Chapter 8

The Riemann-Roch Theorem for Curves

We study a classical problem of algebraic geometry, to determine the rational functions with given poles on a smooth projective curve. This is often difficult. If $p_1, \dots, p_k$ are points of the curve, the rational functions whose poles have orders at most r_i at p_i form a vector space, and one is happy when one can determine the dimension of that space. The most important tool for determining the dimension is the Riemann-Roch Theorem.

8.1. Divisors

Smooth affine curves were discussed in Chapter 5. It was shown there that an affine curve is smooth if its local rings are valuation rings or if its coordinate ring is a normal domain. An arbitrary curve is smooth if it has an open covering by smooth affine curves.

We take a brief look at modules on a smooth curve. Recall that a module M over a domain A is torsion-free if its only torsion element is zero: If $a \in A$ and $m \in M$ are nonzero, then $am \neq 0$. This definition is extended to $\mathcal{O}$-modules by applying it to the affine open subsets.

8.1.1. Lemma. *Let Y be a smooth curve.*

(i) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is locally free if and only if it is torsion-free.*

(ii) *An $\mathcal{O}_Y$-module $\mathcal{M}$ that isn't torsion-free has a nonzero global section.*

Proof of Lemma 8.1.1(i). We may assume that Y is affine, $Y = \operatorname{Spec} B$, and that $\mathcal{M}$ is the $\mathcal{O}$-module associated to a B-module M. Let $\widetilde{B}$ be the local ring of B at a point q, and let $\widetilde{M}$ be the localization of M at q, which is isomorphic to the tensor product $M \otimes_B \widetilde{B}$. If M is a torsion-free B-module, then $\widetilde{M}$ is a torsion-free module over the valuation ring $\widetilde{B}$. It suffices to show that, for every point q of Y, $\widetilde{M}$ is a free $\widetilde{B}$-module (see Subsection 2.6.13). The next sublemma does this.

8.1.2. Sublemma. *A finite, torsion-free module $\widetilde{M}$ over a valuation ring $\widetilde{B}$ is a free module.*

Proof. It is easy to prove this directly. Or, one can use the fact that every finite, torsion-free module over a principal ideal domain is free. A valuation ring is a principal ideal domain because its nonzero ideals are powers of its maximal ideal, and the maximal ideal is a principal ideal. □

Proof of Lemma 8.1.1(ii). If $\mathcal{M}$ isn't torsion-free, then on some affine open subset U, there will be nonzero elements m in $\mathcal{M}(U)$ and a in $\mathcal{O}(U)$, such that $am = 0$. Let Z be the finite set of zeros of a in U. We choose an affine open set V that doesn't contain any points of Z and such that $Y = U \cup V$. The proof of existence of such an open set is Exercise 5.21. Then a is invertible on the intersection $W = U \cap V$, and since $am = 0$, the restriction of m to W is zero.

The open sets U and V cover Y, and the sheaf property for this covering can be written as the exact sequence

$$0 \to \mathcal{M}(Y) \to \mathcal{M}(U) \times \mathcal{M}(V) \xrightarrow{-,+} \mathcal{M}(W)$$

(see Note 6.3.6). In this sequence, the section $(m, 0)$ of $\mathcal{M}(U) \times \mathcal{M}(V)$ maps to zero in $\mathcal{M}(W)$. Therefore it is the image of a nonzero global section of $\mathcal{M}$. □

8.1.3. Lemma. *Let Y be a smooth curve. Every nonzero ideal $\mathcal{I}$ of $\mathcal{O}_Y$ is a product of powers of maximal ideals: $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$.*

Proof. This follows for any smooth curve from the case that the curve is affine, which is Proposition 5.2.11. □

8.1.4. Divisors. A *divisor* on a smooth curve Y is a finite combination

$$D = r_1 q_1 + \cdots + r_k q_k$$

where r_i are integers and q_i are points. It is an element of the abelian group that has the points of Y as a $\mathbb{Z}$-basis.

The *degree* of the divisor D is the sum $r_1 + \cdots + r_k$ of the coefficients. The *support* of D is the set of points q_i such that $r_i \neq 0$.

The *restriction* of a divisor $D = r_1 q_1 + \cdots + r_k q_k$ to an open subset Y' is the divisor obtained from D by deleting points of the support that aren't in Y'. For example, let $D = q$. The restriction to Y' is q if q is in Y', and it is zero if q is not in Y'.

A divisor $D = \sum r_i q_i$ is *effective* if all of its coefficients r_i are nonnegative, and D is *effective on an open set* Y' if its restriction to Y' is effective—if $r_i \geq 0$ when q_i is a point of Y'.

Let $D = \sum r_i p_i$ and $E = \sum s_i p_i$ be divisors. We my write $E \geq D$ if $s_i \geq r_i$ for all i or if $E - D$ is effective. With this notation, $D \geq 0$ means that D is effective.

8.1.5. The divisor of a function. The *divisor of* a nonzero rational function f on a smooth curve Y is

$$\text{div}(f) = \sum_{q \in Y} \text{v}_q(f) q$$

where v_q is the valuation of K associated to the point q. The divisor is written here as a sum over all points q, but it becomes a finite sum when we disregard terms with coefficient zero, because f has finitely many zeros and poles. The coefficients will be zero at all other points.

The map

$$K^\times \xrightarrow{\text{div}} (\text{divisors})^+ \tag{8.1.6}$$

that sends a nonzero rational function to its divisor is a homomorphism from the multiplicative group $K^\times$ of nonzero elements of K to the additive group of divisors:

$$\text{div}(fg) = \text{div}(f) + \text{div}(g).$$

The divisor of a rational function is a *principal divisor*. The image of the map (8.1.6) is the set of principal divisors.

As before, if r is a positive integer, a nonzero rational function f has a *zero of order* r at q if $\text{v}_q(f) = r$, and it has a *pole of order* r at q if $\text{v}_q(f) = -r$. The divisor of f is the difference of two effective divisors:

$$\text{div}(f) = \text{zeros}(f) - \text{poles}(f).$$

A rational function f is regular on Y if and only if $\text{div}(f)$ is effective—if and only if $\text{poles}(f) = 0$.

Every divisor is locally principal: There is an affine covering $\{Y^i\}$ of Y such that the restriction of D is a principal divisor on each Y^i. This is true because the maximal ideals of Y are locally principal. If f is a generator of the maximal ideal at a point q, then $\text{div}(f) = q$.

Two divisors D and E are said to be *linearly equivalent* if their difference $D - E$ is a principal divisor. For instance, the divisors $\text{zeros}(f)$ and $\text{poles}(f)$ of a rational function f are linearly equivalent.

8.1.7. Lemma. *Let f be a rational function on a smooth curve Y. For all complex numbers c, the divisors* $\text{zeros}(f - c)$*, the level sets of f, are linearly equivalent.*

Proof. The functions $f - c$ have the same poles as f. □

8.1.8. Review of terminology. *Divisor*: An integer combination of points: $D = r_1 q_1 + \cdots + r_k q_k$.

Effective divisor: The divisor D is effective if $r_i \geq 0$ for all i.

Principal divisor: The divisor of a rational function. The divisor of f is the sum $\sum_q \mathrm{v}_q(f) q$.

Linearly equivalent divisors: Two divisors D and E are linearly equivalent if $D - E$ is a principal divisor.

Restriction of the divisor D to an open set: The restriction to the open set U is the sum of the terms $r_i q_i$ such that q_i is a point of U.

Support of the divisor D: The points q_i such that $r_i \neq 0$.

Zeros and poles of a divisor: The zeros of the divisor D are the points q_i such that $r_i > 0$. The poles are the points q_i such that $r_i < 0$.

8.1.9. The module $\mathcal{O}(D)$. To analyze functions with given poles on a smooth curve Y, we associate an $\mathcal{O}$-module $\mathcal{O}(D)$ to a divisor D. It is an analog of the module $\mathcal{O}(nH)$ on projective space. The nonzero sections of $\mathcal{O}(D)$ on an open set V are the rational functions f such that the divisor $\mathrm{div}(f) + D$ is effective on V—such that its restriction to V is effective:

$$(8.1.10) \qquad \begin{aligned} (V) &= \{\, f \mid \mathrm{div}(f) + D \text{ is effective on } V \,\} \cup \{0\} \\ &= \{\, f \mid \mathrm{poles}(f) \leq D \text{ on } V \,\} \cup \{0\}. \end{aligned}$$

So $\mathcal{O}(D)$ is a submodule of the function field module $\mathcal{F}$. When the divisor D is effective, the global sections of $\mathcal{O}(D)$ are the rational functions whose poles are bounded by D. They are the solutions of the classical problem mentioned at the beginning of the chapter.

Let D be the divisor $\sum r_i q_i$. If an open set V contains q_i and if $r_i > 0$, a section of $\mathcal{O}(D)$ on V may have a pole of order at most r_i at q_i, and if $r_i < 0$, a section must have a zero of order at least $-r_i$ at q_i. For example, the sections of the module $\mathcal{O}(-q)$ on an open set V that contains q are the regular functions on V that are zero at q. So $\mathcal{O}(-q)$ is the maximal ideal $\mathfrak{m}_q$. Similarly, the sections of $\mathcal{O}(q)$ on an open set V that contains q are the rational functions that have a pole of order at most 1 at q and are regular at every other point of V. The sections of $\mathcal{O}(-q)$ and of $\mathcal{O}(q)$ on an open set V that doesn't contain p are the regular functions on V. Points that aren't in an open set V impose no conditions on sections. A section of $\mathcal{O}(D)$ on V can have arbitrary zeros or poles at points not in V.

The fact that a section of $\mathcal{O}(D)$ is allowed to have a pole at q_i when $r_i > 0$ contrasts with the divisor of a function. If $\mathrm{div}(f) = \sum r_i q_i$, then $r_i > 0$ means that f has a zero at q_i. When $\mathrm{div}(f) = D$, f will be a global section of $\mathcal{O}(-D)$.

8.1.11. Lemma.

(i) *Let D be the principal divisor* $\operatorname{div}(g)$. *Then $\mathcal{O}(D)$ is the free $\mathcal{O}$-module of rank* 1 *with basis g^{-1}.*

(ii) *For any divisor D on a smooth curve, $\mathcal{O}(D)$ is a locally free module of rank* 1.

Proof.

(i) Let D be the divisor of a rational function g. The sections of $\mathcal{O}(D)$ on an open set U are the rational functions f such that $\operatorname{div}(f)+D = \operatorname{div}(fg) \geq 0$ on U—the functions f such that fg is a section of $\mathcal{O}$ on U or such that f is a section of $g^{-1}\mathcal{O}$.

(ii) Every divisor is locally principal. □

8.1.12. Proposition. *Let D and E be divisors on a smooth curve Y.*

(i) *The map $\mathcal{O}(D) \otimes_{\mathcal{O}} \mathcal{O}(E) \to \mathcal{O}(D+E)$ that sends $f \otimes g$ to the product fg is an isomorphism.*

(ii) *$\mathcal{O}(D) \subset \mathcal{O}(E)$ if and only if $E \geq D$.*

Proof.

(i) It is enough to verify this locally, so we may assume that Y is affine and that the supports of D and E contain just one point, say $D = rq$ and $E = sq$. We may also assume that the maximal ideal at q is a principal ideal, generated by an element x. Then $\mathcal{O}(D)$, $\mathcal{O}(E)$, and $\mathcal{O}(D+E)$ will be free modules with bases x^r, x^s, and x^{r+s}, respectively. □

8.1.13. Proposition. *Let Y be a smooth curve.*

(i) *The nonzero ideals of $\mathcal{O}_Y$ are the modules $\mathcal{O}(-E)$, where E is an effective divisor.*

(ii) *The modules $\mathcal{O}(D)$ are the finite $\mathcal{O}$-submodules of the function field module $\mathcal{F}$ of Y.*

(iii) *The function field module $\mathcal{F}$ is the union of the modules $\mathcal{O}(D)$.*

Proof.

(i) Say that $E = r_1q_1 + \cdots + r_kq_k$ and that $r_i \geq 0$ for all i. A rational function f is a section of $\mathcal{O}(-E)$ if $\operatorname{div}(f) - E$ is effective, which happens when $\operatorname{poles}(f) = 0$ and $\operatorname{zeros}(f) \geq E$. The same condition describes the elements of the ideal $\mathcal{I} = \mathfrak{m}_1^{r_1} \cdots \mathfrak{m}_k^{r_k}$.

