotangent path a. The uniqueness of parallel paths shows that τ_a is injective, and so it is a linear isomorphism between the fibers.

Example 11.19. Consider a linear Poisson manifold $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(W)$ be a representation of $\mathfrak{g}$. We view $\mathfrak{g} \subset \Omega^1(\mathfrak{g}^*)$ by interpreting elements in $\mathfrak{g}$ as constant 1-forms. Define a contravariant connection on the trivial bundle $\mathfrak{g}^* \times W \to \mathfrak{g}^*$ by requiring that on constant sections it satisfies

$$\nabla_v w := \rho(v) w.$$

The fact that ρ is a representation implies flatness of ∇.

Since the origin is a zero of $\pi_{\mathfrak{g}}$, any element $v \in T_0^*\mathfrak{g}^* \simeq \mathfrak{g}$ defines the constant cotangent path $a_v(t) = v$ and parallel transport gives a map

$$\tau_{a_v} : W \to W.$$

Exercise 11.20. Show that $\tau_{a_v} = \exp(\rho(v))$.

Chapter 12

Complete Symplectic Realizations

Complete symplectic realizations turn out to play a major role since they provide a bridge between Poisson manifolds and their symplectic groupoids.

Definition 12.1. A symplectic realization $\mu : (S, \omega) \to (M, \pi)$ is called **complete** if for any complete Hamiltonian vector field $X_H \in \mathfrak{X}(M)$ the Hamiltonian vector field $X_{H \circ \mu} \in \mathfrak{X}(S)$ is also complete.

It is not difficult to see that, for symplectic realizations, one has

$$S \text{ is compact} \quad \Longrightarrow \quad \mu \text{ is proper} \quad \Longrightarrow \quad \mu \text{ is complete.}$$

Note that the notion of **complete Poisson map** makes sense for maps between any two Poisson manifolds. This generalizes the notion of a complete Poisson submanifold. The implications above, and in fact many of the results of this chapter, can be adapted to general Poisson maps.

In this chapter, after discussing the infinitesimal action associated to any symplectic realization, we look at examples of complete symplectic realizations for several classes of Poisson manifolds. As we look deeper into these examples, we will slowly unveil the structure of the symplectic groupoid. Inspired by these examples, we then come back to general complete symplectic realizations and clarify the connection with the *Poisson homotopy groupoid*.

12.1. The infinitesimal action

Consider a symplectic realization

$$\mu : (S, \omega) \to (M, \pi).$$

By Libermann's Theorem, we have two foliations on S:

- the **vertical foliation** with tangent distribution $\operatorname{Ker} \mathrm{d}\mu$,
- the **orbit foliation** with tangent distribution $(\operatorname{Ker} \mathrm{d}\mu)^{\perp_\omega}$.

The name *orbit* foliation is due to the fact that it arises from an "action"

> **Definition 12.2.** The **infinitesimal action** associated with a symplectic realization $\mu : (S, \omega) \to (M, \pi)$ is the bundle map
>
> $$\not{a} : \mu^* T^* M \to TS$$
>
> defined by requiring
>
> $$i_{\not{a}(\alpha)} \omega = \mu^* \alpha, \quad \forall\, \alpha \in T^* M.$$

The infinitesimal action can be thought of as follows:

- pointwise, as a linear map $\not{a}_p : T^*_{\mu(p)} M \to T_p S$ for each $p \in S$,
- at the level of sections, as a map $\not{a} : \Omega^1(M) \to \mathfrak{X}(S)$.

There are several reasons for using the name infinitesimal action. A first reason is that at the level of sections $\not{a}$ is a Lie algebra map. This and the main properties of $\not{a}$ are listed in the following:

Proposition 12.3. *Let $\mu : (S, \omega) \to (M, \pi)$ be a symplectic realization. Then $\not{a} : \Omega^1(M) \to \mathfrak{X}(S)$ is a Lie algebra map*

$$\not{a}([\alpha, \beta]_\pi) = [\not{a}(\alpha), \not{a}(\beta)], \quad \forall\, \alpha, \beta \in \Omega^1(M).$$

Moreover, for each $p \in S$, the action $\not{a}_p$ has the following properties:

(i) *It lifts the map $\pi^\sharp$; i.e., the following diagram commutes:*

$$\begin{array}{ccc} & \overset{\not{a}_p}{\nearrow} & T_p S \\ & & \downarrow{\scriptstyle \mathrm{d}\mu} \\ T^*_{\mu(p)} M & \xrightarrow[\pi^\sharp]{} & T_{\mu(p)} M \end{array}$$

(ii) *It is pointwise free; i.e., $\not{a}_p : T^*_{\mu(p)} M \to T_p S$ is injective.*

(iii) *Its image is precisely the orbit foliation:* $\operatorname{Im}(\not{a}_p) = (\operatorname{Ker} \mathrm{d}_p \mu)^{\perp_\omega}$.

(iv) *Its restriction to the isotropy Lie algebra is a linear isomorphism*

$$\not{a}_p : \mathfrak{g}_{\mu(p)} \xrightarrow{\sim} (\operatorname{Ker} \mathrm{d}_p \mu) \cap (\operatorname{Ker} \mathrm{d}_p \mu)^{\perp_\omega}.$$

Proof. Item (i) follows because μ is a Poisson map, and so by the definition of ⱥ_p we have a commutative diagram

$$\begin{array}{ccc} T_p^*S & \xrightarrow{(\omega^\flat)^{-1}} & T_pS \\ {\scriptstyle (\mathrm{d}_p\mu)^*}\uparrow & \overset{\text{ⱥ}_p}{\nearrow} & \downarrow{\scriptstyle \mathrm{d}_p\mu} \\ T^*_{\mu(p)}M & \xrightarrow[\pi^\sharp]{} & T_{\mu(p)}M \end{array}$$

This also shows that ⱥ_p factors as the composition of an injective map and an isomorphism, so it is injective and (ii) follows. For (iii) observe that the image of ⱥ_p is given by

$$\mathrm{Im}(\text{ⱥ}_p) = (\omega^\flat)^{-1}((\mathrm{Ker}\,\mathrm{d}_p\mu)^\circ) = (\mathrm{Ker}\,\mathrm{d}_p\mu)^{\perp_\omega}.$$

Since $\mathfrak{g}_{\mu(p)} = \mathrm{Ker}\,\pi^\sharp_{\mu(p)}$, the diagram gives $\text{ⱥ}_p(\mathfrak{g}_{\mu(p)}) = \mathrm{Ker}\,\mathrm{d}_p\mu \cap \mathrm{Im}(\text{ⱥ}_p)$, and then (iv) follows from (iii).

To see the ⱥ is a Lie algebra homomorphism, one first notes that

$$\text{ⱥ}(f\alpha) = (f\circ\mu)\,\text{ⱥ}(\alpha), \quad \forall\, f\in C^\infty(M), \alpha\in\Omega^1(M).$$

Using this, the Leibniz identity, and (i), the difference $\text{ⱥ}([\alpha,\beta]_\pi) - [\text{ⱥ}(\alpha), \text{ⱥ}(\beta)]$ is $C^\infty(M)$-bilinear. So it is enough to check the identity on exact 1-forms. But for these one has $\text{ⱥ}(\mathrm{d}f) = X_{\mu^*(f)}$ and so, if $\alpha = \mathrm{d}f$ and $\beta = \mathrm{d}g$, then the equation becomes

$$X_{\mu^*(\{f,g\})} = [X_{\mu^*(f)}, X_{\mu^*(g)}],$$

which holds since μ is a Poisson map. □

We will see later how completeness of the symplectic realization can be seen as completeness of the infinitesimal action — similar to the case of Lie algebra actions recalled in Appendix A. An indication of this phenomenon is provided by the following:

Corollary 12.4. *Let $\mu : (S,\omega)\to(M,\pi)$ be a complete symplectic realization, and let $x\in M$. Then the corresponding infinitesimal action restricts to a complete action of the isotropy Lie algebra at x on the fiber of μ above x:*

$$\text{ⱥ} : \mathfrak{g}_x \to \mathfrak{X}(\mu^{-1}(x)).$$

In particular, it integrates to a group action of $\Pi(\mathfrak{g}_x)$ on $\mu^{-1}(x)$ where $\Pi(\mathfrak{g}_x)$ is the 1-connected Lie group with Lie algebra $\mathfrak{g}_x$.

