Preface

These are notes that have been used for an algebraic geometry course at MIT. I had thought about teaching such a course for quite a while, motivated partly by the fact that MIT didn't have very many courses suitable for students who had taken the standard theoretical math classes. I got around to thinking about this seriously twelve years ago and have now taught the subject seven times. I wanted to get to cohomology of $\mathcal{O}$-modules (aka quasicoherent sheaves) in one semester, without presupposing a knowledge of sheaf theory or of much commutative algebra, so it has been a challenge. Fortunately, MIT has many outstanding students who are interested in mathematics. The students and I have made some progress, but much remains to be done. Ideally, one would like the development to be so natural as to seem obvious. Though I haven't tried to put in anything unusual, this has yet to be achieved. And there are too many pages for my taste. To paraphrase Pascal, we haven't had the time to make it shorter.

To cut the material down, I decided to work exclusively with varieties over the complex numbers and to use that restriction freely. Schemes are not discussed. Some people may disagree with these decisions, but I feel that absorbing schemes and general ground fields won't be too difficult for someone who is familiar with complex varieties. Also, I don't go out of my way to state and prove things in their most general form.

If one plans to teach such a course in a single semester, it is essential to keep moving. One can't linger over the topics in the first chapter. To save time, one can replace some proofs with heuristic reasoning or omit them. Proposition 1.8.12 on the order of vanishing of the discriminant is a candidate for some hand-waving, and Lemma 1.9.7 on flex points may be a proof to skip.

Indices can cloud the picture. When that happens, I recommend focussing attention on a low-dimensional case. Schelter's neat proof of Chevalley's Finiteness Theorem is an example. Schelter discovered the proof while studying $\mathbb{P}^1$. That case demonstrates the main point and is a bit easier to follow.

In Chapter 6 on $\mathcal{O}$-modules, all technical points about sheaves are eliminated when one sticks to affine open sets and localizations. Sections over other open sets are important, because one wants the global sections, but the proof that a module extends to arbitrary open sets can safely be put on a back burner.

In Chapter 7, I decided to restrict myself to $\mathcal{O}$-modules when defining cohomology and to characterize the cohomology axiomatically. This was in order to minimize technical points. Simplicial operations are eliminated, though they appear in disguise in the resolution (7.4.13).

The special topics at the ends of Chapters 2, 3, and 4 enrich the subject. I don't recommend skipping them. And, without some of the applications at the end of Chapter 8, the Riemann-Roch Theorem would be pointless.

When I last taught the subject in the spring of 2020, the MIT semester had 39 class hours. I followed this schedule: Chapter 1, 6 hours; Chapters 2–7, roughly 4 hours each; Chapter 8, 7 hours; in-class quizzes, 2 hours. This was a brisk pace. The topics in the notes could be covered comfortably in a one-year course, and there might be time for some extra material.

Great thanks are due to my students. Many of you contributed to these notes by commenting on the drafts or by creating figures. I'm not naming you individually because I'm sure I'd overlook someone important. I hope that you will understand. Edgar Costa, Sam Schiavone, and Raymond van Bommel used the notes and made helpful comments. And I want to thank Arlene O'Sean for her careful and expert editing of the manuscript.

A Note for the Student

The prerequisites are standard undergraduate courses in algebra, analysis, and topology and the definitions of category and functor. We will also use the implicit function theorem for complex variables. But don't worry too much about the prerequisites. There is a review of some points in Chapter 9. You can refer to it as needed or look on the web. And, many points are reviewed briefly in the notes as they come up.

Proofs of some lemmas and propositions are omitted. I have omitted a proof when I consider it simple enough that including it would just clutter up the text or, occasionally, when I feel that it is important for the reader to supply a proof.

As with all mathematics, working exercises and, most importantly, writing up the solutions carefully is, by far, the best way to learn the material well.