(ii) Let $\mathcal{L}$ be a finite $\mathcal{O}$-submodule of $\mathcal{F}$. Since $\mathcal{L}$ is a finite $\mathcal{O}$-module, then because the local ring is a valuation ring, $\mathcal{L}$ will be generated by one element, a rational function f, in some open neighborhood U of a point

q. If D is the divisor of f^{-1} on U, then $\mathcal{L} = \mathcal{O}(D)$ on U. This determines the divisor D uniquely. If D_1 and D_2 are divisors and $D_1 \neq D_2$, then $\mathcal{O}(D_1) \neq \mathcal{O}(D_2)$. So when $\mathcal{L} = \mathcal{O}(D)$ on U and $\mathcal{L} = \mathcal{O}(D')$ on U', D and D' must agree on $U \cap U'$. Therefore there is a divisor D on the whole curve Y such that $\mathcal{L} = \mathcal{O}(D)$ in a suitable neighbohood U of any point q. This implies that $\mathcal{L} = \mathcal{O}(D)$. □

8.1.14. Proposition. *Let D and E be divisors on a smooth curve Y. Multiplication by a rational function f such that* $\operatorname{div}(f) + E - D \geq 0$ *defines a homomorphism of $\mathcal{O}$-modules $\mathcal{O}(D) \to \mathcal{O}(E)$, and every homomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$ is multiplication by such a function.*

Proof. For any $\mathcal{O}$-module $\mathcal{M}$, a homomorphism $\mathcal{O} \to \mathcal{M}$ is multiplication by a global section of $\mathcal{M}$ (see Subsection 6.4.1). So a homomorphism $\mathcal{O} \to \mathcal{O}(E-D)$ will be multiplication by a rational function f such that $\operatorname{div}(f) + E - D \geq 0$. If f is such a function, one obtains a homomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$ by tensoring with $\mathcal{O}(D)$. □

8.1.15. Corollary. *Let D and E be divisors on a smooth curve Y.*

(i) *The modules $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are isomorphic if and only if D and E are linearly equivalent.*

(ii) *Let f be a rational function on Y, and let $D = \operatorname{div}(f)$. Multiplication by f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$.*

Proof. If multiplication by a rational function f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}(E)$, the inverse morphism is defined by f^{-1}. Then $\operatorname{div}(f) + E - D \geq 0$ and also $\operatorname{div}(f^{-1}) + D - E = -\operatorname{div}(f) + D - E \geq 0$, so $\operatorname{div}(f) = D - E$. This proves (i), and (ii) is the special case that $E = 0$. □

8.1.16. Invertible modules. An *invertible* $\mathcal{O}$-module is a locally free module of rank 1—a module that is isomorphic to the module $\mathcal{O}$ in a neighborhood of any point. If D is a divisor on a smooth curve Y, then $\mathcal{O}(D)$ is an invertible $\mathcal{O}$-module. The tensor product $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{M}$ of invertible modules is invertible.

8.1.17. Lemma. *Let $\mathcal{L}$ be an invertible $\mathcal{O}$-module on a smooth curve Y, and let $\mathcal{L}^*$ be the dual module.*

(i) *The canonical map $\mathcal{L}^* \otimes_{\mathcal{O}} \mathcal{L} \to \mathcal{O}$ defined by $\gamma \otimes \alpha \mapsto \gamma(\alpha)$ is an isomorphism.*

(ii) *The map $\mathcal{O} \to {}_{\mathcal{O}}(\mathcal{L},\mathcal{L})$ $\big(= \underline{\operatorname{Hom}}_{\mathcal{O}}(\mathcal{L},\mathcal{L})\big)$ that sends a regular function α to multiplication by α is an isomorphism.*

(iii) *Every nonzero homomorphism $\mathcal{L} \xrightarrow{\varphi} \mathcal{M}$ to a locally free module $\mathcal{M}$ is injective.*

Because of (i), $\mathcal{L}^*$ may be thought of as an inverse to $\mathcal{L}$. This is the reason for the term 'invertible'. The dual of $\mathcal{O}(D)$ is $\mathcal{O}(-D)$.

Proof of Lemma 8.1.17.

(i), (ii) It is enough to verify these assertions in the case that $\mathcal{L}$ is free, isomorphic to $\mathcal{O}$, in which case they are clear.

(iii) The problem is local, so we may assume that the variety is affine, say $Y = \operatorname{Spec} A$, and that $\mathcal{L}$ and $\mathcal{M}$ are free. Then φ becomes a nonzero homomorphism $A \to A^k$, which is injective because A is a domain. □

As Proposition 8.1.13(ii) shows, the only difference between an invertible module $\mathcal{L}$ and a module $\mathcal{O}(D)$ is that $\mathcal{O}(D)$ is a submodule of the function field module $\mathcal{F}$, while $\mathcal{L} \otimes_{\mathcal{O}} \mathcal{F}$ can be any one-dimensional vector space over the function field.

8.1.18. Corollary. *Every invertible $\mathcal{O}$-module $\mathcal{L}$ on a smooth curve Y is isomorphic to one of the form $\mathcal{O}(D)$.* □

If Y is a smooth projective curve, the *degree* of an invertible module $\mathcal{L}$ on Y is defined to be the degree of the divisor D such that $\mathcal{L} \approx \mathcal{O}(D)$.

8.2. The Riemann-Roch Theorem I

Let Y be a smooth projective curve, and let $\mathcal{M}$ be a finite $\mathcal{O}_Y$-module. We have seen that the cohomology $H^q(Y, \mathcal{M})$ is a finite-dimensional vector space for $q = 0, 1$ and that it is zero when $q > 1$ (see Theorem 7.7.3 and Theorem 7.7.1). As before, we denote the dimension of $H^q(Y, \mathcal{M})$ by $\mathbf{h}^q\mathcal{M}$ or $\mathbf{h}^q(Y, \mathcal{M})$.

The Euler characteristic (7.6.6) of a finite $\mathcal{O}$-module $\mathcal{M}$ is

(8.2.1) $$\chi(\mathcal{M}) = \mathbf{h}^0\mathcal{M} - \mathbf{h}^1\mathcal{M}.$$

In particular,

$$\chi(\mathcal{O}_Y) = \mathbf{h}^0\mathcal{O}_Y - \mathbf{h}^1\mathcal{O}_Y.$$

The dimension $\mathbf{h}^1\mathcal{O}_Y$ is the arithmetic genus p_a of Y (see Section 7.6). We will see below that $\mathbf{h}^0\mathcal{O}_Y = 1$ (see Corollary 8.2.9). So

(8.2.2) $$\chi(\mathcal{O}) = 1 - p_a.$$

8.2.3. Theorem (Riemann-Roch Theorem (Version 1)). *Let $D = \sum r_i p_i$ be a divisor on a smooth projective curve Y. Then*

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D \quad \big(= \deg D + 1 - p_a\big).$$

Proof. We analyze the effect on cohomology when a divisor is changed by adding or subtracting a point, by inspecting the inclusion $\mathcal{O}(D-p) \subset \mathcal{O}(D)$.

The cokernel ϵ of the inclusion map is a one-dimensional vector space supported at p, isomorphic to the residue field module κ_p. We'll write the cokernel as κ_p, though the identification of ϵ with κ_p requires choosing a basis of the one-dimensional module ϵ.

So there is a short exact sequence

$$0 \to \mathcal{O}(D-p) \to \mathcal{O}(D) \to \kappa_p \to 0. \tag{8.2.4}$$

This sequence can be obtained by tensoring the sequence

$$0 \to \mathfrak{m}_p \to \mathcal{O} \to \kappa_p \to 0 \tag{8.2.5}$$

with the invertible module $\mathcal{O}(D)$, because $\mathfrak{m}_p = \mathcal{O}(-p)$.

Since the support of κ_p has dimension zero, $H^1(\kappa_p) = 0$, and $H^0(\kappa_p) = \mathbb{C}$. Let's denote the one-dimensional vector space $H^0(Y, \kappa_p)$ by [1]. The cohomology sequence associated to (8.2.4) is

$$\begin{aligned} 0 \to H^0(Y, \mathcal{O}(D-p)) \to H^0(Y, \mathcal{O}(D)) &\xrightarrow{\gamma} [1] \\ \xrightarrow{\delta} H^1(Y, \mathcal{O}(D-p)) \to H^1(Y, \mathcal{O}(D)) &\to 0. \end{aligned} \tag{8.2.6}$$

In this exact sequence of vector spaces, one of the maps γ or δ must be zero. Either

- γ is zero and δ is injective. In this case

$$\mathbf{h}^0\mathcal{O}(D-p) = \mathbf{h}^0\mathcal{O}(D) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D) + 1 \quad \text{or}$$

- δ is zero and γ is surjective. In this case

$$\mathbf{h}^0\mathcal{O}(D-p) = \mathbf{h}^0\mathcal{O}(D) - 1 \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D).$$

In either case,

$$\chi(\mathcal{O}(D)) = \chi(\mathcal{O}(D-p)) + 1. \tag{8.2.7}$$

We also have $\deg D = \deg(D-p)+1$. The Riemann-Roch Theorem follows. It is true for the module $\mathcal{O}$, and we can get from $\mathcal{O}$ to $\mathcal{O}(D)$ by a finite number of operations, each of which changes the divisor by adding or subtracting a point. Therefore it is true for all D. □

Because $\mathbf{h}^0 \geq \mathbf{h}^0 - \mathbf{h}^1 = \chi$, this version of the Riemann-Roch Theorem gives reasonably good control of H^0. It is less useful for controlling H^1. For that, one wants the full Riemann-Roch Theorem, which we call Version 2. The full theorem requires some preparation, so we have put it into Section 8.7. However, Version 1 has important consequences:

8.2.8. Corollary. *Let p be a point of a smooth projective curve Y. The dimension $\mathbf{h}^0(Y, \mathcal{O}(np))$ tends to infinity with n. There exist rational functions on Y that have a pole of some order at a single point p and no other poles.*

Proof. The sequence (8.2.6) shows that when we go from $\mathcal{O}(np)$ to $\mathcal{O}((n+1)p)$, either $\mathbf{h}^0$ increases or else $\mathbf{h}^1$ decreases. Since $\mathbf{h}^1\mathcal{O}(np)$ is finite, the second possibility can occur only finitely often, as n tends to ∞. □

8.2.9. Corollary. *Let Y be a smooth projective curve.*

(i) *The divisor of a rational function on Y has degree 0: The number of zeros is equal to the number of poles.*
(ii) *Linearly equivalent divisors on Y have equal degrees.*
(iii) *A nonconstant rational function on Y takes every value, including infinity, the same number of times, when counted with multiplicity.*
(iv) *A rational function on Y that is regular at every point of Y is a constant: $H^0(Y,\mathcal{O}) = \mathbb{C}$.*

Proof.

(i) Let f be a nonzero rational function and let $D = \operatorname{div}(f)$. Multiplication by f defines an isomorphism $\mathcal{O}(D) \to \mathcal{O}$ (see Corollary 8.1.15), so $\chi(\mathcal{O}(D)) = \chi(\mathcal{O})$. On the other hand, by Riemann-Roch, $\chi(\mathcal{O}(D)) = \chi(\mathcal{O}) + \deg D$. Therefore $\deg D = 0$.
(ii) If D and E are linearly equivalent divisors, $\mathcal{O}(D)$ and $\mathcal{O}(E)$ are isomorphic.
(iii) The divisor of zeros of the function $f - c$ is linearly equivalent to the divisor of poles of f.
(iv) According to (iii), a nonconstant function must have a pole. □

8.2.10. Corollary. *Let D be a divisor on Y. If $\deg D \geq p_a$, then $\mathbf{h}^0\mathcal{O}(D) > 0$. If $\mathbf{h}^0\mathcal{O}(D) > 0$, then $\deg D \geq 0$.*

When the degree d of D is in the range $0 < d < p_a$, $\mathbf{h}^0$ depends on the particular divisor.

Proof of Corollary 8.2.10. If $\deg D \geq p_a$, then $\chi(\mathcal{O}(D)) = \deg D + 1 - p_a \geq 1$, and $\mathbf{h}^0 \geq \mathbf{h}^0 - \mathbf{h}^1 = \chi$. If $\mathcal{O}(D)$ has a nonzero global section f, a rational function such that $\operatorname{div}(f) + D$ is effective, then $\deg(\operatorname{div}(f) + D) \geq 0$, and because the degree of $\operatorname{div}(f)$ is 0, $\deg D \geq 0$. □

8.2.11. Theorem. *With its classical topology, a smooth projective curve Y is a connected, compact, orientable two-dimensional manifold.*

Proof. We prove connectedness here. The other points have been explained before in the case of a plane curve (Theorem 1.7.21), and the proofs for any projective curve are no different.

A topological space is connected if it isn't the union of two disjoint, nonempty, closed subsets. Suppose that, in the classical topology, Y is the union of disjoint, nonempty closed subsets Y_1 and Y_2. Both Y_1 and Y_2 will be compact, two-dimensional manifolds. Let p be a point of Y_1. There is a nonconstant rational function f whose only pole is at p (Corollary 8.2.8). Then f will be a regular function on the complement of p and therefore a regular function on the entire compact manifold Y_2.

For review: A point q of the smooth curve Y has a neighborhood V that is analytically equivalent to an open subset U of the affine line X. If a function g on V is analytic, the function on U that corresponds to g is an analytic function of one variable. The maximum principle for analytic functions asserts that, on an open region of the complex plane, the absolute value of a nonconstant analytic function has no maximum. This applies to the open set U and therefore also to the neighborhood V of q. Since q can be any point of Y_2, a nonconstant analytic function cannot have a maximum anywhere on Y_2. On the other hand, since Y_2 is compact, a continuous function does have a maximum. So a function that is analytic on Y_2 must be a constant.

Going back to the rational function f with a single pole p, the restriction of f to Y_2 will be analytic and therefore constant. When we subtract that constant from f, we obtain a nonconstant rational function on Y with a single pole p that is zero on Y_2. But a rational function on a curve has finitely many zeros. This is a contradiction. □

8.3. The Birkhoff-Grothendieck Theorem

This theorem describes the finite, torsion-free modules on the projective line $X = \mathbb{P}^1$.