Proof. Write $v\in\mathfrak{g}_x$ as d_xH with H compactly supported. Then $\text{ⱥ}|_{\mathfrak{g}_x} : \mathfrak{g}_x \to \mathfrak{X}(\mu^{-1}(x))$ sends v to $X_{H\circ\mu}|_{\mu^{-1}(x)}$, which is complete by assumption. The action integrates to one of the Lie group $\Pi(\mathfrak{g}_x)$ — see Proposition A.3. □

Property (iii) shows that the leaves of the orbit foliation can be thought of as the orbits of the infinitesimal action and will therefore be called **orbits**. The compatibility with the brackets immediately gives:

Corollary 12.5. *The fiberwise inverse of the infinitesimal action induces a Lie algebroid map*

$$\Psi : \mathrm{Im}(\mathscr{a}) = (\mathrm{Ker}\, \mathrm{d}\mu)^{\perp_\omega} \to T^*M, \quad \Psi(v) := \mathscr{a}^{-1}(v).$$

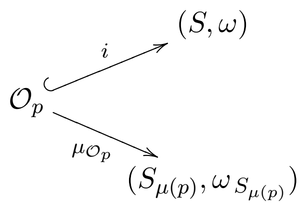
Hence, for any orbit $\mathcal{O}_p \subset S$ of the action, Ψ restricts to a cotangent map

$$\Psi_p : T\mathcal{O}_p \to T^*M. \tag{12.1}$$

We also deduce that the orbits of the infinitesimal action are related to the symplectic leaves as follows:

Proposition 12.6. *Let $\mu : (S, \omega) \to (M, \pi)$ be a symplectic realization. Let $\mathcal{O}_p$ be the orbit of the action through $p \in S$. Then μ maps $\mathcal{O}_p$ to the symplectic leaf $S_{\mu(p)}$ through $\mu(p) \in M$ and $\mu|_{\mathcal{O}_p} : \mathcal{O}_p \to S_{\mu(p)}$ is a submersion.*

Moreover, we have the following diagram:



$$i^*\omega = \mu_{\mathcal{O}_p}^* \omega_{S_{\mu(p)}}.$$

Proof. Corollary 12.5 implies the first part. The second part follows because the Poisson condition gives

$$\omega(\mathscr{a}(\alpha), \mathscr{a}(\beta)) = -\pi(\alpha, \beta) = \omega_{S_{\mu(p)}}(\pi^\sharp(\alpha), \pi^\sharp(\beta)). \qquad \square$$

12.2. Case study: Linear Poisson structures

We start by looking at general symplectic realizations of a linear Poisson structure $\pi_{\mathfrak{g}}$. It was mentioned already in Example 1.35 that moment maps for $\mathfrak{g}$-Hamiltonian spaces correspond to Poisson maps $(S, \omega) \to (\mathfrak{g}^*, \pi_{\mathfrak{g}})$. We will now complete that discussion.

Consider a symplectic realization

$$\mu : (S, \omega) \to (\mathfrak{g}^*, \pi_{\mathfrak{g}}).$$

The associated infinitesimal action $\mathscr{a} : \Omega^1(\mathfrak{g}^*) \to \mathfrak{X}(S)$ restricts to an infinitesimal $\mathfrak{g}$-action $\mathscr{a} : \mathfrak{g} \to \mathfrak{X}(S)$ by interpreting elements in $\mathfrak{g}$ as constant 1-forms. We recover the Lie algebra action from Example 1.35. Notice that the fact that this is a Lie algebra action follows from Proposition 12.3. On the other hand, the moment map condition amounts to the definition of the action $\mathscr{a} : \Omega^1(\mathfrak{g}^*) \to \mathfrak{X}(S)$ and we have the following:

Proposition 12.7. *Let $\mathfrak{g}$ be a Lie algebra. There is a 1-to-1 correspondence*

$$\left\{\begin{array}{c}\text{symplectic realizations}\\ \mu : (S,\omega) \to (U, \pi_{\mathfrak{g}}|_U)\\ \text{with } U \subset \mathfrak{g}^* \text{ open}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{infinitesimally free}\\ \mathfrak{g}\text{-Hamiltonian}\\ \text{spaces } (S,\omega)\end{array}\right\}.$$

Moreover, μ is a complete realization if and only if the infinitesimal action of $\mathfrak{g}$ comes from an action of the 1-connected Lie group G integrating $\mathfrak{g}$. In particular, one obtains a 1-to-1 correspondence

$$\left\{\begin{array}{c}\text{complete symplectic realizations}\\ \mu : (S,\omega) \to (U, \pi_{\mathfrak{g}}|_U)\\ \text{with } U \subset \mathfrak{g}^* \text{ open } G\text{-invariant}\end{array}\right\} \xleftrightarrow{\sim} \left\{\begin{array}{c}\text{locally free}\\ G\text{-Hamiltonian}\\ \text{spaces } (S,\omega)\end{array}\right\}.$$

Proof. For the first 1-to-1 correspondence we already observed that by Proposition 12.3 a symplectic realization yields a Lie algebra action and this action is infinitesimally free since $\mathscr{a}$ is injective. The $\mathfrak{g}$-equivariance also follows from (i) in Proposition 12.3. For the opposite direction, we observe that the moment map of a $\mathfrak{g}$-Hamiltonian action is a submersion iff the Lie algebra action is infinitesimally free.

Assume now that $\mu : (S,\omega) \to (U, \pi_{\mathfrak{g}}|_U)$ is a complete realization of an open G-invariant subset $U \subset \mathfrak{g}^*$. Given $v \in \mathfrak{g}$, the evaluation $\mathrm{ev}_v : \mathfrak{g}^* \to \mathbb{R}$ yields a Hamiltonian vector field $X_{\mathrm{ev}_v} \in \mathfrak{X}(\mathfrak{g}^*)$, which coincides with the coadjoint action ad^*_v, and hence it is a complete vector field. It follows that

$$\mathscr{a}(v) = X_{\mu_v} = X_{\mu^*(\mathrm{ev}_v)}$$

is also complete. Therefore, $\mathscr{a} : \mathfrak{g} \to \mathfrak{X}(S)$ is a complete Lie algebra action, so it integrates to a locally free G-action of the 1-connected Lie group G — see Proposition A.3.

