Preface

What is this book about?

This book is about **ultrafilters**. So what is an ultrafilter? Given a set X, an ultrafilter on X is simply a "sensible" division of all of the subsets of X into two categories: small and large. For this division to be sensible, one should impose some axioms:

- X should be a large subset of X, while $\emptyset$ should be a small subset of X.
- If Y is a large subset of X and $Y \subseteq Z \subseteq X$, then Z should also be large; that is, a set containing a large set should also be large.
- If Y and Z are two large subsets of X, then so is $Y \cap Z$.

The last axiom is perhaps not entirely intuitive, but becomes more intuitive when stated in terms of small sets: the union of two small sets is once again small. The axioms also imply that a set is large precisely when its complement is small.

Why write a book about such a seemingly simple notion? It turns out that this notion is very useful for describing limits of various objects. For example, much to the chagrin of many calculus students, one knows that there are many sequences $(a_n)_{n\in\mathbb{N}}$ from $[0,1]$ that have no limit. However, limit in the usual sense is very restrictive in that it requires a_n to be close to the limit for a large number of n, where large here means for all but finitely many n. Note that this restrictive notion of largeness does not lead to an ultrafilter on $\mathbb{N}$ as there are certainly sets that are infinite and which have infinite complement. However, if one works with a notion of largeness as given by an ultrafilter, then all of a sudden *every sequence in* $[0,1]$ *has a* *limit!* This fact can be used as a powerful tool in analytic and topological endeavors.

The notion of ultrafilter also allows one to consider limits of families of structures like groups, rings, graphs, or Banach spaces. The limiting structures alluded to here are called **ultraproducts** and will become a central part of this book. These limiting objects can be very useful in solving problems, for often various desirable properties are approximately true in the individual structures of the family, while in the limit they become exactly true.

Who should read this book?

The short answer is: everyone! More precisely, the thesis of this book is that, while ultrafilters and ultraproducts are often relegated to graduate-level courses in logic, we believe that this practice is entirely misguided. Indeed, the notion of ultrafilter and ultraproduct are entirely accessible to an advanced undergraduate or beginning graduate student in mathematics (the target audience of this book). Moreover, as we will see throughout the course of this book, ultrafilters and ultraproducts have had numerous applications to nearly every area of mathematics. Thus, no matter what area of mathematics the reader is interested in, it is quite likely that ultrafilters and ultraproducts have made an impact in that area. An attempt has been made to present as diverse a sample of such applications as possible.

That being said, this book is being written by a logician, and ultrafilters present numerous fascinating foundational concerns, many of which are discussed in this book. If the reader is purely interested in mathematical applications, they may safely skip the portions of this book discussing these metamathematical issues.

What is in this book?

Let us briefly summarize the contents of this book. Part 1 is entirely devoted to ultrafilters. Chapter 1 introduces the basic facts about ultrafilters, including what it means for them to be isomorphic and how many of them there are. Chapter 2 provides one with a first application of ultrafilters, namely to a proof of Arrow's theorem on fair voting. This application is nice in the sense that it requires little to no mathematical background and yet exemplifies a perfect use of ultrafilters. Chapter 3 introduces the use of ultrafilters in topology, including the aforementioned facts about generalized limits. This chapter also shows how ultrafilters can be used to describe the important Stone-Čech compactification construction. Chapter 4 is a brief introduction to how ultrafilters can be used in certain parts of combinatorics; a much more detailed investigation of that line of research can be found in the book [**42**], written by the author with Mauro Di Nasso and Martino Lupini. Chapter 5, the last chapter in Part 1 of the book, discusses many of the interesting foundational issues presented by the existence of ultrafilters.