8.3.1. Theorem (Birkhoff-Grothendieck Theorem). *A finite, locally free (or torsion-free) $\mathcal{O}$-module on the projective line X is isomorphic to a direct sum of twisting modules: $\mathcal{M} \approx \bigoplus \mathcal{O}(n_i)$.*

This theorem was proved by Grothendieck using cohomology. It had been proved earlier by Birkhoff, in the following equivalent form:

Birkhoff Factorization Theorem. *Let $A_0 = \mathbb{C}[u]$, $A_1 = \mathbb{C}[u^{-1}]$, and $A_{01} = \mathbb{C}[u, u^{-1}]$. Let P be an invertible A_{01}-matrix. There exist an invertible A_0-matrix Q_0 and an invertible A_1-matrix Q_1 such that $Q_0^{-1}PQ_1$ is diagonal and its diagonal entries are integer powers of u.*

Proof of the Birkhoff-Grothendieck Theorem. This is Grothendieck's proof.

According to Theorem 7.5.5, the cohomology of the twisting modules on the projective line X is $\mathbf{h}^0\mathcal{O} = 1$, $\mathbf{h}^1\mathcal{O} = 0$, and if r is a positive integer,

$$\mathbf{h}^0\mathcal{O}(r) = r + 1, \quad \mathbf{h}^1\mathcal{O}(r) = 0, \quad \mathbf{h}^0\mathcal{O}(-r) = 0, \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(-r) = r - 1.$$

8.3.2. Lemma. *Let $\mathcal{M}$ be a finite, locally free $\mathcal{O}$-module on X. For sufficiently large r,*

(i) *the only homomorphism $\mathcal{O}(r) \to \mathcal{M}$ is the zero map, and*

(ii) $\mathbf{h}^0(X, \mathcal{M}(-r)) = 0$.

Proof.

(i) A nonzero homomorphism $\mathcal{O}(r) \xrightarrow{\varphi} \mathcal{M}$ from the twisting module $\mathcal{O}(r)$ to the locally free module $\mathcal{M}$ will be injective (see Lemma 8.1.17), and the associated map $H^0(X, \mathcal{O}(r)) \to H^0(X, \mathcal{M})$ will be injective too, so $\mathbf{h}^0(X, \mathcal{O}(r)) \le \mathbf{h}^0(X, \mathcal{M})$. Since $\mathbf{h}^0(X, \mathcal{O}(r)) = r + 1$, r is bounded by the integer $\mathbf{h}^0(X, \mathcal{M}) - 1$.

(ii) A global section of $\mathcal{M}(-r)$ defines a map $\mathcal{O} \to \mathcal{M}(-r)$. Its twist by r will be a map $\mathcal{O}(r) \to \mathcal{M}$. By (i), r is bounded. □

We go to the proof now. We use induction on the rank. We suppose that $\mathcal{M}$ has rank r, that $r > 0$, and that the theorem has been proved for locally free $\mathcal{O}$-modules of rank less than r. The plan is to show that $\mathcal{M}$ has a twisting module as a direct summand, so that $\mathcal{M} = \mathcal{W} \oplus \mathcal{O}(n)$ for some $\mathcal{W}$. Then induction on the rank, applied to $\mathcal{W}$, will prove the theorem.

Twisting is compatible with direct sums, so we may replace $\mathcal{M}$ by a twist $\mathcal{M}(n)$. Instead of showing that $\mathcal{M}$ has a twisting module $\mathcal{O}(n)$ as a direct summand, we show that, after we replace $\mathcal{M}$ by a suitable twist, the structure sheaf $\mathcal{O}$ will be a direct summand.

The twist $\mathcal{M}(n)$ will have a nonzero global section when n is sufficiently large (see Theorem 6.7.21), and it will have no nonzero global section when n is sufficiently negative (see Lemma 8.3.2(ii)). Therefore, when we replace $\mathcal{M}$ by a suitable twist, we will have $H^0(X, \mathcal{M}) \neq 0$ but $H^0(X, \mathcal{M}(-1)) = 0$. We assume that this is true for $\mathcal{M}$.

We choose a nonzero global section m of $\mathcal{M}$ and consider the injective multiplication map $\mathcal{O} \xrightarrow{m} \mathcal{M}$. Let $\mathcal{W}$ be its cokernel, so that we have a short exact sequence

(8.3.3) $$0 \to \mathcal{O} \xrightarrow{m} \mathcal{M} \xrightarrow{\pi} \mathcal{W} \to 0.$$

8.3.4. Lemma. *Let $\mathcal{W}$ be the $\mathcal{O}$-module that appears in the sequence* (8.3.3)*.*

(i) $H^0(X, \mathcal{W}(-1)) = 0$.

(ii) *$\mathcal{W}$ is torsion-free and therefore locally free.*

(iii) *$\mathcal{W}$ is isomorphic to a direct sum $\bigoplus_{i=1}^{r-1} \mathcal{O}(n_i)$ of twisting modules on $\mathbb{P}^1$, with $n_i \le 0$.*

Proof.

(i) This follows from the cohomology sequence associated to the twisted sequence

$$0 \to \mathcal{O}(-1) \to \mathcal{M}(-1) \to \mathcal{W}(-1) \to 0$$

because $H^0(X, \mathcal{M}(-1)) = 0$ and $H^1(X, \mathcal{O}(-1)) = 0$.

(ii) If the torsion submodule of $\mathcal{W}$ were nonzero, the torsion submodule of $\mathcal{W}(-1)$ would also be nonzero, and then $\mathcal{W}(-1)$ would have a nonzero global section (see Lemma 8.1.1).

(iii) The fact that $\mathcal{W}$ is a direct sum of twisting modules follows by induction on the rank: $\mathcal{W} \approx \bigoplus \mathcal{O}(n_i)$. Since $H^0(X, \mathcal{W}(-1)) = 0$, we must have $H^0(X, \mathcal{O}(n_i - 1)) = 0$. Therefore $n_i - 1 < 0$, and $n_i \le 0$. □

We go back to the proof of Theorem 8.3.1 and dualize the sequence (8.3.3). Because $\mathcal{O}^* = \mathcal{O}$, the dual sequence is an exact sequence

$$0 \to \mathcal{W}^* \xrightarrow{\pi^*} \mathcal{M}^* \xrightarrow{m^*} \mathcal{O} \to 0.$$

In it, $\mathcal{W}^* \approx \bigoplus \mathcal{O}(n_i)^* \approx \bigoplus \mathcal{O}(-n_i)$, and $-n_i \ge 0$ for all i. Therefore $\mathbf{h}^1 \mathcal{W}^* = 0$. The map $H^0(\mathcal{M}^*) \to H^0(\mathcal{O})$ is surjective. Let α be a global section of $\mathcal{M}^*$ whose image in $\mathcal{O}$ is 1. Multiplication by α defines a map $\mathcal{O} \xrightarrow{\alpha} \mathcal{M}^*$ that splits the sequence, i.e., such that $m^*\alpha$ is the identity map on $\mathcal{O}$. Then $\mathcal{M}^*$ is the direct sum $\operatorname{im}(\alpha) \oplus \ker(m^*) \approx \mathcal{O} \oplus \mathcal{W}^*$. Therefore $\mathcal{M} \approx \mathcal{W} \oplus \mathcal{O}$. □

8.4. Differentials

We introduce differentials and branched coverings, because they will be used in Version 2 of the Riemann-Roch Theorem. Why differentials enter into the Riemann-Roch Theorem is something of a mystery, but one important fact is the *Residue Theorem*, which controls the poles of a rational differential. Proofs of Riemann-Roch are often based on the Residue Theorem. We recommend reading about it, though we won't use it. [1]

Try not to get bogged down in the preliminary disussions here. Give the next pages a quick read to learn the terminology. You can look back as needed. Begin to read carefully when you get to Section 8.6.

Let A be an algebra and let M be an A-module. A *derivation* $A \xrightarrow{\delta} M$ is a $\mathbb{C}$-linear map that satisfies the product rule for differentiation, a map that has these properties:

(8.4.1) $$\delta(ab) = a\,\delta b + b\,\delta a, \quad \delta(a+b) = \delta a + \delta b, \quad \text{and} \quad \delta c = 0$$

[1] See one of the books by Fulton, Miranda, or Mumford in the bibliography, or, for a general treatment, Tate, J., Residues of differentials on curves, Ann. Sci. ENS 1968.

for all a, b in A and all c in $\mathbb{C}$. The fact that δ is $\mathbb{C}$-linear, i.e., that it is a homomorphism of vector spaces, follows: Since $\delta c = 0$, $\delta(cb) = c\,\delta b + b\,\delta c = c\,\delta b$.

For example, differentiation $\frac{d}{dt}$ is a derivation $\mathbb{C}[t] \to \mathbb{C}[t]$.

8.4.2. Lemma.

(i) *Let B be an algebra, let $M \xrightarrow{g} N$ be a homomorphism of B-modules, and let $B \xrightarrow{\delta} M$ be a derivation. The composition $B \xrightarrow{g\delta} N$ is a derivation.*

(ii) *Let $A \xrightarrow{\varphi} B$ be an algebra homomorphism, and let $B \xrightarrow{\delta} M$ be a derivation. The composition $A \xrightarrow{\delta\varphi} M$ is a derivation.*

(iii) *Let $A \xrightarrow{\varphi} B$ be a surjective algebra homomorphism, let $B \xrightarrow{h} M$ be a map to a B-module M, and let $d = h\varphi$. If $A \xrightarrow{d} M$ is a derivation, then h is a derivation.* □

The *module of differentials* Ω_A of an algebra A is an A-module generated by elements that are denoted by da, one for each element a of A. The elements of Ω_A are (finite) combinations $\sum b_i da_i$, with a_i and b_i in A. The defining relations among the generators da are the ones that make the map $A \xrightarrow{d} \Omega_A$ that sends a to da a derivation: For all a, b in A and all c in $\mathbb{C}$,

$$d(ab) = a\,db + b\,da, \quad d(a+b) = da + db, \quad \text{and} \quad dc = 0. \tag{8.4.3}$$

The elements of Ω_A are the *differentials.*

8.4.4. Lemma.

(i) *When we compose a homomorphism $\Omega_A \xrightarrow{f} M$ of $\mathcal{O}$-modules with the derivation $A \xrightarrow{d} \Omega_A$, we obtain a derivation $A \xrightarrow{fd} M$. This composition with d defines a bijection between module homomorphisms $\Omega_A \to M$ and derivations $A \xrightarrow{\delta} M$.*

(ii) *Ω is a functor: An algebra homomorphism $A \xrightarrow{u} B$ induces a homomorphism $\Omega_A \xrightarrow{v} \Omega_B$ that is compatible with the ring homomorphism u, and that makes a diagram*

$$\begin{array}{ccc} B & \xrightarrow{d} & \Omega_B \\ {\scriptstyle u}\uparrow & & \uparrow{\scriptstyle v} \\ A & \xrightarrow{d} & \Omega_A \end{array}$$

(Recall that, when ω is an element of Ω_A and α is an element of A, compatibility of v with u means that $v(\alpha\omega) = u(\alpha)v(\omega)$.)

Proof.

(ii) When Ω_B is made into an A-module by restriction of scalars, the composed map $A \overset{u}{\to} B \overset{d}{\to} \Omega_B$ will be a derivation to which (i) applies. □

8.4.5. Lemma. *Let R be the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. The module of differentials Ω_R is a free R-module with basis $dx_1, \dots, dx_n$.*

Proof. The formula $df = \sum \frac{df}{dx_i} dx_i$ follows from the defining relations. It shows that the elements $dx_1, \dots, dx_n$ generate Ω_R. Let V be a free R-module with basis $v_1, \dots, v_n$. The product rule for differentiation shows that the map $\delta : R \to V$ defined by $\delta(f) = \frac{\partial f}{\partial x_i} v_i$ is a derivation. It induces a module homomorphism $\Omega_A \xrightarrow{\varphi} V$ that sends dx_i to v_i (see Lemma 8.4.4). Since $dx_1, \dots, dx_n$ generate Ω_R and $v_1, \dots, v_n$ is a basis of V, φ is an isomorphism. □

8.4.6. Proposition. *Let I be an ideal of an algebra R, let A be the quotient algebra R/I, and let dI denote the set of differentials df with f in I. The subset $N = dI + I\Omega_R$ of Ω_R is a submodule, and Ω_A is isomorphic to the quotient Ω_R/N.*

The proposition can be interpreted this way: Suppose that an ideal I is generated by elements $f_1, \dots, f_r$ of R, and let $A = R/I$. Then Ω_A is the quotient of Ω_R that is obtained by introducing these two rules:

- $df_i = 0$, and
- multiplication by f_i is zero.

These rules hold in Ω_A because the elements f_i are zero in A.

8.4.7. Example. Let R be the polynomial ring $\mathbb{C}[y]$ in one variable. So Ω_R is a free module with basis dy. Let I be the principal ideal (y^2), and let $A = R/I$. In this case, $y^2 dy$ generates the module $I\Omega_A$, and dI is also an R-module. It is generated by the element $2y\,dy$. So N is generated by ydy. If $\bar{y}$ denotes the residue of y in A, $\Omega_A = \Omega_R/N$ is generated by an element $d\bar{y}$, with the relation $\bar{y}\,d\bar{y} = 0$. It isn't the zero module. □

Proof of Proposition 8.4.6. First, $I\Omega_R$ is a submodule of Ω_R, and dI is an additive subgroup of Ω_R. To show that N is a submodule, we must show that scalar multiplication by an element of R maps dI to N, i.e., that if g is in R and f is in I, then $g\,df$ is in N. By the product rule, $g\,df = d(fg) - f\,dg$. Since I is an ideal, fg is in I. Then $d(fg)$ is in dI, and $f\,dg$ is in $I\Omega_R$. So $g\,df$ is in N.