Conversely, if $\mu : (S,\omega) \to \mathfrak{g}^*$ is a locally free Hamiltonian G-space, then $\mathscr{a} : \mathfrak{g} \to \mathfrak{X}(S)$ is injective so the action is infinitesimally free and μ is a submersion. It follows that its image is an open G-invariant subset $U \subset \mathfrak{g}^*$. We show now that the symplectic realization $\mu : (S,\omega) \to (U, \pi_{\mathfrak{g}}|_U)$ is complete. Let $H \in C^\infty(U)$ be a smooth function with complete Hamiltonian vector field. We show that for any $p \in S$ the integral curve of $X_{H\circ\mu}$ starting at p is defined on $[0,1]$. Let $x := \mu(p)$ and denote

$$\gamma(t) := \phi^t_{X_H}(x) : [0,1] \to U \quad \text{and} \quad a(t) := \mathrm{d}_{\gamma(t)}H : [0,1] \to T^*_{\gamma(t)}\mathfrak{g}^* \simeq \mathfrak{g}.$$

By Lemma 10.32, there exists a unique path $g : [0,1] \to G$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t) = \mathrm{d}L_{g(t)}(a(t)), \quad g(0) = e.$$

We claim that

$$\gamma(t) = \mathrm{Ad}^*_{g(t)^{-1}}(x).$$

This follows from the calculation

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\,\mathrm{Ad}^*_{g(t)}(\gamma(t)) &= \mathrm{Ad}^*_{g(t)}\Big(-\,\mathrm{ad}^*_{a(t)}\,|_{\gamma(t)} + \dot{\gamma}(t)\Big)\\ &= \mathrm{Ad}^*_{g(t)}\Big(-\,\pi^\sharp_{\mathfrak{g}}|_{\gamma(t)}(a(t)) + X_H|_{\gamma(t)}\Big) = 0.\end{aligned}$$

The result will now follow by showing that the integral curve of $X_{H\circ\mu}$ starting at p is given by

$$\tilde{\gamma}(t) := g(t)^{-1}\cdot p.$$

Since μ is G-equivariant, $\tilde{\gamma}(t)$ covers $\gamma(t)$. Its derivative is

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}(t) &= \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t}(g(t)^{-1}g(s))^{-1}\cdot\tilde{\gamma}(t)\\ &= \not{a}_{\tilde{\gamma}(t)}(a(t))\\ &= \not{a}_{\tilde{\gamma}(t)}(\mathrm{d}_{\gamma(t)}H) = X_{H\circ\mu}|_{\tilde{\gamma}(t)}.\end{aligned}$$

This concludes the proof. □

Remark 12.8. Notice that the assumption about freeness of the $\mathfrak{g}$-action is equivalent to the property that μ is a submersion. Omitting that assumption, the first part of the proof establishes the 1-to-1 correspondence stated in Example 1.35.

As a summary of this case study, keep in mind that *any complete symplectic realization of* $(\mathfrak{g}^*, \pi_{\mathfrak{g}})$ *comes with a Lie group action*

$$\begin{array}{c} G \circlearrowright (S,\omega) \\ \big\downarrow \mu \\ (\mathfrak{g}^*, \pi_{\mathfrak{g}}) \end{array}$$

12.3. Case study: The zero Poisson structure

We look at an arbitrary manifold M endowed with the zero Poisson structure $\pi \equiv 0$. We already know that it admits the canonical symplectic realization $\mathrm{pr} : (T^*M, \omega_{\mathrm{can}}) \to (M, 0)$, and we now look at more general ones.

Consider an arbitrary symplectic realization $\mu : (S,\omega) \to (M, 0)$. Items (iii) and (iv) of Proposition 12.3 imply that the infinitesimal action satisfies

$$\mathrm{Im}(\not{a}_p) = (\mathrm{Ker}\,\mathrm{d}_p\mu)^{\perp_\omega} \subset \mathrm{Ker}\,\mathrm{d}_p\mu.$$

So the fibers of μ are coisotropic submanifolds. The converse also holds.

Proposition 12.9. *A symplectic realization $\mu : (S, \omega) \to (M, 0)$ is the same thing as a surjective submersion $\mu : (S, \omega) \to M$ with coisotropic fibers. In particular, if $\dim S = 2 \dim M$, μ is a symplectic realization if and only if its fibers are Lagrangian submanifolds.*

Proof. We already know that one implication holds. For the other one we can invoke, e.g., Libermann's Theorem. □

For each point $x \in M$, the infinitesimal action restricts to a linear map

$$\mathscr{a} : T_x^*M \to \mathfrak{X}(\mu^{-1}(x)). \tag{12.2}$$

Again Proposition 12.3 shows that this map is a Lie algebra action of the abelian Lie algebra T_x^*M on the fiber $\mu^{-1}(x)$. Completeness amounts to integrability of this action.

Proposition 12.10. *A symplectic realization $\mu : (S, \omega) \to (M, 0)$ is complete if and only if for each $x \in M$ the Lie algebra action* (12.2) *integrates to an action of the abelian group $(T_x^*M, +)$ on the fiber $\mu^{-1}(x)$.*

Proof. The symplectic realization is complete if and only if all the vector fields $X_{f\circ\mu} = \mathscr{a}(\mathrm{d}f)$, with $f \in C^\infty(M)$, are complete. Since these vector fields are vertical, this is equivalent to the vector fields $\mathscr{a}(\xi) \in \mathfrak{X}(\mu^{-1}(x))$ being complete, for all $x \in M$ and $\xi \in T_x^*M$. By Proposition A.3, this is equivalent to the Lie algebra actions (12.2) integrating to Lie group actions. □

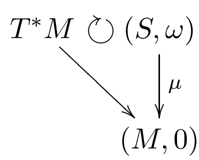
Example 12.11. For the symplectic realization $\mu : (T^*M, \omega_{\text{can}}) \to (M, 0)$, the associated infinitesimal action is given by

$$\mathscr{a}_\beta(\alpha) = -\alpha \quad (\alpha, \beta \in T_xM).$$

The minus sign is due to our convention for the canonical symplectic form: $\omega_{\text{can}} = -\mathrm{d}\theta_L$. Therefore, this realization is complete and, by our convention (A.6) for differentiating actions, the resulting group action is given by

$$(T_x^*M, +) \times T_x^*M \to T_x^*M, \quad \alpha \cdot \beta = \alpha + \beta.$$

Summarizing this case study, note that *the actions from the proposition fit together into a global "action" of the bundle of abelian groups T^*M:*



12.4. Case study: Nondegenerate Poisson structures

Next, we consider the other extreme case, when (M, π) is nondegenerate; hence π is obtained by inverting a symplectic form $\omega \in \Omega^2(M)$. Of course, the identity map $\mathrm{Id} : M \to M$ is a symplectic realization, and so is any surjective local diffeomorphism $\mu : S \to M$ with the pullback symplectic form $\mu^*\omega$. These are not necessarily complete, and in fact we have:

Proposition 12.12. *If M is a nondegenerate Poisson manifold and $\mu : S \to M$ is a symplectic realization with $\dim S = \dim M$, then*

$$\mu \text{ is complete} \iff \mu \text{ is a covering map.}$$

This will soon become clear. Note that other symplectic realizations can be obtained by taking products of (M, ω) with another symplectic manifold.