Part 2 of the book is concerned with the classical ultraproduct construction. As alluded to above, this construction allows one to take the limit of families of objects such as groups, rings, graphs, etc., ... The lengthy Chapter 6 introduces this construction and proves the Fundamental Theorem of Ultraproducts (otherwise known as Łoś's theorem), which states that the truth of a first-order sentence in an ultraproduct is determined by whether or not the sentence is true in a large (as measured by the ultrafilter) number of the individual structures. This chapter includes many other important facts about ultraproducts, including cardinalities of ultraproducts and a discussion of what happens when one tries to iterate the ultraproduct construction.

Chapter 7 gives one a first look at how ultraproducts can be used "in practice." The applications in this chapter are all algebraic in nature, and include Ax's theorem on polynomial functions and the Ax-Kochen theorem relating the rings $\mathbb{Z}_p$ of p-adic integers with the power series rings $\mathbb{F}_p[[T]]$. One important feature of ultraproducts is that they are often very "rich" in the precise sense of being saturated. Chapter 8 gives a detailed discussion of exactly how saturated ultraproducts can be. Chapter 9 gives a brief introduction to nonstandard analysis. While nonstandard analysis is a subject of its own, it is often presented using ultraproducts and we discuss this approach here. This chapter is far from a complete story on nonstandard analysis and we refer the interested reader to [**42**] for a more thorough discussion. Chapter 10 discusses the class of subgroups of nonstandard (in the sense of Chapter 9) free groups; the finitely generated such subgroups are called limit groups and have become a widely studied class of groups in geometric group theory.

The ultraproduct construction referred to above is suitable for discrete spaces such as those arising in algebra and combinatorics, but is not very useful for structures appearing in analysis. Part 3 of the book is concerned with a modification of the ultraproduct construction for structures based on metric spaces. Chapter 11 introduces this metric ultraproduct and discusses some of its basic properties. That chapter also includes a discussion of a relatively new logic, aptly called continuous logic, which is the logic naturally connected to this metric ultraproduct construction.

The remainder of Part 3 details several applications of the metric ultraproduct construction. Chapter 12 describes a fascinating theorem of Gromov from geometric group theory, where the key ingredient to the proof is a particular metric ultraproduct called an asymptotic cone. Chapter 13 discusses the class of sofic groups, which can be defined in terms of metric ultraproducts of symmetric groups. Chapter 14, the final chapter of Part 3, discusses some applications of metric ultraproducts to functional analysis. One might argue that functional analysis is an area of mathematics where ultraproducts have played an increasingly more important role. Unfortunately, the mathematical background needed by the reader is much larger in this area of mathematics and thus this section cannot quite do justice to the importance of ultraproducts in functional analysis.

Part 4, the last part of this book, is devoted to three advanced topics. Chapter 15 discusses a question that often arises to many people seeing ultraproducts for the first time: does the ultraproduct depend on the ultrafilter being used? The answer to this question is surprisingly subtle and a more or less complete answer to a specific case of this question is discussed. Chapter 16 discusses the fantastic Keisler-Shelah theorem, which shows how elementary equivalence, a notion from logic, can be reformulated in terms of isomorphic ultrapowers, a purely algebraic notion. This chapter also includes a few applications of the Keisler-Shelah theorem. Chapter 17, the final chapter of the book, shows how the study of large cardinals in set theory can be recast in terms of ultrafilters satisfying certain extra properties. This part of the book might require a bit more maturity and/or background from the reader.

What are the prerequisites for reading this book?