The rules displayed above show that N is contained in the kernel of the surjective map $\Omega_R \xrightarrow{v} \Omega_A$ defined by the homomorphism $R \to A$. Let $\bar{\Omega}$ denote the quotient Ω_R/N. It is an A-module, and v defines a surjective map of A-modules $\bar{\Omega} \xrightarrow{\bar{v}} \Omega_A$, because $N \subset \ker v$. We show that $\bar{v}$ is an isomorphism.

Let r be an element of R and let $\overline{dr}$ be its image in $\bar{\Omega}$. The composed map $R \xrightarrow{d} \Omega_R \xrightarrow{\pi} \bar{\Omega}$ is a derivation that sends r to $\overline{dr}$, and I is in its kernel. It defines a derivation $R/I = A \xrightarrow{\delta} \bar{\Omega}$, and if a is the residue of r in A, then $\delta(a) = \overline{dr}$. This derivation corresponds to a homomorphism of A-modules $\Omega_A \to \bar{\Omega}$ that sends da to $\overline{dr}$ and that inverts $\bar{v}$. □

8.4.8. Corollary. *If A is a finite-type algebra, then Ω_A is a finite A-module.*

This follows from Proposition 8.4.6, because the module of differentials of the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ is a finite module. □

8.4.9. Lemma. *Let S be a multiplicative system in a domain A. The module $\Omega_{S^{-1}A}$ of differentials of $S^{-1}A$ is canonically isomorphic to the module of fractions $S^{-1}\Omega_A$. In particular, if K is the field of fractions of A, then $K \otimes_A \Omega_A \approx \Omega_K$.*

We have moved the symbol S^{-1} to the left for clarity. The lemma shows that a finite $\mathcal{O}$-module Ω_Y of differentials on a variety Y is defined such that, when $U = \operatorname{Spec} A$ is an affine open subset of Y, $\Omega_Y(U) = \Omega_A$.

Proof of Lemma 8.4.9. The composed map $A \to S^{-1}A \xrightarrow{d} \Omega_{S^{-1}A}$ is a derivation. It defines an A-module homomorphism $\Omega_A \to \Omega_{S^{-1}A}$ which extends to an $S^{-1}A$-homomorphism $S^{-1}\Omega_A \xrightarrow{\varphi} \Omega_{S^{-1}A}$ because scalar multiplication by the elements of S is invertible in $\Omega_{S^{-1}A}$. The relation $ds^{-k} = -ks^{-k-1}\, ds$ follows from the definition of a differential, and it shows that φ is surjective. The quotient rule

$$\delta(s^{-k}a) = -ks^{-k-1}a\, ds + s^{-k}\, da$$

defines a derivation $S^{-1}A \xrightarrow{\delta} S^{-1}\Omega_A$ that corresponds to a homomorphism $\Omega_{S^{-1}A} \to S^{-1}\Omega_A$ and that inverts φ. Here, one must show that δ is well-defined, that $\delta(s_1^{-k}a_1) = \delta(s_2^{-\ell}a_2)$ if $s_1^{-\ell}a_1 = s_2^{-k}a_2$, and that δ is a derivation. You will be able to do this. □

8.4.10. Proposition. *Let y be a local generator for the maximal ideal at a point q of a smooth curve Y. In a suitable neighborhood of q, the module Ω_Y of differentials is a free $\mathcal{O}$-module with basis dy. Therefore Ω_Y is an invertible module.*

Proof. We may assume that Y is affine, say $Y = \operatorname{Spec} B$. Let q be a point of Y, and let y be an element of B with $\mathrm{v}_q(y) = 1$. To show that dy generates Ω_B locally, we may localize, so we may suppose that y generates the maximal ideal $\mathfrak{m}$ at q. We must show that after we localize B once more, every differential df with f in B will be a multiple of dy. Let $c = f(q)$, so that $f = c + yg$ for some g in B, and because $dc = 0$, $df = g\, dy + y\, dg$. Here $g\, dy$ is in $B\, dy$ and $y\, dy$ is in $\mathfrak{m}\Omega_B$. This shows that

$$\Omega_B = B\, dy + \mathfrak{m}\Omega_B.$$

An element β of Ω_B can be written as $\beta = b\,dy + \gamma$, with b in B and γ in $\mathfrak{m}\Omega_B$. If W denotes the quotient module $\Omega_B/(B\,dy)$, then $W = \mathfrak{m}W$. The Nakayama Lemma tells us that there is an element z in $\mathfrak{m}$ such that $s = 1-z$ annihilates W. When we replace B by its localization B_s, we will have $W = 0$ and $\Omega_B = B\,dy$, as required.

We must still verify that the generator dy of Ω_B isn't a torsion element. Suppose that $b\,dy = 0$, with $b \neq 0$. Then Ω_B will be zero except at the finite set of zeros of b in Y. We choose for q a point at which Ω_B is zero, keeping the rest of the notation unchanged. Let $A = \mathbb{C}[y]/(y^2)$. As was noted in Example 8.4.7, Ω_A isn't the zero module. Proposition 5.2.10 tells us that, at our point q, the algebra $B/\mathfrak{m}_q^2$ is isomorphic to A, and Proposition 8.4.6 tells us that Ω_A is a quotient of Ω_B. Since Ω_A isn't zero, neither is Ω_B. Therefore dy isn't a torsion element. □

8.5. Branched Coverings

By a *branched covering*, we mean an integral morphism $Y \xrightarrow{\pi} X$ of smooth curves. Chevalley's Finiteness Theorem, Theorem 4.6.5, shows that, when Y is projective, every morphism $Y \to X$ will be a branched covering, unless it maps Y to a point.

Let $Y \to X$ be a branched covering. The function field K of Y will be a finite extension of the function field F of X. The *degree* $[Y\!:\!X]$ of the covering is defined to be the degree $[K\!:\!F]$ of that field extension. If $X' = \operatorname{Spec} A$ is an affine open subset of X, its inverse image Y' will be an affine open subset $Y' = \operatorname{Spec} B$ of Y, and B will be a locally free A-module, of rank equal to $[Y\!:\!X]$.

To describe the fibre of a branched covering $Y \xrightarrow{\pi} X$ over a point p of X, we may localize. We assume that X and Y are affine, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and that the maximal ideal $\mathfrak{m}_p$ of A at a point p is a principal ideal, generated by an element x of A. If a point q of Y lies over p, the *ramification index* at q is defined to be $\mathrm{v}_q(x)$, where v_q is the valuation of the function field K that corresponds to q. We will denote the ramification index by e. Then if y is a local generator for the maximal ideal $\mathfrak{m}_q$ of B at q, we will have

$$x = uy^e$$

where u is a *local unit*—a rational function on Y that is regular and invertible on some open neighborhood of q.

Points of Y whose ramification indices are greater than 1 are called *branch points*. We also call a point of X a *branch point* of the covering if it is the image of a branch point of Y. A branched covering $Y \to X$ has finitely many branch points.

8.5.1. Corollary. *A branched covering $Y \xrightarrow{\pi} X$ of degree* 1 *is an isomorphism.*

Proof. When $[Y{:}X] = 1$, the function fields of Y and X will be equal. Then because $Y \to X$ is an integral morphism and X is normal, $Y = X$. □

The next lemma follows from the Chinese Remainder Theorem.

8.5.2. Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. Suppose that the maximal ideal $\mathfrak{m}_p$ at a point p of X is a principal ideal, generated by an element x. Let $q_1, \ldots, q_k$ be the points of Y that lie over p and let $\mathfrak{m}_i$ and e_i be the maximal ideal and ramification index at q_i, respectively.*

(i) *The extended ideal $\mathfrak{m}_p B = xB$ is the product ideal $\mathfrak{m}_1^{e_1} \cdots \mathfrak{m}_k^{e_k}$.*

(ii) *Let $\bar{B}_i = B/\mathfrak{m}_i^{e_i}$. The quotient $\bar{B} = B/xB$ is isomorphic to the product $\bar{B}_1 \times \cdots \times \bar{B}_k$.*

(iii) *The degree $[Y{:}X]$ of the covering is the sum $e_1 + \cdots + e_k$ of the ramification indices at the points q_i.*

(iv) *If p isn't a branch point, the fibre over p consists of n points with ramification indices equal to 1, n being the degree $[Y{:}X]$.* □

8.5.3. Proposition. *Let f be a nonconstant rational function on a smooth projective curve Y, and let d be the degree of the divisor of poles of f. For generic scalars c, the level set $f = c$ has no multiple point.*

Proof. We map Y to $X = \mathbb{P}^1$ by the functions $(1, f)$, so that Y becomes a branched covering of X. The level sets of f are the fibres of that morphism. For all but finitely many points of X, the fibres consist of d points of multiplicity one, with $d = [Y{:}X]$ (see Example 4.2.7). □

8.5.4. Local analytic structure. In the classical topology, the analytic structure of a branched covering $Y \xrightarrow{\pi} X$ is very simple. We describe it here because it is useful and helpful for intuition.

8.5.5. Proposition. *Locally in the classical topology, Y is analytically isomorphic to the e-th root covering $y^e = x$.*

Proof. Let q be a point of Y, let p be its image in X, let x and y be local generators for the maximal ideals $\mathfrak{m}_p$ of $\mathcal{O}_X$ and $\mathfrak{m}_q$ of $\mathcal{O}_Y$, respectively. Let e be the ramification index at q. So $x = uy^e$, where u is a local unit at q. In a neighborhood of q in the classical topology, u will have an analytic e-th root w. The element $y_1 = wy$ also generates $\mathfrak{m}_q$ locally, and $x = y_1^e$. We replace y by y_1. The Implicit Function Theorem tells us that x and y are local analytic coordinate functions on X and Y (see Section 9.2). □

8.5.6. Corollary. *Let $Y \xrightarrow{\pi} X$ be a branched covering, let $\{q_1, \ldots, q_k\}$ be the fibre over a point p of X, and let e_i be the ramification index at q_i. As a point p' of X approaches p, e_i points that lie over p' approach q_i.* □

8.5.7. Suppressing notation for the direct image. When considering a branched covering $Y \xrightarrow{\pi} X$ of smooth curves, we will often pass between an $\mathcal{O}_Y$-module $\mathcal{M}$ and its direct image $\pi_*\mathcal{M}$, and we may want to work primarily on X. Recall that if X' is an open subset X' of X and Y' is its invere image, then

$$[\pi_*\mathcal{M}](X') = \mathcal{M}(Y').$$

One can think of the direct image $\pi_*\mathcal{M}$ as working with $\mathcal{M}$ but looking only at the open subsets Y' of Y that are inverse images of open subsets of X. If we look only at such subsets, the only significant difference between $\mathcal{M}$ and its direct image will be that, when X' is open in X and $Y' = \pi^{-1}X'$, the $\mathcal{O}_Y(Y')$-module $\mathcal{M}(Y')$ is made into an $\mathcal{O}_X(X')$-module by restriction of scalars.

To simplify notation, we will often drop the symbol π_* and write $\mathcal{M}$ instead of $\pi_*\mathcal{M}$. If X' is an open subset of X, $\mathcal{M}(X')$ will stand for $\mathcal{M}(\pi^{-1}X')$. When denoting the direct image of an $\mathcal{O}_Y$-module $\mathcal{M}$ by the same symbol $\mathcal{M}$, we may refer to it as an $\mathcal{O}_X$-module. In accordance with this convention, we may also write $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$, but we must include the subscript Y.

This abbreviated terminology is analogous to the one used for restriction of scalars in a module. When $A \to B$ is an algebra homomorphism and M is a B-module, the B-module $_BM$ and the A-module $_AM$ obtained from it by restriction of scalars are usually denoted by the same letter M.