Let us look at the geometry of an arbitrary symplectic realization of a nondegenerate Poisson structure

$$\mu : (S, \omega) \to (M, \pi). \tag{12.3}$$

By Proposition 12.3, the image of the action $\mathrm{Im}(a) = (\mathrm{Ker}\, \mathrm{d}\mu)^{\perp_\omega}$ is an *Ehresmann connection* for $\mu : S \to M$, i.e., a complement to the vertical distribution

$$TS = \mathrm{Ker}\, \mathrm{d}\mu \oplus (\mathrm{Ker}\, \mathrm{d}\mu)^{\perp_\omega}. \tag{12.4}$$

By Libermann's Theorem, this connection is *flat*; i.e., $(\mathrm{Ker}\, \mathrm{d}\mu)^{\perp_\omega} \subset TS$ is an involutive distribution. The infinitesimal action can be reinterpreted as the *horizontal lift* with respect to this Ehresmann connection

$$\mathrm{Hor}_p : T_{\mu(p)}M \to T_pS, \qquad \mathrm{Hor}_p(\pi^\sharp \xi) = a(\xi).$$

An Ehresmann connection allows one to lift paths from M to S. Given

$$\gamma : [0,1] \to M$$

and a point $p \in \mu^{-1}(\gamma(0))$ there is a unique horizontal path $\tilde{\gamma}^p$ starting at p and covering γ; i.e.,

$$\tilde{\gamma}^p : I \to S, \qquad \text{such that} \qquad \begin{cases} \frac{\mathrm{d}\tilde{\gamma}^p}{\mathrm{d}t}(t) = \mathrm{Hor}_{\tilde{\gamma}^p(t)}\left(\dot{\gamma}(t)\right), \\ \tilde{\gamma}^p(0) = p, \end{cases}$$

where, in general, $0 \in I \subset [0,1]$ is a small interval. The Ehresmann connection is called **complete** if for every curve $\gamma : [0,1] \to M$ and any $p \in \mu^{-1}(\gamma(0))$ the horizontal lift $\tilde{\gamma}^p$ is defined on the whole interval $[0,1]$. The following is a particular case of Theorem 12.22, which we will see later:

Proposition 12.13. *A symplectic realization $\mu : (S, \omega) \to (M, \pi)$ of a nondegenerate Poisson manifold is complete if and only if the Ehresmann connection $(\mathrm{Ker}\, \mathrm{d}\mu)^{\perp_\omega}$ is complete.*

For a complete Ehresmann connection, given any path γ with $\gamma(0) = x_0$ and $\gamma(1) = x_1$ we have a parallel transport map

$$\tau_\gamma : \mu^{-1}(x_0) \to \mu^{-1}(x_1), \quad p \mapsto \tilde{\gamma}^p(1).$$

Smoothness of this map follows from standard results on smooth dependence on the parameters of solutions of ODEs. This map is actually a diffeomorphism because, if $\bar{\gamma}(t) = \gamma(1-t)$ denotes the reverse path, we find

$$\tau_{\bar{\gamma}} \circ \tau_\gamma = \mathrm{Id}_{\mu^{-1}(x_0)}, \quad \tau_\gamma \circ \tau_{\bar{\gamma}} = \mathrm{Id}_{\mu^{-1}(x_1)}.$$

Also, if a smooth path is the concatenation of two paths, parallel transport is transformed into composition of parallel transports:

$$\tau_{\delta\circ\gamma} = \tau_\delta \circ \tau_\gamma.$$

Even more, since the connection is flat, it follows that path-homotopic paths induce the same parallel transport:

$$\gamma_0 \sim \gamma_1 \quad \Longrightarrow \quad \tau_{\gamma_0} = \tau_{\gamma_1}.$$

If you are not familiar with these properties, the proofs are similar to the ones given in Section 12.5 for parallel transport along cotangent paths — however, our discussion there does not depend on these results.

The decomposition (12.4) also implies that the two complementary foliations are symplectic; i.e., the fibers of μ and the orbits of the infinitesimal action a are symplectic submanifolds. Moreover, the parallel transport is by symplectomorphisms. Therefore, under mild topological conditions on M, the geometry of complete symplectic realizations can be made completely explicit:

Proposition 12.14. *Let (M, π) be a nondegenerate, 1-connected Poisson manifold. Then any complete symplectic realization $\mu : (S, \omega) \to (M, \pi)$ is isomorphic to a product; i.e., there exists a symplectomorphism*

$$\Phi : (S, \omega) \xrightarrow{\sim} (M, \pi) \times (F, \omega_F)$$

under which μ becomes the projection.

Proof. Fix $x_0 \in M$, and set $F = \mu^{-1}(x_0)$ and $\omega_F := \omega|_F$. Define

$$\Phi : S \to M \times F, \quad \Phi(p) = (\mu(p), \tau_\gamma(p)),$$

where γ is any path in M starting at $\mu(p)$ and ending at x_0. Since M is 1-connected, this is well-defined. We leave it as an exercise to check that this is the desired symplectomorphism. □

Exercise 12.15. Modify the previous proof to deduce Proposition 12.12.

To summarize this discussion in a manner similar to the previous two case studies, we rephrase it as follows. First of all, for any two points $x, y \in M$, we consider path-homotopy classes of paths starting at x and ending at y:

$$\Pi(M, y, x) := \frac{\text{paths in } M \text{ from } x \text{ to } y}{\text{path-homotopy}}.$$

Concatenation of paths induces a group-like multiplication

$$\begin{gathered} \Pi(M, z, y) \times \Pi(M, y, x) \to \Pi(M, z, x), \\ ([\delta], [\gamma]) \mapsto [\delta] \circ [\gamma] := [\delta \circ \gamma]. \end{gathered}$$

All these together form the so-called **homotopy groupoid** of M

$$\Pi(M) := \frac{\text{paths in } M}{\text{path-homotopy}} \overset{\mathbf{t}}{\underset{\mathbf{s}}{\rightrightarrows}} M$$

where the maps $\mathbf{s}$ (for “source”) and $\mathbf{t}$ (for “target”) give the initial and the end points of a path:

$$\begin{aligned} \mathbf{s} : \Pi(M) \to M, &\quad [\gamma] \mapsto \gamma(0), \\ \mathbf{t} : \Pi(M) \to M, &\quad [\gamma] \mapsto \gamma(1). \end{aligned}$$

The multiplication $[\delta] \circ [\gamma]$ is defined only when $\mathbf{s}([\delta]) = \mathbf{t}([\gamma])$.

Now, the previous discussion concerning parallel transport yields an action of $\Pi(M)$ on S along the μ-fibers. For any $x, y \in M$, one sets

$$\Pi(M, y, x) \times \mu^{-1}(x) \to \mu^{-1}(y), \quad ([\gamma], p) \mapsto [\gamma] \cdot p := \tau_\gamma(p),$$

and these satisfy the following action-like properties:

(i) The class of the constant path $\gamma(t) \equiv x$ acts as an identity:

$$[x] \cdot p = p, \quad \forall\, p \in \mu^{-1}(x).$$

(ii) Whenever $[\delta], [\gamma] \in \Pi(M)$ are composable one has

$$([\delta] \circ [\gamma]) \cdot p = [\delta] \cdot ([\gamma] \cdot p).$$

In the next chapter we will discuss groupoids in depth and this will be referred to as an *action of* $\Pi(M) \rightrightarrows M$ *on the map* $\mu : S \to M$.

Therefore, summarizing this discussion, *any complete symplectic realization of a nondegenerate Poisson manifold carries a canonical action of the homotopy groupoid:*

$$\begin{array}{ccc} \Pi(M) & \circlearrowleft & (S, \omega) \\ & \searrow\!\!\searrow & \downarrow \mu \\ & & (M, \pi) \end{array}$$

12.5. Completeness

Note that item (i) in Proposition 12.3 yields the commutative diagram

$$\begin{array}{ccc} \mu^*T^*M & \xrightarrow{\not{a}} & TS \\ \big\downarrow & & \big\downarrow \mathrm{d}\mu \\ T^*M & \xrightarrow[\pi^\sharp]{} & TM \end{array}$$

This suggests that one should think of the infinitesimal action as a "horizontal lift" of covectors in M to tangent vectors in S. Pursuing this point of view, one is led to an operation of "horizontal lift" of cotangent paths. We formalize this as follows:

Definition 12.16. Let $\mu : (S, \omega) \to (M, \pi)$ be a symplectic realization. A **lift of a cotangent path** $a : I \to T^*M$ to S is any path $\tilde{\gamma}_a : I \to S$, such that

$$\frac{\mathrm{d}\tilde{\gamma}_a}{\mathrm{d}t}(t) = \not{a}_{\tilde{\gamma}_a(t)}(a(t)), \quad \forall\, t \in I.$$