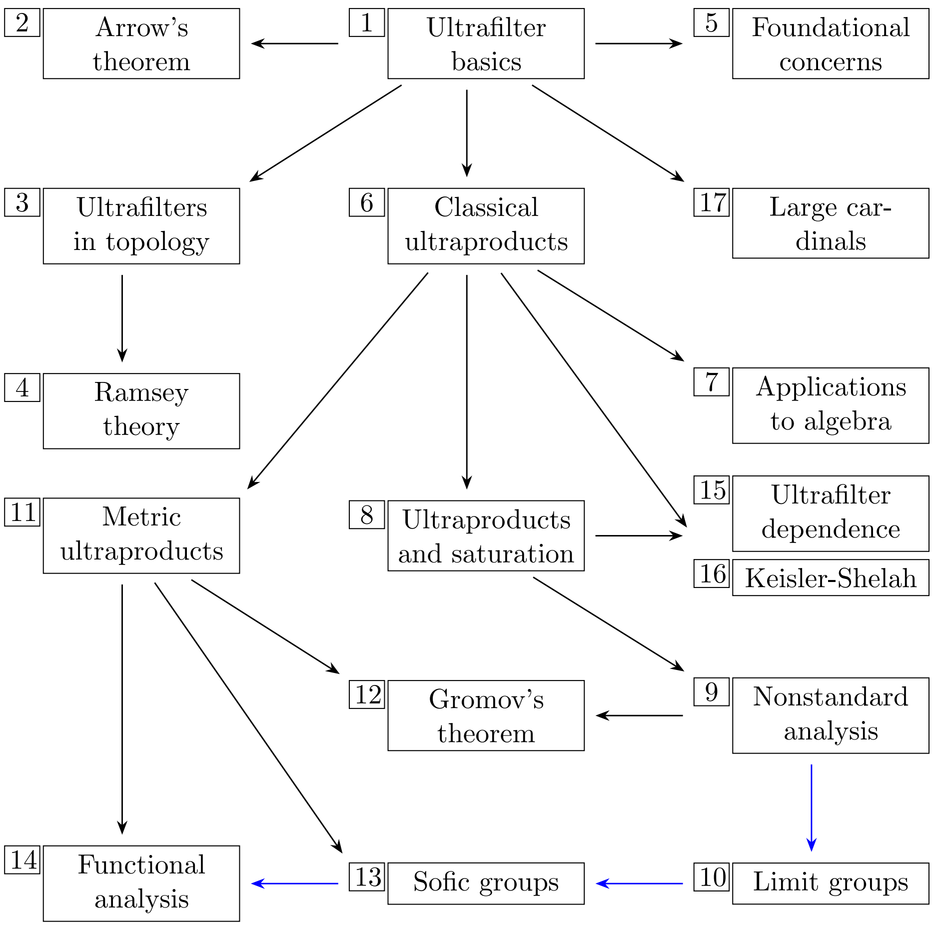
We have no illusions that any one student has all of the prerequisites necessary to read the entire book. However, this fact is by design! As discussed above, we are trying to convey to the reader that ultrafilters and ultraproducts are applicable in most areas of mathematics and thus we have tried to describe a wide variety of applications.

That being said, we have assumed that the reader is familiar with some basic facts from real analysis, topology, and algebra. Any facts that we believe are not part of the usual curricula from those disciplines are often described in full detail here. Sometimes certain topics are outside of the scope of this book and we provide references to the reader for places in the literature where they can learn more. It is also our hope that a reader interested in, for example, algebra sees the chapter on, say, functional analysis, and finds the general idea interesting enough that they decide to learn more about this area. In today's mathematical world, breadth is everything and an aspiring mathematician should keep their eyes open to all areas of mathematics.

In discussing ultrafilters, one cannot hide the fact that logic and set theory play an important role. Moreover, there is a high probability that the average reader might not have the requisite knowledge in these areas to follow the main parts of this book. For the reader's convenience, appendices on these subjects are included in this book. Also, occasionally in the text, very basic parts of category theory are needed and the necessary facts from category theory are collected in the final appendix.

How to read this book

Some later chapters rely somewhat heavily on earlier chapters. The following flowchart lists some of these dependencies. The blue arrows indicate dependencies that are not strictly necessary but possibly helpful.



Exercises

Rather than ending each section or chapter with a list of exercises, we have instead sprinkled them throughout the text itself. Some of the exercises are simply checks for understanding, but others are more involved. Often, the exercises themselves will be used in the proofs of later results. We recommend that the reader stop reading when they encounter an exercise and attempt a solution at that moment. Solutions to a handful of exercises appear in Appendix D but we urge the reader not to consult these solutions unless the situation becomes dire!

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