8.5.8. Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, of degree $n = [Y:X]$. With notation as above:*

(i) *The direct image of $\mathcal{O}_Y$, also denoted by $\mathcal{O}_Y$, is a locally free $\mathcal{O}_X$-module of rank n.*

(ii) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is a torsion $\mathcal{O}_Y$-module if and only if its direct image is a torsion $\mathcal{O}_X$-module.*

(iii) *A finite $\mathcal{O}_Y$-module $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module if and only if its direct image is a locally free $\mathcal{O}_X$-module. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module of rank r, then its direct image is a locally free $\mathcal{O}_X$-module of rank nr.* □

8.6. Trace of a Differential

8.6.1. Trace of a function. Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, and let F and K be the function fields of X and Y, respectively. The trace map $K \xrightarrow{\text{trace}} F$ for a field extension of finite degree has been defined before (see Lemma 4.3.11). If α is an element of K, multiplication by α on the F-vector space K is a linear operator, and $\text{trace}(k)$ is the trace of that operator. The trace is F-linear: If f_i are in F and α_i are in K, then $\text{trace}(\sum f_i\alpha_i) = \sum f_i\,\text{trace}(\alpha_i)$. Moreover, the trace carries regular functions to regular functions: If $X' = \operatorname{Spec} A'$ is an affine open subset of X, with inverse image $Y' = \operatorname{Spec} B'$, then because A' is normal, the trace of an element of B' will be in A' (see Lemma 4.3.7). Using our abbreviated notation $\mathcal{O}_Y$ for $\pi_*\mathcal{O}_Y$, the trace defines a homomorphism of $\mathcal{O}_X$-modules

$$\mathcal{O}_Y \xrightarrow{\text{trace}} \mathcal{O}_X. \tag{8.6.2}$$

Analytically, this trace can be described as a sum over the sheets of the covering. Let $n = [Y:X]$. When a point p of X isn't a branch point, there will be n points $q_1, \ldots, q_n$ of Y lying over p. If U is a small neighborhood of p in X in the classical topology, its inverse image V will consist of disjoint neighborhoods V_i of q_i, each of which maps bijectively to U. The ring of analytic functions on V_i will be isomorphic to the ring $\mathcal{A}$ of analytic functions on U, and the ring of analytic functions on V is isomorphic to the direct sum $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ of n copies of $\mathcal{A}$. If a rational function g on Y is regular on V, its restriction to V can be written as $g = g_1 \oplus \cdots \oplus g_n$, with g_i in $\mathcal{A}_i$. The matrix of left multiplication by g on $\mathcal{A}_1 \oplus \cdots \oplus \mathcal{A}_n$ is the diagonal matrix with entries $g_1, \ldots, g_n$, and

$$\text{trace}(g) = g_1 + \cdots + g_n. \tag{8.6.3}$$

8.6.4. Lemma. *Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves, let p be a point of X, let $q_1, \ldots, q_k$ be the fibre over p, and let e_i be the ramification index at q_i. If a rational function g on Y is regular at the points $q_1, \ldots, q_k$, its trace is regular at p. Its value at p is $[\text{trace}(g)](p) = e_1 g(q_1) + \cdots + e_k g(q_k)$.*

Proof. The regularity was discussed above. If p isn't a branch point, we will have $k = n$ and $e_i = 1$ for all i. In this case, the lemma follows by evaluating (8.6.3). It follows by continuity for any point p. As a point p' approaches p, e_i points q' of Y approach q_i (see Corollary 8.5.6). For each point q' that approaches q_i, the limit of $g(q')$ will be $g(q_i)$. □

8.6.5. Trace of a differential. The structure sheaf is naturally contravariant. A branched covering $Y \xrightarrow{\pi} X$ corresponds to a homomorphism of $\mathcal{O}_X$-modules $\mathcal{O}_Y \leftarrow \mathcal{O}_X$. The trace map for functions is a homomorphism in the opposite direction: $\mathcal{O}_Y \xrightarrow{\text{trace}} \mathcal{O}_X$.

Differentials are also naturally contravariant. A morphism $Y \xrightarrow{\pi} X$ induces an $\mathcal{O}_X$-module homomorphism $\Omega_X \to \Omega_Y$ that sends a differential dx on X to a differential on Y that we may denote by dx too (see Lemma 8.4.4). As is true for functions, there is a trace map for differentials in the opposite direction. It is defined below, in (8.6.7), and it will be denoted by τ:

$$\Omega_Y \xrightarrow{\tau} \Omega_X.$$

But first, a lemma about the natural contravariant map $\Omega_X \to \Omega_Y$:

8.6.6. Lemma. *Let $Y \to X$ be a branched covering.*

(i) *Let p be the image in X of a point q of Y, let x and y be local generators for the maximal ideals of X and Y at p and q, respectively, and let e be the ramification index at q. As a differential on Y, $dx = vy^{e-1}\,dy$, where v is a local unit at q.*

(ii) *The canonical homomorphism $\Omega_X \to \Omega_Y$ is injective.*

Proof. As we have noted before, $x = uy^e$, for some local unit u. Since dy generates Ω_Y locally, there is a rational function z that is regular at q, such that $du = z\,dy$. Let $v = yz + eu$. Then

$$dx = d(uy^e) = y^e z\,dy + ey^{e-1}u\,dy = vy^{e-1}\,dy.$$

Since yz is zero at q and eu is a local unit, v is a local unit. See Lemma 8.1.17(iii) for part (ii). □

To define the trace for differentials, we begin with differentials of the function fields F and K of X and Y, respectively. The modules of differentials of those fields are defined in the same way as for any algebra. The $\mathcal{O}_Y$-module Ω_Y is invertible (see Proposition 8.4.10), and when $Y = \operatorname{Spec} B$, the module Ω_K of differentials of K is the localization $S^{-1}\Omega_B$, where S is the multiplicative system of nonzero elements of B. So Ω_K is a free K-module of rank 1. Any nonzero differential will form a K-basis. We choose as basis a nonzero F-differential α. Its image in Ω_K, which we denote by α too, will be a K-basis for Ω_K. We could, for instance, take $\alpha = dx$, where x is a local coordinate function on X.

$$\begin{array}{ccc} K & Y & \Omega_Y \\ \uparrow & \downarrow & \downarrow{\scriptstyle\tau} \\ F & X & \Omega_X \end{array}$$

Since α is a basis, elements β of Ω_K can be written uniquely, as

$$\beta = g\alpha$$

where g is an element of K. The *trace* $\Omega_K \xrightarrow{\tau} \Omega_F$ is defined by

$$\tau(\beta) = \text{trace}(g)\alpha \qquad (8.6.7)$$

where trace(g) is the trace of the function g. Since the trace for functions is F-linear, τ is F-linear.

8.6.8. Corollary. *Let x be a local generator for the maximal ideal $\mathfrak{m}_p$ at a point p of X. If the degree $[Y:X]$ of Y over X is n, then when we regard dx as a differential on Y, $\tau(dx) = n\,dx$.* □

We need to check that τ is independent of the choice of α. Let α' be another nonzero F-differential. Then $f\alpha' = \alpha$ for some nonzero f in F, and $gf\alpha' = g\alpha$. Since trace is F-linear, $\operatorname{trace}(gf) = f\operatorname{trace}(g)$. Then

$$\tau(gf\alpha') = \operatorname{trace}(gf)\alpha' = f\operatorname{trace}(g)\alpha' = \operatorname{trace}(g)f\alpha' = \operatorname{trace}(g)\alpha = \tau(g\alpha).$$

Using α' in place of α gives the same value for the trace.

A differenial of the function field K of Y is a *rational differential* on Y. A rational differential β is *regular* at a point q of Y if there is an affine open neighborhood $Y' = \operatorname{Spec} B$ of q such that β is an element of Ω_B. If y is a local generator for the maximal ideal $\mathfrak{m}_q$ and if $\beta = g\,dy$, the differential β is regular at q if and only if the rational function g is regular at q.

Let $Y \to X$ be a branched covering of affine varieties, with $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let p be a point of X. Suppose that the maximal ideal at p is generated by an element x of A and so that the differential dx generates Ω_A locally. Let $q_1, \ldots, q_k$ be the points of Y that lie over p, and let e_i be the ramification index at q_i.

8.6.9. Corollary. *With notation as above:*

(i) *When dx is viewed as a differential on Y, it has a zero of order $e_i - 1$ at q_i.*

(ii) *When a differential β on Y that is regular at q_i is written as $\beta = g\,dx$, where g is a rational function on Y, then g has a pole of order at most $e_i - 1$ at q_i.*

This follows from Lemma 8.6.6(i). □

8.6.10. Lemma (Main Lemma). *Let $Y \xrightarrow{\pi} X$ be a branched covering, let p be a point of X, let $q_1, \ldots, q_k$ be the points of Y that lie over p, and let β be a rational differential on Y.*

(i) *If β is regular at the points $q_1, \ldots, q_k$, its trace $\tau(\beta)$ is regular at p.*

(ii) *If β has a simple pole at q_i and is regular at q_j for all $j \neq i$, then $\tau(\beta)$ is not regular at p.*

Proof.

(i) Let x be a local generator for the maximal ideal at p. We write β as $g\,dx$, where g is a rational function on Y. Suppose that β is regular at the points q_i. Corollary 8.6.9 tells us that g has poles of orders at most $e_i - 1$ at the points q_i. Since x has a zero of order e_i at q_i, the function xg is regular at the points q_i, and its value there is 0. Then $\operatorname{trace}(xg)$ is regular at p, and its value at p is 0 (see Lemma 8.6.4). So $x^{-1}\operatorname{trace}(xg)$ is a regular function at p. Since trace is F-linear and x is in F, $x^{-1}\operatorname{trace}(xg) = \operatorname{trace}(g)$. So $\operatorname{trace}(g)$ and $\tau(\beta) = \operatorname{trace}(g)\,dx$ are regular at p.

(ii) In this case, g has poles of orders at most $e_j - 1$ at the points q_j when $j \neq i$, and it has a pole of order e_i at q_i. The function xg will be regular at all of the points q_j. Its value at q_j will be zero when $j \neq i$, and not zero when $j = i$. Then trace(xg) will be regular at p, but not zero there (see Lemma 8.6.4). Therefore $\text{trace}(g) = x^{-1}\,\text{trace}(xg)$ and $\tau(\beta) = \text{trace}(g)\,dx$ won't be regular at p. □

8.6.11. Corollary. *The trace map* (8.6.7) *defines a homomorphism of $\mathcal{O}_X$-modules $\Omega_Y \xrightarrow{\tau} \Omega_X$.* □

Let $Y \xrightarrow{\pi} X$ be a branched covering. As is true for any $\mathcal{O}_Y$-module, Ω_Y is isomorphic to the module of homomorphisms ${}_Y(\mathcal{O}_Y, \Omega_Y)$. The homomorphism $\mathcal{O}_Y \to \Omega_Y$ that corresponds to a section β of Ω_Y on an open set U sends a regular function f on U to $f\beta$. We denote that homomorphism by β too: $\mathcal{O}_Y \xrightarrow{\beta} \Omega_Y$.

8.6.12. Lemma. *Composition with the trace $\Omega_Y \xrightarrow{\tau} \Omega_X$ defines a homomorphism of $\mathcal{O}_X$-modules*

(8.6.13) $$\Omega_Y \approx {}_Y(\mathcal{O}_Y, \Omega_Y) \xrightarrow{\tau} {}_X(\mathcal{O}_Y, \Omega_X).$$

Proof. An $\mathcal{O}_Y$-linear map becomes an $\mathcal{O}_X$-linear map by restriction of scalars. When we compose an $\mathcal{O}_Y$-linear map β with τ, then because τ is $\mathcal{O}_X$-linear, the result will be $\mathcal{O}_X$-linear. It will be a homomorphism of $\mathcal{O}_X$-modules. □

8.6.14. Theorem.

(i) *The homomorphism of Lemma* 8.6.12 *is an isomorphism.*

(ii) *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. Composition with the trace defines an isomorphism of $\mathcal{O}_X$-modules*

(8.6.15) $${}_Y(\mathcal{M}, \Omega_{\mathcal{O}_Y}) \xrightarrow{\tau\circ} {}_X(\mathcal{M}, \Omega_{\mathcal{O}_X}).$$

When one looks carefully, this theorem follows from the Main Lemma.

Note. The domain and range (see (8.6.15)) are to be interpreted as modules on X. When we put the symbols Hom and π_* that we have suppressed into the notation, the map (8.6.15) becomes an isomorphism

$$\pi_*\big(\underline{\text{Hom}}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_Y)\big) \xrightarrow{\tau\circ} \underline{\text{Hom}}_{\mathcal{O}_X}(\pi_*\mathcal{M}, \Omega_X).$$

It suffices to verify the theorem locally, because it concerns modules on X. So we may suppose that X and Y are affine, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$. When the theorem is stated in terms of algebras and modules, it becomes this:

8.6.16. Theorem. *Let $Y \to X$ be a branched covering, with $Y = \operatorname{Spec} B$ and $X = \operatorname{Spec} A$.*

(i) *The trace map $\Omega_B = {}_B(B, \Omega_B) \xrightarrow{\tau\circ} {}_A(B, \Omega_A)$ is an isomorphism of A-modules.*

(ii) *For any locally free B-module M, composition with the trace defines an isomorphism of A-modules ${}_B(M, \Omega_B) \xrightarrow{\tau\circ} {}_A(M, \Omega_A)$.*

The B-modules Ω_B and ${}_B(M, \Omega_B)$ become A-modules by restriction of scalars, and in ${}_A(M, \Omega_A)$ the B-module M is interpreted as an A-module by restriction of scalars too.