Equivalently, by the definition of the infinitesimal action, the equation for the lift can be written as

$$i_{\frac{\mathrm{d}\tilde{\gamma}_a}{\mathrm{d}t}}\omega = (\mathrm{d}_{\tilde{\gamma}_a}\mu)^* a.$$

Exercise 12.17. Show that a path $\tilde{\gamma} : I \to S$ is a lift of some (necessarily unique!) cotangent path $a : I \to T^*M$ if and only if

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}t}(t) \in (\mathrm{Ker}\, \mathrm{d}\mu)^{\perp_\omega}, \quad \forall\, t \in I.$$

As for classical Ehresmann connections, we have:

Proposition 12.18. *Let $\mu : (S, \omega) \to (M, \pi)$ be a symplectic realization. Given a cotangent path $a : [0,1] \to T^*M$ and an initial point $p \in \mu^{-1}(\gamma_a(0))$, there exists a unique maximal lift $\tilde{\gamma}_a^p : I \to S$ of a starting at p, which is defined on some interval $0 \in I \subset [0,1]$.*

Proof. By Lemma 10.3, there exists a smooth family of functions $\{H_t\}_{t\in[0,1]}$ such that $a(t) = \mathrm{d}H_t|_{\gamma_a(t)}$ and $\gamma_a(t)$ is an integral curve of X_{H_t}; i.e., $\gamma_a(t) = \phi^t_{X_H}(\gamma_a(0))$. Let $\tilde{\gamma}_a^p(t) = \phi^t_{X_{H\circ\mu}}(p) : I \to S$ be the maximal integral curve of the time-dependent Hamiltonian vector field $X_{H_t\circ\mu}$ starting at p. Since $X_{H_t\circ\mu}$ projects to X_{H_t}, it follows that $\mu \circ \tilde{\gamma}_a^p(t) = \gamma_a(t)$. We have that $\tilde{\gamma}_a^p$ is a lift of a:

$$\frac{\mathrm{d}\tilde{\gamma}_a^p}{\mathrm{d}t}(t) = X_{H_t\circ\mu}|_{\tilde{\gamma}_a^p(t)} = \not{a}_{\tilde{\gamma}_a^p(t)}(\mathrm{d}H_t|_{\gamma_a(t)}) = \not{a}_{\tilde{\gamma}_a^p(t)}(a(t)).$$

These equations, read in a different order, reveal that *any* lift $\widetilde{\gamma} : J \to S$ of a is an integral curve of $X_{H_t \circ \mu}$:

$$\frac{\mathrm{d}\tilde{\gamma}}{\mathrm{d}t}(t) = \not{a}_{\tilde{\gamma}(t)}(a(t)) = \not{a}_{\tilde{\gamma}(t)}(\mathrm{d}H_t|_{\gamma_a(t)}) = X_{H_t\circ\mu}|_{\tilde{\gamma}(t)}.$$

So the uniqueness of the maximal lift follows from the corresponding property of integral curves of time-dependent vector fields. □

Exercise 12.19. If $a : [0,1] \to S$ is a cotangent path, show that the pullback bundle $\gamma_a^* S \to [0,1]$ has an induced Ehresman connection and that the lifts of a can be interpreted as the parallel transport with respect to this connection — see Exercise 11.15 for the linear version.

It is hard not to notice the striking similarity between the operations of lifting of cotangent paths for a symplectic realization, from the previous proposition, and the lifting of ordinary paths for an Ehresmann connection, discussed in the last case study. These are indeed instances of a very general notion of lifting operation for nonlinear connections on Lie algebroids, as we explain in the following remark.

Remark 12.20 (Nonlinear connections)**.** Let $(A, [\cdot,\cdot]_A, \rho)$ be a Lie algebroid, and let $p : P \to M$ be a surjective submersion. A **nonlinear** A-**connection** on P is a vector bundle map covering Id_P,

$$h_P : p^*A \equiv A \times_M P \to TP,$$

that makes the following diagram commute:

$$\begin{array}{ccc} p^*A & \xrightarrow{h_P} & TP \\ {\scriptstyle \mathrm{pr}}\downarrow & & \downarrow{\scriptstyle \mathrm{d}p} \\ A & \xrightarrow[\rho]{} & TM \end{array} \tag{12.5}$$

Given a nonlinear A-connection h_P, let $a : [0,1] \to A$ be an A-path with base path $\gamma_a : [0,1] \to M$ — see Problem 10.1. For a point $x \in P_{\gamma_a(0)}$, one defines the **horizontal lift** $\tilde{\gamma}_a^x : [0,\varepsilon) \to P$ to be the unique path over γ_a that satisfies the ODE

$$\begin{cases} \frac{\mathrm{d}\tilde{\gamma}_a^x}{\mathrm{d}t}(t) = h_P(a(t), \tilde{\gamma}_a^x(t)), \\ \tilde{\gamma}_a^x(0) = x. \end{cases}$$

One calls h_P a **complete** nonlinear A-connection if the horizontal lifts of any cotangent path are defined up to time 1. For a complete connection one defines a **parallel transport** map between the fibers of p:

$$\tau_a : P_{\gamma_a(0)} \to P_{\gamma_a(1)}, \quad x \mapsto \tilde{\gamma}_a^x(1).$$

For example, if $p : P \to M$ is proper, then any nonlinear connection h_P is complete.

With these notions at hand, one can explain the similarity between the operations of lifts of paths and cotangent paths. These lifting operations are obtained by considering the appropriate Lie algebroids:

(i) When $A = TM$ with $\rho = \mathrm{Id}$ it follows from the diagram (12.5) that a nonlinear connection is completely determined by its image, which is a distribution complementary to the vertical distribution $\operatorname{Ker} dp$. So for $A = TM$, nonlinear connections are the same as Ehresmann connections.

(ii) When $A = T^*M$ is the contangent algebroid of a Poisson manifold (M, π), we have $\rho = \pi^\sharp$. In general, $\pi^\sharp$ has kernel and a nonlinear connection
$$h_P : p^*T^*M \to TP$$
is not anymore determined by its image. In this case, we call h_P a **contravariant nonlinear connection**.

(iii) Any symplectic realization $\mu : (S, \omega) \to (M, \pi)$ comes with a canonical contravariant nonlinear connection, namely the infinitesimal action
$$h_S = \not{a} : \mu^*T^*M \to TS.$$

You may also wonder about the name *nonlinear* connection. This is explained in the following exercise.

Exercise 12.21. Assume that $P = E$ is a vector bundle $p : E \to M$. Then note that both vertical arrows in (12.5) are naturally vector bundles — e.g., fiber addition on $dp : TE \to TM$ is obtained by differentiating that on $p : E \to M$. An A-connection h_E is called **linear** if h_E is a vector bundle map for these vector bundle structures. Show the following:

(a) A linear connection h_E is always complete.

(b) Parallel transport τ_a is a linear isomorphism.

For an A-path $a : [0,1] \to A$ and a path $c : [0,1] \to E$ above $\gamma_a : [0,1] \to M$ one defines
$$D_a c(t) := \lim_{h \to 0} \frac{1}{h}\left((\tau_a^{t,t+h})^{-1} c(t+h) - c(t)\right),$$
where $\tau_a^{t,t+h}$ denotes parallel transport along the restriction $a|_{[t,t+h]}$. Show that D_a is the derivative along A-paths associated to a unique A-connection ∇ on $E \to M$ in the sense of Problem 11.8.

Completeness of symplectic realizations can be equivalently characterized in terms of the completeness of the corresponding nonlinear connection.

Theorem 12.22. *A symplectic realization $\mu : (S, \omega) \to (M, \pi)$ is complete if and only if, for every cotangent path $a : [0,1] \to T^*M$ and every initial point $p \in \mu^{-1}(\gamma_a(0))$, the maximal lift $\tilde{\gamma}_a^p$ is defined on $[0,1]$.*

Proof. Assume first that all maximal lifts are defined on $[0,1]$. Consider a function $H \in C^\infty(M)$ with complete Hamiltonian vector field X_H. Then for each $x \in M$ the flow of X_H yields a cotangent path $a(t) := \mathrm{d}_{\gamma(t)}H$, where $\gamma(t) = \phi^t_{X_H}(x)$. By the proof of Proposition 12.18, the lift $\tilde{\gamma}^p_a : [0,1] \to S$ of a starting at $p \in \mu^{-1}(x)$ is precisely the integral curve of $X_{H\circ\mu}$ starting at p. Since lifts exist, $X_{H\circ\mu}$ is complete. So the realization is complete.

To prove the converse, i.e., that cotangent paths can be lifted, we start by making a few remarks:

- It is enough to prove existence of lifts for all symplectic realizations $\mu : (S,\omega) \to M$ satisfying the (apparently) weaker property

$$H \in C^\infty(M) \text{ compactly supported} \quad \Longrightarrow \quad X_{H\circ\mu} \text{ is complete.}$$

  Moreover, if $\mu : S \to M$ satisfies this property, then for any open $U \subset M$ the restriction $\mu : \mu^{-1}(U) \to U$ still satisfies this property.
- It suffices to show that each point in M has a neighborhood U over which cotangent paths can be lifted. Indeed, given any cotangent path $a : [0,1] \to T^*M$, one can cover the base path γ_a with a finite number of open sets U_i where lifts exist, and then the lifts of a over each U_i glue smoothly to a lift of a defined on $[0,1]$.

Hence, we can replace M by the domain of a splitting chart

$$U = (L, \pi_{\mathrm{can}}) \times (X, \pi_X), \quad 0 \in L \subset \mathbb{R}^{2n},\ 0 \in X \subset \mathbb{R}^q,$$

where π_X vanishes at 0. Here $L \equiv L \times \{0\} \hookrightarrow M$ is an embedded symplectic leaf and the splitting allows us to identify the isotropy Lie algebras at all points of L:

$$\mathfrak{g} := \mathrm{Ker}\, \pi^\sharp_{y,0} \quad (y \in L).$$

Restricting the infinitesimal action $\mathscr{a}$, we obtain a Lie algebra action of $\mathfrak{g}$ on $\mu^{-1}(0)$, which we denote by the same symbol:

$$\mathscr{a} : \mathfrak{g} \to \mathfrak{X}(\mu^{-1}(0)), \quad \mathscr{a}_p(\xi) := \mathscr{a}_p(0,\xi) \quad (p \in \mu^{-1}(0), \xi \in \mathfrak{g}).$$

The splitting also gives a flat Ehresmann connection on $\mu^{-1}(L) \to L$, with horizontal lift defined by

(12.6)
$$\mathrm{Hor} : \mu^*(TL) \to T\mu^{-1}(L), \quad \mathrm{Hor}_p(\pi^\sharp_{\mathrm{can}}\alpha) := \mathscr{a}_p(\alpha, 0) \quad (\alpha \in T^*_{\mu(p)}L).$$

By Proposition 12.3(i) this is an Ehresmann connection. It is flat because both $\pi^\sharp_{\mathrm{can}} : \Omega^1(L) \xrightarrow{\sim} \mathfrak{X}(L)$ and $\mathscr{a} : \Omega^1(M) \to \mathfrak{X}^1(S)$ preserve the Lie brackets — see Proposition 12.3 — and so does the map $\alpha \mapsto (\alpha, 0)$.

Next, we claim that there exists a trivialization Φ of μ over an open set $0 \in V \subset L$,

$$\begin{array}{ccc} V \times \mu^{-1}(0) & \xrightarrow{\quad\Phi\quad} & \mu^{-1}(V) \\ & {\scriptstyle \mathrm{pr}_L} \searrow \quad \swarrow {\scriptstyle \mu} & \\ & V & \end{array}$$

which has the property that it trivializes the action, i.e., that satisfies for all $y \in V$ and $p \in \mu^{-1}(V)$

$$\mathrm{d}_{(y,p)}\Phi\big(\pi^\sharp_{\mathrm{can}}\alpha, \mathscr{a}_p(\xi)\big) = \mathscr{a}_{\Phi(y,p)}(\alpha, \xi) \quad (\alpha \in T^*_y L,\, \xi \in \mathfrak{g}). \tag{12.7}$$

To see this, fix a convex neighborhood $V \subset L$ of $0 \in L$ and for each $y \in V$ denote by H^y the unique linear function H^y on (L, π_{can}) whose Hamiltonian flow sends 0 to y. We extend these functions to $L \times X$ to be constant in the second variable. Using a bump function in $L \times X$ which equals 1 in a neighborhood of $V \times \{0\}$ we make all the H^y with compact support. Hence, keeping the same notation, the vector fields $X_{H^y \circ \mu}$ are complete and we can define

$$\Phi : V \times \mu^{-1}(0) \to \mu^{-1}(L), \quad \Phi(y,p) := \phi^1_{X_{H^y \circ \mu}}(p). \tag{12.8}$$

Notice that the integral curves of $X_{H^y \circ \mu}$ cover the integral curves of X_{H^y} so Φ fits in the previous commutative diagram. Hence, Φ is a diffeomorphism onto $\mu^{-1}(V)$ with inverse

$$\Phi^{-1}(q) = \phi^{-1}_{X_{H^y \circ \mu}}(q), \quad \text{where } \ y := \mu(q).$$

We still need to show that Φ satisfies (12.7) for any (α, ξ):

- Assume $\xi = 0$: By definition (12.6) of the flat Ehresmann connection, we have $X_{H^y \circ \mu} = \mathrm{Hor}\, X_{H^y}$ in $\mu^{-1}(V)$. It follows that $\Phi(V \times \{p\})$ is included in a leaf of the corresponding horizontal foliation. Therefore $\mathrm{d}_{(y,p)}\Phi(\pi^\sharp_{\mathrm{can}}\alpha, 0)$ is horizontal, and by (12.8) this vector projects to $\pi^\sharp_{\mathrm{can}}\alpha$. We conclude that

$$\mathrm{d}_{(y,p)}\Phi\big(\pi^\sharp_{\mathrm{can}}\alpha, 0\big) = \mathrm{Hor}_{\Phi(y,p)}(\pi^\sharp_{\mathrm{can}}\alpha) = \mathscr{a}_{\Phi(y,p)}(\alpha, 0).$$

- Assume $\alpha = 0$: Given $\xi \in \mathfrak{g}$, write $\xi = \mathrm{d}_0 f$ for some $f \in C^\infty(X)$. Extending f to $L \times X$ as a constant function in the first variable, we have $\{H^y, f\} = 0$, for all $y \in L$. Since μ is a Poisson map, the vector fields $X_{H^y \circ \mu}$ and $X_{f \circ \mu}$ commute. This implies that the flow of $X_{H^y \circ \mu}$ preserves $X_{f \circ \mu} = \mathscr{a}(\mathrm{d}f)$, and so

$$\begin{aligned} \mathrm{d}_{(y,p)}\Phi\big(0, \mathscr{a}_p(\xi)\big) &= \mathrm{d}_{(y,p)}\Phi\big(0, \mathscr{a}_p(\mathrm{d}_0 f)\big) \\ &= \mathscr{a}_{\Phi(y,p)}(0, \mathrm{d}_0 f) = \mathscr{a}_{\Phi(y,p)}(0, \xi). \end{aligned}$$

This shows that Φ has the desired properties. Note that, by assumption, the action of $\mathfrak{g}$ on $\mu^{-1}(0)$ is complete (see Corollary 12.4). Therefore, by Proposition A.3 it comes from a Lie group action $G \times \mu^{-1}(0) \to \mu^{-1}(0)$ of a 1-connected Lie group G with Lie algebra $\mathfrak{g}$.