8.6.17. Lemma. *Let $\mathcal{L} \subset \mathcal{M}$ be an inclusion of invertible modules on a smooth curve Y, let q be a point in the support of the quotient $\mathcal{M}/\mathcal{L}$, and let V be an affine open subset of Y that contains q. Suppose that a rational function f has a simple pole at q and is regular at all other points of V. If α is a section of $\mathcal{L}$ on V, then $f^{-1}\alpha$ is a section of $\mathcal{M}$ on V.*

Proof. Working locally, we may assume that $\mathcal{L} = \mathcal{O}$. Then $\mathcal{M} = \mathcal{O}(D)$ for some effective divisor D. Since q is in the support of $\mathcal{M}/\mathcal{L}$, the coefficient of q in D is positive. Therefore $\mathcal{O}(q) \subset \mathcal{O}(D) = \mathcal{M}$. Then α is a section of $\mathcal{O}$, and f^{-1} is a section of $\mathcal{O}(q)$. So $f^{-1}\alpha$ will be a section of $\mathcal{O}(q)$ and therefore a section of $\mathcal{O}(D)$. □

8.6.18. Lemma. *Let $A \subset B$ be rings, let M be a B-module, and let N be an A-module. The module ${}_A(M,N)$ of homomorphisms has the structure of a B-module.*

Proof. We must define scalar multiplication of a homomorphism $M \xrightarrow{\varphi} N$ of A-modules by an element b of B. The definition is $[b\varphi](m) = \varphi(bm)$. One must show that the map $b\varphi$ is a homomorphism of A-modules $M \to N$ and check the axioms for a B-module. You will be able to do this. □

Proof of Theorem 8.6.14. Since the theorem is local, we are allowed to localize. We use the algebra version, Theorem 8.6.16, of the theorem.

(i) Both B and Ω_B are torsion-free and therefore locally free A-modules. Localizing as needed, we may assume that they are free A-modules and that Ω_A is a free A-module of rank 1 with basis of the form dx. Then ${}_A(B, \Omega_A)$ will be a free A-module too. Let's denote ${}_A(B, \Omega_A)$ by Θ. Lemma 8.6.18 tells us that Θ is a B-module. Because B and Ω_A are free A-modules, Θ

is a free A-module and a locally free B-module (see Lemma 8.5.8). Since Ω_A has A-rank 1, the A-rank of Θ is the same as the A-rank of B. Therefore the B-rank of Θ is the same as the B-rank of B, which is 1. So Θ is an invertible B-module.

If x is a local coordinate on X, then $\tau(dx) \neq 0$ (see Corollary 8.6.8). The trace map $\Omega_B \xrightarrow{\tau} \Theta$ isn't the zero map. Since domain and range are invertible B-modules, it is an injective homomorphism. Its image, which is isomorphic to Ω_B, is an invertible submodule of the invertible B-module Θ.

To show that $\Omega_B = \Theta$, we apply Lemma 8.6.17 to show that the quotient $\bar{\Theta} = \Theta/\Omega_B$ is the zero module. Suppose not, and let q be a point in the support of $\bar{\Theta}$. Let p be the image of q in X and let $q_1, \ldots, q_k$ be the fibre over p, with $q = q_1$.

We choose a differential α that is regular at all of the points q_i. If y is a local generator for the maximal ideal at q_1, then $\alpha = g\,dy$, where g is a regular function at q_1. We assume also that α has been chosen so that $g(q_1) \neq 0$.

Let f be a rational function that is regular on an affine open subset V of Y that contains the points $q_1, \ldots, q_k$ and such that $f(q_1) = 0$ and $f(q_i) \neq 0$ when $i > 1$. Lemma 8.6.17 tells us that $\beta = f^{-1}\alpha$ is a section of Θ on V, but the Main Lemma 8.6.10 tells us that $\tau(\beta)$ isn't regular at p. This contradiction proves part (i) of the theorem. □

Proof of Theorem 8.6.14.

(ii) We are to show that if $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, composition with the trace defines an isomorphism ${}_{\mathcal{O}_Y}(\mathcal{M}, \Omega_{\mathcal{O}_Y}) \to {}_{\mathcal{O}_X}(\mathcal{M}, \Omega_{\mathcal{O}_X})$. Part (i) of the theorem tells us that this is true when $\mathcal{M} = \mathcal{O}_Y$. Therefore it is also true when $\mathcal{M}$ is a free module $\mathcal{O}_Y^k$. And, since (ii) is a statement about $\mathcal{O}_X$-modules, it suffices to prove it locally on X.

8.6.19. Lemma. *Let $q_1, \ldots, q_k$ be points of a smooth curve Y, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. There is an open subset V of Y that contains the points $q_1, \ldots, q_k$, such that $\mathcal{M}$ is free on V.*

We assume the lemma and complete the proof of the theorem. Let $\{q_1, \ldots, q_k\}$ be the fibre over a point p of X and let V be as in the lemma. The complement $D = Y - V$ is a finite set whose image Z in X is also finite, and Z doesn't contain p. If U is the complement of Z in X, its inverse image W will be a subset of V that contains the points q_i and on which $\mathcal{M}$ is free. □

Proof of the lemma. With terminology as in Lemma 8.5.2, let $\mathfrak{m}_i$ be the maximal ideal of B at q_i, and let $\bar{B}_i = B/\mathfrak{m}_i^{e_i}$. The quotient $\bar{B} = B/xB$ is isomorphic to the product $\bar{B}_1 \times \cdots \times \bar{B}_k$. Since $\mathcal{M}$ is locally free, $M/\mathfrak{m}_i M = \bar{M}_i$ is a free $\bar{B}_i$-module. Its dimension is the rank r of the B-module M.

If M has rank r, there will be a set of elements $m = (m_1, \dots, m_r)$ in M whose residues form a basis of $\bar{M}_i$ for every i. This follows from the Chinese Remainder Theorem. The set m will generate M locally at each of the points. Let M' be the B-submodule of M generated by m. The cokernel of the map $M' \to M$ is zero at the points $q_1, \dots, q_k$, and therefore it's support, which is a finite set, is disjoint from those points. When we localize to delete this finite set from X, the set m becomes a basis for M. □

Note. Theorem 8.6.14 is subtle. Unfortunately the proof, though understandable, doesn't give an intuitive explanation of the fact that Ω_Y is isomorphic to $_X(\mathcal{O}_Y, \Omega_X)$. To get more insight into that, we would need a better understanding of differentials. My father, Emil Artin, said: "One doesn't really understand differentials, but one can learn to work with them."

8.7. The Riemann-Roch Theorem II

8.7.1. The Serre dual. Let Y be a smooth projective curve, and let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module. The *Serre dual* of $\mathcal{M}$ is defined to be the module

(8.7.2) $$\mathcal{M}^S = {}_Y(\mathcal{M}, \Omega_Y) \qquad \big(= \underline{\mathrm{Hom}}_{\mathcal{O}_Y}(M, \Omega_Y)\big).$$

Its sections on an open subset U are the homomorphisms of $\mathcal{O}_Y(U)$-modules $\mathcal{M}(U) \to \Omega_Y(U)$, and it can also be written as $\mathcal{M}^* \otimes_{\mathcal{O}} \Omega_Y$, where $\mathcal{M}^*$ is the ordinary dual ${}_Y(\mathcal{M}, \mathcal{O}_Y)$. The invertible module Ω_Y is locally isomorphic to $\mathcal{O}_Y$. So the Serre dual $\mathcal{M}^S$ is a locally free module of the same rank as $\mathcal{M}$. The Serre bidual $(\mathcal{M}^S)^S$ is isomorphic to $\mathcal{M}$:

$$(\mathcal{M}^S)^S \approx (\mathcal{M}^* \otimes_{\mathcal{O}} \Omega_Y)^* \otimes_{\mathcal{O}} \Omega_Y \approx \mathcal{M}^{**} \otimes_{\mathcal{O}} \Omega_Y^* \otimes_{\mathcal{O}} \Omega_Y \approx \mathcal{M}^{**} \approx \mathcal{M}.$$

(See Lemma 8.1.17(i).) For example, $\mathcal{O}_Y^S = \Omega_Y$ and $\Omega_Y^S = \mathcal{O}_Y$.

8.7.3. Theorem (Riemann-Roch Theorem, Version 2). *Let $\mathcal{M}$ be a locally free $\mathcal{O}_Y$-module on a smooth projective curve Y, and let $\mathcal{M}^S$ be its Serre dual. Then $\mathbf{h}^0\mathcal{M} = \mathbf{h}^1\mathcal{M}^S$ and $\mathbf{h}^1\mathcal{M} = \mathbf{h}^0\mathcal{M}^S$.*

The second assertion follows from the first when one replaces $\mathcal{M}$ by $\mathcal{M}^S$. Thus $\mathbf{h}^1\Omega_Y = \mathbf{h}^0\mathcal{O}_Y = 1$ and $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = p_a$. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, then

(8.7.4) $$\chi(\mathcal{M}) = \mathbf{h}^0\mathcal{M} - \mathbf{h}^0\mathcal{M}^S.$$

A more precise statement of the Riemann-Roch Theorem is that $H^1(Y, \mathcal{M})$ and $H^0(Y, \mathcal{M}^S)$ are dual spaces. This becomes important when one wants to apply the theorem to a cohomology sequence, but the fact that the dimensions are equal is enough for many applications. We omit the proof of the duality.

Our plan is to prove Theorem 8.7.3 directly for the projective line $\mathbb{P}^1$. This will be easy, because the structure of locally free modules on $\mathbb{P}^1$ is very simple. We derive it for an arbitrary smooth projective curve by projection to $\mathbb{P}^1$. Projection to projective space is a method that was used by Grothendieck in the proof of his general Riemann-Roch Theorem.

Let $X = \mathbb{P}^1$, let Y be a smooth projective curve, and let $Y \xrightarrow{\pi} X$ be a branched covering. Let the Serre dual of $\mathcal{M}$, a locally free $\mathcal{O}_Y$-module, be

$$\mathcal{M}_1^S = {}_Y(\mathcal{M}, \Omega_Y).$$

The direct image of $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module that we denote by $\mathcal{M}$, and we can form the Serre dual on X. Let

$$\mathcal{M}_2^S = {}_X(\mathcal{M}, \Omega_X).$$

8.7.5. Corollary. *The direct image $\pi_*\mathcal{M}_1^S$, also denoted by $\mathcal{M}_1^S$, is isomorphic to $\mathcal{M}_2^S$.*

Proof. This is Theorem 8.6.14. □

The corollary allows us to drop the subscripts from $\mathcal{M}^S$. Because a branched covering $Y \xrightarrow{\pi} X$ is an affine morphism, the cohomology of $\mathcal{M}$ and of its Serre dual $\mathcal{M}^S$ can be computed, either on Y or on X. If $\mathcal{M}$ is a locally free $\mathcal{O}_Y$-module, then $H^q(Y, \mathcal{M}) \approx H^q(X, \mathcal{M})$ and $H^q(Y, \mathcal{M}^S) \approx H^q(X, \mathcal{M}^S)$ (see Proposition 7.4.22).

Thus it is enough to prove Riemann-Roch for the projective line.

8.7.6. Riemann-Roch for the projective line. The Riemann-Roch Theorem for the projective line $X = \mathbb{P}^1$ is a consequence of the Birkhoff-Grothendieck Theorem, which tells us that a locally free $\mathcal{O}_X$-module $\mathcal{M}$ on X is a direct sum of twisting modules. To prove Riemann-Roch for the projective line X, it suffices to prove it for the twisting modules $\mathcal{O}_X(k)$.

8.7.7. Lemma. *The module of differentials Ω_X on the projective line X is isomorphic to the twisting module $\mathcal{O}_X(-2)$.*

Proof. Let $\mathbb{U}^0 = \operatorname{Spec} \mathbb{C}[x]$ and $\mathbb{U}^1 = \operatorname{Spec} \mathbb{C}[z]$ be the standard open subsets of $\mathbb{P}^1$, with $z = x^{-1}$. On $\mathbb{U}^0$, the module of differentials is free, with basis dx, and $dx = d(z^{-1}) = -z^{-2}\,dz$ describes the differential dx on $\mathbb{U}^1$. The point at infinity is $p_\infty : \{x_0 = 0\}$, and dx has a pole of order 2 there. It is a global section of $\Omega_X(2p_\infty)$, and as a section of that module, it isn't zero anywhere. Multiplication by dx defines an isomorphism $\mathcal{O} \to \Omega_X(2p_\infty)$ that sends 1 to dx. Tensoring with $\mathcal{O}(-2p_\infty)$ shows that $\mathcal{O}(-2p_\infty)$ is isomorphic to Ω_X. □

8.7.8. Lemma. *Let $\mathcal{M}$ and $\mathcal{N}$ be locally free $\mathcal{O}$-modules on the projective line X. Then ${}_X(\mathcal{M}(r),\mathcal{N})$ is canonically isomorphic to ${}_X(\mathcal{M},\mathcal{N}(-r))$.*

Proof. When we tensor a homomorphism $\mathcal{M}(r) \xrightarrow{\varphi} \mathcal{N}$ with $\mathcal{O}(-r)$, we obtain a homomorphism $\mathcal{M} \to \mathcal{N}(-r)$, and tensoring with $\mathcal{O}(r)$ is the inverse operation. □

The Serre dual $\mathcal{O}(n)^S$ of $\mathcal{O}(n)$ is therefore

$$\mathcal{O}(n)^S \approx {}_X(\mathcal{O}(n),\mathcal{O}(-2)) \approx \mathcal{O}(-2-n).$$

To prove Riemann-Roch for $X = \mathbb{P}^1$, we must show that

$$\mathbf{h}^0\mathcal{O}(n) = \mathbf{h}^1\mathcal{O}(-2-n) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(n) = \mathbf{h}^0\mathcal{O}(-2-n).$$

This follows from Theorem 7.5.5, which computes the cohomology of the twisting modules.

8.8. Using Riemann-Roch

8.8.1. Genus. There are three closely related numbers associated to a smooth projective curve Y: its *topological genus* g, its *arithmetic genus* $p_a = \mathbf{h}^1\mathcal{O}_Y$, and the *degree* δ of the module of differentials Ω_Y (see Corollary 8.1.18).

8.8.2. Theorem. *The topological genus g and the arithmetic genus p_a of a smooth projective curve Y are equal, and the degree δ of the module Ω_Y is $2g-2$, which is equal to $2p_a-2$.*

Thus the Riemann-Roch Theorem 8.2.3 can we written as

$$\chi(\mathcal{O}(D)) = \deg D + 1 - g. \tag{8.8.3}$$

We'll write it this way when the theorem is proved.

Proof. Let $Y \xrightarrow{\pi} X$ be a branched covering with $X = \mathbb{P}^1$ and of degree n. The topological Euler characteristic $e(Y) = 2-2g$ can be computed in terms of the branching data for the covering, as in (1.7.24). Let q_i be the ramification points in Y, and let e_i be the ramification index at q_i. Then e_i sheets of the covering come together at q_i. This decreases the number of points in the fibre of π that contains q_i by $e_i - 1$. So

$$2-2g = e(Y) = ne(X) - \sum(e_i - 1) = 2n - \sum(e_i-1). \tag{8.8.4}$$

We compute the degree δ of Ω_Y in two ways. First, the Riemann-Roch Theorem tells us that $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = p_a$ and $\mathbf{h}^1\Omega_Y = \mathbf{h}^0\mathcal{O}_Y = 1$. So $\chi(\Omega_Y) = -\chi(\mathcal{O}_Y) = p_a - 1$. The Riemann-Roch Theorem also tells us that $\chi(\Omega_Y) = \delta + 1 - p_a$. Therefore

$$\delta = 2p_a - 2. \tag{8.8.5}$$

Next, we compute δ by computing the divisor of the differential dx on Y, x being a coordinate on the projective line.

Let e_i be the ramification index at a ramification point q_i. Then dx has a zero of order $e_i - 1$ at q_i. At the point of X at infinity, dx has a pole of order 2. Let's choose coordinates so that the point at infinity isn't a branch point. Then there will be n points of Y at which dx has a pole of order 2, n being the degree of Y over X. The degree of Ω_Y is therefore

$$\delta = \text{zeros} - \text{poles} = \sum(e_i - 1) - 2n. \tag{8.8.6}$$

Combining (8.8.6) with (8.8.4), one sees that $\delta = 2g - 2$. Since we also have $\delta = 2p_a - 2$, we conclude that $g = p_a$. □

8.8.7. Canonical divisors. Because the module Ω_Y of differentials on a smooth curve Y is invertible, it is isomorphic to $\mathcal{O}(K)$ for some divisor K (Proposition 8.1.13). Such a divisor is called a *canonical divisor*. The degree of a canonical divisor is $2g-2$, the same as the degree of Ω_Y (see Theorem 8.8.2). It is often convenient to represent Ω_Y as a module $\mathcal{O}(K)$, though the canonical divisor K isn't unique. It is determined only up to linear equivalence.

When written in terms of a canonical divisor K, the Serre dual of an invertible module $\mathcal{O}(D)$ is

$$\mathcal{O}(D)^S = {}_Y(\mathcal{O}(D), \mathcal{O}(K)) \approx \mathcal{O}(D)^* \otimes_{\mathcal{O}} \mathcal{O}(K) \approx \mathcal{O}(-D) \otimes_{\mathcal{O}} \mathcal{O}(K) \approx \mathcal{O}(K - D) \tag{8.8.8}$$

(see Proposition 8.1.12 and Corollary 6.4.25). With this notation, the Riemann-Roch Theorem for $\mathcal{O}(D)$ becomes

$$\mathbf{h}^0\mathcal{O}(D) = \mathbf{h}^1\mathcal{O}(K - D) \quad \text{and} \quad \mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(K - D). \tag{8.8.9}$$ □

8.8.10. Curves of genus 0. Every smooth projective curve of genus 0 is isomorphic to the projective line $\mathbb{P}^1$. The proof is an exercise.

A *rational curve* is a curve, smooth or not, whose function field is isomorphic to the field $\mathbb{C}(t)$ of rational functions in one variable. A smooth projective curve of genus 0 is a rational curve.

8.8.11. Curves of genus 1. A smooth projective curve of genus $g = 1$ is called an *elliptic curve*.

Let Y be an elliptic curve. The Riemann-Roch Theorem tells us that, on X,

$$\chi(\mathcal{O}(D)) = \deg D$$

and that $\mathbf{h}^0\Omega_Y = \mathbf{h}^1\mathcal{O}_Y = 1$. So Ω_Y has a nonzero global section ω. Also, Ω_Y has degree 0 (see Theorem 8.8.2). Therefore the global section doesn't vanish anywhere, and multiplication defines an isomorphism $\mathcal{O} \xrightarrow{\omega} \Omega_Y$.

8.8.12. Lemma. *Let D be a divisor of positive degree r on an elliptic curve Y. Then $\mathbf{h}^0\mathcal{O}(D) = r$, and $\mathbf{h}^1\mathcal{O}(D) = 0$. In particular, $\mathbf{h}^0(\mathcal{O}(rp)) = r$ and $\mathbf{h}^1(\mathcal{O}(rp)) = 0$.*

Proof. Since Ω_Y is isomorphic to $\mathcal{O}$, $K = 0$ is a canonical divisor, and the Serre dual of $\mathcal{O}(D)$ is $\mathcal{O}(-D)$. Then $\mathbf{h}^1\mathcal{O}(D) = \mathbf{h}^0\mathcal{O}(-D)$, which is 0 when the degree of D is positive. □

Now, since $H^0(Y, \mathcal{O}_Y) \subset H^0(Y, \mathcal{O}_Y(p))$ and since both spaces have dimension 1, they are equal. So (1) is a basis for $H^0(Y, \mathcal{O}_Y(p))$. We choose a basis $(1, x)$ for the two-dimensional space $H^1(Y, \mathcal{O}_Y(2p))$. Then x isn't a section of $\mathcal{O}(p)$. It has a pole of order precisely 2 at p, and no other pole. Next, we choose a basis $(1, x, y)$ for $H^1(Y, \mathcal{O}_Y(3p))$. So y has a pole of order 3 at p.

The point $(1, x, y)$ of $\mathbb{P}^2$ with values in K determines a morphism $Y \xrightarrow{\varphi} \mathbb{P}^2$. Let u, v, w be coordinates in $\mathbb{P}^2$. The map φ sends a point q distinct from p to $(u, v, w) = (1, x(q), y(q))$. Since Y has dimension 1, φ is a finite morphism, and its image Y' is a plane curve (see Corollary 5.5.4).

To determine the image of the point p, we multiply $(1, x, y)$ by $\lambda = y^{-1}$, obtaining the equivalent vector $(y^{-1}, xy^{-1}, 1)$. The rational function y^{-1} has a zero of order 3 at p, and xy^{-1} has a simple zero there. Evaluating at p, we see that the image of p is the point $(0, 0, 1)$.

Let ℓ be a generic line $\{au + bv + cw = 0\}$ in $\mathbb{P}^2$. The rational function $a + bx + cy$ on Y has a pole of order 3 at p and no other pole. It takes every value, including zero, three times, and the three points of Y at which $a+bx+cy$ is zero form the inverse image of ℓ. The only possibilities for the degree of Y' are 1 and 3. Since 1, x, y are independent, they don't satisfy a homogeneous linear equation. So Y' isn't a line. It is a cubic curve (see Corollary 1.3.10). The rational functions x, y satisfy a cubic relation f.

We determine the form of the polynomial f. The seven monomials 1, x, y, x^2, xy, y^2, x^3 have poles of orders 0, 2, 3, 4, 5, 6, 6 at p, respectively. They are global sections of $\mathcal{O}(6p)$. Riemann-Roch tells us that $\mathbf{h}^0\mathcal{O}_Y(6p) = 6$. So those monomials are linearly dependent. The dependency relation gives us a cubic equation among x and y, of the form

$$cy^2 + (a_1x + a_3)y + (a_0x^3 + a_2x^2 + a_4x + a_6) = 0. \tag{8.8.13}$$

There can be no linear relation among functions whose orders of pole at p are distinct. So when we delete either x^3 or y^2 from the list of seven monomials, we obtain an independent set—a basis for the six-dimensional space $H^0(Y, \mathcal{O}(6p))$. In the cubic relation, the coefficients c and a_0 aren't zero. We normalize them to 1. Then we eliminate the linear term in y from the relation by substituting $y - \frac{1}{2}(a_1x + a_3)$ for y, and we eliminate the quadratic term in x in the resulting polynomial by substituting $x - \frac{1}{3}a_2$ for x. Bringing the terms in x to the other side of the equation, we are left with a cubic relation of the form

$$y^2 = x^3 + a_4 x + a_6.$$

The coefficients a_4 and a_6 have been changed, of course.

This derivation seems magical, but one can obtain the result using only the fact that, because the curve is a cubic, there is some relation f among the monomials in x, y of degree ≤ 3. Among those monomials, y^3 has the largest order of pole, namely 9. If y^3 had nonzero coefficient in f, we could solve for it as a combination of the other monomials. But because its order of pole is greater than the other orders of pole, this is impossible. The coefficient of y^3 in f is 0. This being so, the monomial xy^2 becomes the one with largest order of pole, namely 8. So the coefficient of xy^2 in f is zero, and then similarly, the coefficient of x^2y, with a pole of order 7, is also zero. Since the relation f is a cubic polynomial, the only remaining monomial of degree 3, which is x^3, has a nonzero coefficient. Both x^3 and y^2 have poles of order 6, and the remaining monomials $1, x, y, x^2, xy$ have orders of pole less than 6. So y^2 also has a nonzero coefficient, and f has the form (8.8.13) in which c and a_0 are nonzero. The rest of the reduction is the same.

The cubic curve Y' defined by the homogenized equation $y^2z = x^3 + a_4xz^2 + a_6z^3$ is the image of Y. This curve meets a generic line $ax+by+cz = 0$ in three points and, as we saw above, its inverse image in Y consists of three points too. Therefore the morphism $Y \xrightarrow{\varphi} Y'$ is generically injective, and Y is the normalization of Y'. Projection from a singular point shows that a singular curve of degree 3 is a rational curve. Therefore Y' is smooth and is isomorphic to Y.

8.8.14. The group law on an elliptic curve. The points of an elliptic curve Y form an abelian group, once one fixes a point as the identity element.

We choose an identity element and label it o. We write the law of composition as $p \oplus q$, to make a clear distinction between the product in the group, which is a point of Y, and the divisor $p + q$. To define $p \oplus q$, we compute the cohomology of $\mathcal{O}_Y(p+q-o)$. Lemma 8.8.12 tells us that $\mathbf{h}^0\mathcal{O}_Y(p+q-o) = 1$. So there is a nonzero function f, unique up to scalar factor, with simple poles at p and q and a simple zero at o. This function has exactly one additional zero. That zero is defined to be the sum $p \oplus q$. In terms of linearly equivalent divisors, $p \oplus q$ is the unique point s such that the divisor $p+q$ is linearly equivalent to $o + s$.

8.8.15. Proposition. *With the law of composition $\oplus$ defined above, an elliptic curve becomes an abelian group.*

The proof makes a good exercise. □

8.8.16. Mapping a curve to projective space. Let Y be a smooth projective curve. We have seen that any set $(f_0, \dots, f_n)$ of rational functions on Y, not all zero, defines a morphism $Y \xrightarrow{\varphi} \mathbb{P}^n$ (see Proposition 5.2.6). For review: Let q be a point of Y and let $g_j = f_j/f_i$, where i is an index such that the value $v_q(f_i)$ is a minimum, for $k = 0, \dots, n$. The rational functions g_j are regular at q for all j, and the morphism φ sends the point q to $(g_0(q), \dots, g_n(q))$. For example, the inverse image $\varphi^{-1}(\mathbb{U}^0)$ of the standard open set $\mathbb{U}^0$ is the set of points of Y at which the functions $g_j = f_j/f_0$ are regular. If q is such a point, then $\varphi(q) = (1, g_1(q), \dots, g_n(q))$.

8.8.17. Lemma. *Let Y be a smooth projective curve, and let $Y \xrightarrow{\varphi} \mathbb{P}^n$ be the morphism to projective space defined by a set $(f_0, \dots, f_n)$ of rational functions on Y.*

(i) *If the space spanned by $\{f_0, \dots, f_n\}$ has dimension at least 2, then φ isn't a constant morphism to a point.*

(ii) *If $f_0, \dots, f_n$ are linearly independent, the image of Y isn't contained in a hyperplane.* □

8.8.18. Base points. Let D be a divisor on the smooth projective curve Y, and suppose that $\mathbf{h}^0\mathcal{O}(D) = k > 1$. A basis $(f_0, \dots, f_k)$ of global sections of $\mathcal{O}(D)$ defines a morphism $Y \to \mathbb{P}^{k-1}$. This is a common way to construct a morphism to projective space, though one could use any set of rational functions that aren't all zero.