We can now show that any cotangent path $a : [0,1] \to T^*_L M$ can be lifted. We decompose the path according to the splitting

$$a(t) = (\alpha(t), \xi(t)), \quad \text{where} \quad \pi^\sharp_{\mathrm{can}}\alpha(t) = \frac{\mathrm{d}\gamma_a}{\mathrm{d}t}(t), \quad \xi : [0,1] \to \mathfrak{g}.$$

By Lemma 10.32, there exists a unique path $g : [0,1] \to G$ such that

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} g(t)^{-1}g(s) = \xi(t), \quad g(0) = e.$$

We claim that for $q \in \mu^{-1}(\gamma_a(0))$ the lift of a at q is given by

$$\tilde{\gamma}^q_a(t) := \Phi(\gamma_a(t), g(t)^{-1} \cdot p), \quad \text{where } q = \Phi(\gamma_a(0), p).$$

This follows by a computation. First of all,

$$\frac{\mathrm{d}}{\mathrm{d}t} g(t)^{-1} \cdot p = \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} (g(t)^{-1}g(s))^{-1} \cdot (g(t)^{-1} \cdot p) = \not{a}_{g(t)^{-1}\cdot p}(\xi(t)).$$

Then, using (12.7) we obtain that $\tilde{\gamma}^q_a(t)$ is indeed the lift of a:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(\gamma_a(t), g(t)^{-1} \cdot p) &= \mathrm{d}_{\Phi(\gamma_a(t), g(t)^{-1}\cdot p)}\Phi\big(\pi^\sharp_V \alpha(t), \not{a}_{g(t)^{-1}\cdot p}\xi(t)\big) \\ &= \not{a}_{\Phi(\gamma_a(t), g(t)^{-1}\cdot p)}(\alpha(t), \xi(t)) \\ &= \not{a}_{\Phi(\gamma_a(t), g(t)^{-1}\cdot p)}(a(t)).\end{aligned}$$

Moreover, this shows that the lift is defined for all $t \in [0,1]$. □

In the literature, complete symplectic realizations are defined in various ways. We prove now that these different approaches are all equivalent:

Corollary 12.23. *For a symplectic realization $\mu : (S, \omega) \to (M, \pi)$ the completeness assumption is equivalent to any of the following conditions:*

(i) $\{X_{H_t}\}_{t\in[0,1]}$ *is complete* $\Rightarrow$ $\{X_{H_t\circ\mu}\}_{t\in[0,1]}$ *is complete,*

(ii) $\{X_{H_t}\}_{t\in[0,1]}$ *is compactly supported* $\Rightarrow$ $\{X_{H_t\circ\mu}\}_{t\in[0,1]}$ *is complete,*

(iii) $\{\pi^\sharp\alpha_t\}_{t\in[0,1]}$ *is complete* $\Rightarrow$ $\{\not{a}(\alpha_t)\}_{t\in[0,1]}$ *is complete,*

(iv) $\{\pi^\sharp\alpha_t\}_{t\in[0,1]}$ *is compactly supported* $\Rightarrow$ $\{\not{a}(\alpha_t)\}_{t\in[0,1]}$ *is complete,*

where $H_t \in C^\infty(M)$ denotes any smooth family of functions, $\alpha_t \in C^\infty(M)$ denotes any smooth family of 1*-forms, and compactly supported time-dependent vector fields are defined as in Section* A.3.

Proof. Any compactly supported time-dependent vector field is complete — see Proposition A.12. Therefore, (i) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv). By taking

$\alpha_t = \mathrm{d}H_t$, we clearly also have that (iii) $\Rightarrow$ (i) and (iv) $\Rightarrow$ (ii). So, we are left with proving the implications

$$\text{(ii)} \quad \Longrightarrow \quad \text{completeness} \quad \Longrightarrow \quad \text{(iii)}.$$

Assume that (ii) holds, and we check the equivalent condition for completeness from Theorem 12.22. By Lemma 10.3, for any cotangent path $a : [0,1] \to T^*M$ there exists a smooth family of functions $\{H_t\}_{t\in[0,1]}$, all supported in the same compact set, such that $a(t) = \mathrm{d}H_t|_{\gamma_a(t)}$ and $\gamma_a(t) = \phi^t_{X_H}(\gamma_a(0))$. By (ii), $X_{H_t\circ\mu}$ is a complete vector field, so its integral curve $\tilde{\gamma}^p_a(t) = \phi^t_{X_{H\circ\mu}}(p)$ starting at $p \in \mu^{-1}(\gamma_a(0))$ exists for all $t \in [0,1]$. By the proof of Proposition 12.18, this is precisely the lift of a starting at p. We obtained completeness.

Assume now that the connection is complete. To check (iii), consider a time-dependent section $\alpha_t \in \Omega^1(M)$ such that $\pi^\sharp\alpha_t$ is a complete vector field. We need to show that for any $p \in S$ the integral curve of $\mathcal{a}(\alpha_t)$ starting at p exists for all $t \in [0,1]$. The integral curve $\gamma(t)$ of $\pi^\sharp\alpha_t$ starting at $\mu(p)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\gamma(t) = \pi^\sharp\alpha_t|_{\gamma(t)}.$$

Therefore, $a(t) := \alpha_t|_{\gamma(t)} : [0,1] \to T^*M$ is a cotangent path. Since the connection is complete, Theorem 12.22 implies that $a(t)$ has a complete lift $\tilde{\gamma}^p_a : [0,1] \to S$ that starts at p. Note that $\tilde{\gamma}^p_a$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\gamma}^p_a(t) = \mathcal{a}_{\tilde{\gamma}^p_a(t)}(\alpha_t),$$

and so it is the integral curve of $\mathcal{a}(\alpha_t)$ starting at p. Hence, (iii) holds. □

12.6. The Poisson homotopy groupoid

The previous section shows that, given a complete symplectic realization $\mu : (S,\omega) \to (M,\pi)$, any cotangent path $a : [0,1] \to T^*M$ yields an operation of **parallel transport**

$$\tau_a : \mu^{-1}(\gamma_a(0)) \to \mu^{-1}(\gamma_a(1)), \quad p \mapsto \tilde{\gamma}^p_a(1).$$

Next, we show that parallel transport is invariant under cotangent path-homotopy.

Theorem 12.24. *Let $\mu : (S,\omega) \to (M,\pi)$ be a complete symplectic realization. Let $a, b : [0,1] \to T^*M$ be cotangent paths with $\gamma_a(0) = \gamma_b(0) =: x$, and fix $p \in \mu^{-1}(x)$. Let $\mathcal{O}_p$ be the orbit of the infintesimal action through p. Then a and b are cotangent path-homotopic if and only if their lifts $\tilde{\gamma}^p_a, \tilde{\gamma}^p_b : [0,1] \to S$ are path-homotopic inside $\mathcal{O}_p$.*

Remark 12.25. The theorem gives a geometric interpretation of cotangent path-homotopy on a Poisson manifold in terms of ordinary path-homotopy, once one finds a *complete* symplectic realization. This raises the important question of finding complete symplectic realizations — the symplectic realizations constructed in Theorem 11.43 are rarely complete. We will come back to this problem in the next chapters.

Proof. We assume first that $\tilde{\gamma}_a^p$ and $\tilde{\gamma}_b^p$ are path-homotopic inside $\mathcal{O}_p$ as in the statement. Fix a path-homotopy $H : [0,1] \times [0,1] \to \mathcal{O}_p$, so that

$$H(t,0) = \tilde{\gamma}_a^p(t), \quad H(t,1) = \tilde{\gamma}_b^p(t), \quad H(0,\cdot) = \mathrm{const}_0, \quad H(1,\cdot) = \mathrm{const}_1.$$

Viewing $\mathrm{d}H : T([0,1]\times[0,1]) \to T\mathcal{O}_p$ as a Lie algebroid map and composing it with the map $\Psi_p : T\mathcal{O}_p \to T^*M$ from (12.1), we obtain a cotangent map

$$\Phi = \Phi_1\,\mathrm{d}t + \Phi_2\,\mathrm{d}\varepsilon : T([0,1]\times[0,1]) \to T^*M.$$

The boundary conditions in Definition 10.18 hold because, for $i \in \{0,1\}$,

$$H(i,\cdot) = \mathrm{const}_i \quad \Longrightarrow \quad \frac{\mathrm{d}H}{\mathrm{d}\varepsilon}(i,\varepsilon) = 0 \quad \Longrightarrow \quad \Phi_2(i,\cdot) = 0$$

and

$$H(t,0) = \tilde{\gamma}_a^p(t) \quad \Longrightarrow \quad \Phi_1(t,0) = \Psi_p\Big(\dot{\tilde{\gamma}}_a^p(t)\Big) = a(t),$$
$$H(t,1) = \tilde{\gamma}_b^p(t) \quad \Longrightarrow \quad \Phi_1(t,1) = \Psi_p\Big(\dot{\tilde{\gamma}}_b^p(t)\Big) = b(t).$$

Therefore, Φ is a cotangent path-homotopy between a and b.

In order to prove the converse, let

$$\Phi = \Phi_1\,\mathrm{d}t + \Phi_2\,\mathrm{d}\varepsilon$$

be a cotangent path-homotopy between the cotangent paths a and b, covering a path-homotopy $\gamma : [0,1]\times[0,1] \to M$. For each ε, $s \mapsto \Phi_1(s,\varepsilon)$ is a cotangent path that can be lifted to a path $H(\cdot,\varepsilon)$ in $\mathcal{O}_p$, starting at p. We get a smooth map

$$H : [0,1]\times[0,1] \to \mathcal{O}_p$$

sitting above $\gamma : [0,1]\times[0,1] \to M$ and which, by construction, satisfies

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t,\varepsilon) = \alpha_{H(t,\varepsilon)}\Phi_1(t,\varepsilon), \quad H(0,\varepsilon) = p,$$

and by definition

$$H(t,0) = \tilde{\gamma}_a^p(t) \quad H(t,1) = \tilde{\gamma}_b^p(t).$$

As in the first part, consider the cotangent map $\Phi' := \Psi_p \circ \mathrm{d}H$. We know that $\Phi_1' = \Phi_1$. We claim that $\Phi_2' = \Phi_2$. For this, note that the equation in Proposition 10.17(ii) is satisfied both by Φ and Φ'. Therefore, the difference

$$D_\varepsilon(t) := \Phi_2(t,\varepsilon) - \Phi_2'(t,\varepsilon)$$

satisfies for each ε and each vector field X the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle X, D_\varepsilon(t)\rangle = \mathrm{d}_\pi(X)(\Phi_1, D_\varepsilon(t)).$$

Writing this for a local basis of vector fields, we obtain that D_ε satisfies locally a linear ODE in t. Therefore, if D_ε vanishes at a point, then it must vanish around that point. Since the vanishing set is also closed and $D_\varepsilon(0) = \Phi_2(0,\varepsilon) - \Phi_2'(0,\varepsilon) = 0 - 0$, we must have $D_\varepsilon = 0$. So $\Phi_2 = \Phi_2'$.

Finally, since $0 = \Phi_2(1,\varepsilon) = \Phi_2'(1,\varepsilon)$ and Ψ_p is a fiberwise isomorphism, it follows that $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}H(1,\varepsilon) = 0$. Hence $H(1,\varepsilon)$ is constant, showing that H is a path-homotopy between the lifts $\tilde{\gamma}_a^p$ and $\tilde{\gamma}_b^p$. □

Corollary 12.26. *Let $\mu : (S,\omega) \to (M,\pi)$ be a complete symplectic realization. If $a, b : [0,1] \to T^*M$ are cotangent path-homotopic, then they induce the same parallel transport: $\tau_a = \tau_b$.*

Remark 12.27 (Flat nonlinear connections)**.** When a complete Ehresmann connection is *flat*, i.e., its horizontal distribution is involutive, then path-homotopic paths induce the same parallel transport. The previous corollary states that a similar fact holds for the infinitesimal action associated with a complete symplectic realization. As we discussed in Remark 12.20, both of these are instances of nonlinear connections on a Lie algebroid.

In general, a **flat nonlinear A-connection** $h_S : p^*A \to TS$ is a nonlinear connection for which the induced map at the level of sections

$$h_S : \Gamma(A) \to \mathfrak{X}(S),$$

preserves Lie brackets

$$h_S([\alpha_1, \alpha_2]_A) = [h_S(\alpha_1), h_S(\alpha_2)] \qquad (\alpha_1, \alpha_2 \in \Gamma(A)).$$

A flat nonlinear A-connection is often called an **infinitesimal action of the Lie algebroid** A. Generalizing our previous results, parallel transport for a flat nonlinear connection is invariant under the appropriate notion of A-path-homotopy — see Problem 10.1.

This suggests proceeding as in the study case of nondegenerate Poisson structures. To that end, we introduce the **Poisson homotopy groupoid**

$$\Pi(M,\pi) := \frac{\text{cotangent paths}}{\text{cotangent path-homotopy}} \overset{\mathbf{t}}{\underset{\mathbf{s}}{\rightrightarrows}} M$$

where the maps $\mathbf{s}$ (for "source") and $\mathbf{t}$ (for "target") give the initial and end points of the base path:

$$\mathbf{s} : \Pi(M,\pi) \to M, \quad [a] \mapsto \gamma_a(0),$$
$$\mathbf{t} : \Pi(M,\pi) \to M, \quad [a] \mapsto \gamma_a(1),$$

and multiplication is defined by concatenation of cotangent paths:

$$[a] \circ [b] := [a \circ b] \quad \text{if} \quad \mathbf{s}([a]) = \mathbf{t}([b]).$$

We conclude the following:

Proposition 12.28. *The Poisson homotopy groupoid* $\Pi(M, \pi)$ *acts canonically on any complete symplectic realization* $\mu : (S, \omega) \to (M, \pi)$*:*

$$[a] \cdot p := \tau_a(p) \quad \textit{if} \quad \mathbf{s}([a]) = \mu(p) \qquad \begin{array}{ccc} \Pi(M,\pi) & \circlearrowright & (S,\omega) \\ & \searrow\!\!\searrow & \downarrow \mu \\ & & (M,\pi) \end{array}$$

It satisfies the following action-like properties:

(i) *The constant cotangent path* $a(t) \equiv 0_x$ *acts as an identity:*

$$[0_x] \cdot p = p, \quad \forall\, p \in \mu^{-1}(x).$$

(ii) *Whenever* $[a], [b] \in \Pi(M, \pi)$ *are composable one has*