If a global section of $\mathcal{O}(D)$ vanishes at a point p of Y, it is a global section of $\mathcal{O}(D-p)$. A *base point* of $\mathcal{O}(D)$ is a point of Y at which *every* global section of $\mathcal{O}(D)$ vanishes. A base point can be described in terms of the usual exact sequence

$$0 \to \mathcal{O}(D-p) \to \mathcal{O}(D) \to \kappa_p \to 0.$$

The point p is a base point if $\mathbf{h}^0\mathcal{O}(D-p) = \mathbf{h}^0\mathcal{O}(D)$ or if $\mathbf{h}^1\mathcal{O}(D-p) = \mathbf{h}^1\mathcal{O}(D) - 1$.

The *degree* of a nonconstant morphism $Y \xrightarrow{\varphi} \mathbb{P}^n$ is defined to be the number of points of the inverse image $\varphi^{-1}H$ of a generic hyperplane H in $\mathbb{P}^n$.

8.8.19. Lemma. *Let D be a divisor on a smooth projective curve Y with $\mathbf{h}^0\mathcal{O}(D) = n+1$, $n > 0$, and let $Y \xrightarrow{\varphi} \mathbb{P}^n$ be the morphism defined by a basis of global sections.*

(i) *The image of φ isn't contained in any hyperplane.*

(ii) *If $\mathcal{O}(D)$ has no base point, the degree of φ is equal to the degree of D. If there are base points, the degree is lower.*

Proof.

(ii) We may assume that D is an effective divisor of degree d. If D has no base point, then for every point p in the support of D, $H^0(\mathcal{O}(D-p)) < H^0(\mathcal{O}(D))$. A generic section f of $\mathcal{O}(D)$ will not vanish at any point of D. Then poles(f) will be equal to D, and zeros(f) will have degree d. Proposition 8.5.3 shows that zeros(f) consists of d points. □

8.8.20. Proposition. *Let K be a canonical divisor on a smooth projective curve Y of genus $g > 0$.*

(i) *$\mathcal{O}(K)$ has no base point.*

(ii) *Every point p of Y is a base point of $\mathcal{O}(K+p)$.*

Proof.

(i) Let p be a point of Y. We apply Riemann-Roch to the exact sequence

$$0 \to \mathcal{O}(K-p) \to \mathcal{O}(K) \to \kappa_p \to 0.$$

The Serre duals of $\mathcal{O}(K)$ and $\mathcal{O}(K-p)$ are $\mathcal{O}(K)^S = \mathcal{O}$ and $\mathcal{O}(K-p)^S = \mathcal{O}(p)$, respectively. They form an exact sequence

$$0 \to \mathcal{O} \to \mathcal{O}(p) \to \kappa_p \to 0.$$

Because Y has positive genus, there is no rational function on Y with just one simple pole. So $\mathbf{h}^0\mathcal{O} = \mathbf{h}^0\mathcal{O}(p) = 1$. Riemann-Roch tells us that $\mathbf{h}^1\mathcal{O}(K-p) = \mathbf{h}^1\mathcal{O}(K) = 1$. The cohomology sequence

$$0 \to H^0(\mathcal{O}(K-p)) \to H^0(\mathcal{O}(K)) \to [1] \to H^1(\mathcal{O}(K-p)) \to H^1(\mathcal{O}(K)) \to 0$$

shows that $\mathbf{h}^0\mathcal{O}(K-p) = \mathbf{h}^0\mathcal{O}(K) - 1$. So p is not a base point.

(ii) Here, the relevant sequence is

$$0 \to \mathcal{O}(K) \to \mathcal{O}(K+p) \to \kappa_p \to 0.$$

The Serre dual of $\mathcal{O}(K+p)$ is $\mathcal{O}(-p)$, which has no global section. Therefore $\mathbf{h}^1\mathcal{O}(K+p) = 0$, while $\mathbf{h}^1\mathcal{O}(K) = \mathbf{h}^0\mathcal{O} = 1$. The cohomology sequence

$$0 \to \mathbf{h}^0\mathcal{O}(K) \to \mathbf{h}^0\mathcal{O}(K+p) \to [1] \to \mathbf{h}^1\mathcal{O}(K) \to \mathbf{h}^1\mathcal{O}(K+p) \to 0$$

shows that

$$H^0(\mathcal{O}(K+p)) = H^0(\mathcal{O}(K)).$$

So p is a base point of $\mathcal{O}(K+p)$. □

8.8.21. Hyperelliptic curves. A *hyperelliptic curve* Y is a smooth projective curve of genus $g \geq 2$ that can be represented as a branched double covering of the projective line—such that there exists a morphism $Y \xrightarrow{\pi} X = \mathbb{P}^1$ of degree 2. The term 'hyperelliptic' comes from the fact that every elliptic curve can be represented (though not uniquely) as a double cover of $\mathbb{P}^1$. The global sections of $\mathcal{O}(2p)$, where p is any point of an elliptic curve Y, define a map to $\mathbb{P}^1$ of degree 2.

The topological Euler characteristic of a hyperelliptic curve Y can be computed in terms of the double covering $Y \to X$. The covering will be branched at a finite set of points of X. The branch points are those such that the fibre consists of one point. If there are n branch points, the Euler characteristic is $e(Y) = 2e(X) - n = 4 - n$. Since we know that $e(Y) = 2 - 2g$, the number of branch points is $n = 2g + 2$. When $g = 3$, $n = 8$.

It would take some experimentation to guess that the next remarkable theorem might be true and some time to find a proof.

8.8.22. Theorem. *Let Y be a hyperelliptic curve, and let $Y \xrightarrow{\pi} X = \mathbb{P}^1$ be a branched covering of degree 2. The morphism $Y \xrightarrow{\psi} \mathbb{P}^{g-1}$ defined by the global sections of $\Omega_Y = \mathcal{O}(K)$ factors through π. There is a unique morphism $X \xrightarrow{u} \mathbb{P}^{g-1}$ such that ψ is the composed map $Y \xrightarrow{\pi} X \xrightarrow{u} \mathbb{P}^{g-1}$:*

$$\begin{array}{ccc} Y & \xrightarrow{\pi} & X \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle u} \\ \mathbb{P}^{g-1} & = & \mathbb{P}^{g-1} \end{array}$$

8.8.23. Corollary. *A curve of genus $g \geq 2$ can be presented as a branched covering of $\mathbb{P}^1$ of degree 2 in at most one way.* □

Proof of Theorem 8.8.22. Let x be an affine coordinate in X, so that the standard affine open subset $\mathbb{U}^0$ of X is $\operatorname{Spec} \mathbb{C}[x]$. Suppose that the point of X at infinity isn't a branch point of the covering π. The open set $Y^0 = \pi^{-1}\mathbb{U}^0$ will be described by an equation of the form $y^2 = f(x)$, where f is a polynomial of degree $n = 2g+2$ with simple roots, and there will be two points of Y above the point at infinity. They are interchanged by the automorphism $y \to -y$. Let's call those points q_1 and q_2.

We start with the differential dx, which we view as a rational differential on Y. Then $2y\,dy = f'(x)\,dx$. Since f has simple roots at the branch points, f' doesn't vanish at any of them. Therefore dx has simple zeros on Y at the points at which $y = 0$, the points above the branch points of f on X. We also have a regular function on Y^0 with simple roots at those points, the function y.

Therefore the differential $\omega = \frac{dx}{y}$ is regular and nowhere zero on Y^0. Because the degree of a differential on Y is $2g-2$, ω has a total of $2g-2$ zeros at infinity. By symmetry, ω has zeros of order $g-1$ at q_1 and at q_2. Then $K = (g-1)q_1 + (g-1)q_2$ is a canonical divisor on Y.

Since K has zeros of order $g-1$ at infinity, the rational functions $1, x, x^2, \dots, x^{g-1}$, when viewed as functions on Y, are among the global sections of $\mathcal{O}_Y(K)$. They are independent, and there are g of them. Since $\mathbf{h}^0\mathcal{O}_Y(K) = g$, they form a basis of $H^0(\mathcal{O}_Y(K))$. The map $Y \to \mathbb{P}^{g-1}$ defined by the global sections of $\mathcal{O}_Y(K)$ evaluates these powers of x, so it factors through X. □

When Y is a hyperelliptic curve of genus 3, its image via the canonical map will be a plane conic, and if the genus of a hyperelliptic curve is 4, its image will be a twisted cubic in $\mathbb{P}^3$.

8.8.24. Canonical embedding. Let Y be a smooth projective curve of genus $g \geq 2$, and let K be a canonical divisor on Y. Its global sections define a morphism $Y \to \mathbb{P}^{g-1}$. This morphism is called the *canonical map*. We denote the canonical map by ψ, as before. Since $\mathcal{O}(K)$ has no base point, the degree of ψ is the degree $2g-2$ of the canonical divisor. Theorem 8.8.22 shows that, when Y is hyperelliptic, the image of the canonical map is isomorphic to $\mathbb{P}^1$.

8.8.25. Theorem. *Let Y be a smooth projective curve of genus g at least 2. If Y is not hyperelliptic, the canonical map embeds Y as a closed subvariety of projective space $\mathbb{P}^{g-1}$.*

Proof. We show first that if the canonical map $Y \xrightarrow{\psi} \mathbb{P}^{g-1}$ isn't injective, then Y is hyperelliptic.

Let p and q be distinct points of Y with the same image: $\psi(p) = \psi(q)$. We choose an effective canonical divisor K whose support doesn't contain p or q, and we inspect the global sections of $\mathcal{O}(K-p-q)$. Since $\psi(p) = \psi(q)$, any global section of $\mathcal{O}(K)$ that vanishes at p vanishes at q too. Therefore $\mathcal{O}(K-p)$ and $\mathcal{O}(K-p-q)$ have the same global sections, and q is a base point of $\mathcal{O}(K-p)$. We've computed the cohomology of $\mathcal{O}(K-p)$ before: $\mathbf{h}^0\mathcal{O}(K-p) = g-1$ and $\mathbf{h}^1\mathcal{O}(K-p) = 1$. Therefore $\mathbf{h}^0\mathcal{O}(K-p-q) = g-1$ and $\mathbf{h}^1\mathcal{O}(K-p-q) = 2$. The Serre dual of $\mathcal{O}(K-p-q)$ is $\mathcal{O}(p+q)$, so by Riemann-Roch, $\mathbf{h}^0\mathcal{O}(p+q) = 2$. For any divisor D of degree 1 on a curve of positive genus, $\mathbf{h}^0(\mathcal{O}(D)) \leq 1$. So $\mathcal{O}(p+q)$ has no base point, and the global sections of $\mathcal{O}(p+q)$ define a morphism $Y \to \mathbb{P}^1$ of degree 2. The curve Y is hyperelliptic.

If Y isn't hyperelliptic, the canonical map is injective, so Y is mapped bijectively to its image Y' in $\mathbb{P}^{g-1}$. This almost proves the theorem, but: Can Y' have a cusp? We must show that the bijective map $Y \xrightarrow{\psi} Y'$ is an isomorphism. We go over the computation made above for a pair of points p, q, this time taking

$q = p$. The computation is the same. Since Y isn't hyperelliptic, p isn't a base point of $\mathcal{O}_Y(K-p)$. Therefore $\mathbf{h}^0\mathcal{O}_Y(K-2p) = \mathbf{h}^0\mathcal{O}_Y(K-p)-1$. This tells us that there is a global section f of $\mathcal{O}_Y(K)$ that has a zero of order exactly 1 at p. When properly interpreted, this fact shows that ψ doesn't collapse any tangent vector at p and that ψ is an isomorphism. Since we haven't discussed tangent vectors, we prove this directly.

Since ψ is bijective, the function fields of Y and its image Y' are equal, and Y is the normalization of Y'. Moreover, ψ is an isomorphism except on a finite set. We work locally at a point p' of Y', denoting the unique point of Y that maps to p' by p. When we restrict the global section f of $\mathcal{O}_Y(K)$ found above to the image Y', we obtain an element of the maximal ideal $\mathfrak{m}_{p'}$ of $\mathcal{O}_{Y'}$ at p' that we denote by x. On Y, x has a zero of order 1 at p. Therefore it is a local generator for the maximal ideal $\mathfrak{m}_p$ of $\mathcal{O}_Y$. We apply the Local Nakayama Lemma 5.1.1. Let R' and R denote the local rings at p. We regard R as a finite R'-module. Since x is in $\mathfrak{m}_{p'}$, $R/\mathfrak{m}_{p'}R$ is the residue field of R, which is spanned, as R'-module, by the element 1. The Local Nakayama Lemma, with $V = R$ and $M = \mathfrak{m}_{p'}$, tells us that R is spanned by 1, and this shows that $R = R'$. □

8.8.26. Some curves of low genus.

Curves of genus 2. When Y is a smooth projective curve of genus 2. The canonical map ψ is a map from Y to $\mathbb{P}^1$, of degree $2g-2 = 2$. Every smooth projective curve of genus 2 is hyperelliptic.

Curves of genus 3. Let Y be a smooth projective curve of genus 3. The canonical map ψ is a morphism of degree 4 from Y to $\mathbb{P}^2$. If Y isn't hyperelliptic, its image will be a plane curve of degree 4 that is isomorphic to Y. This agrees with the fact that the genus of a smooth projective curve of degree 4 is equal to 3 (see (1.7.26)).

There is another way to arrive at the same result. We go through it because the method can be used for curves of genus 4 or 5.

Let K be a canonical divisor on a smooth curve of genus $g > 1$. When $d > 1$,

$$\mathbf{h}^1\mathcal{O}(dK) = \mathbf{h}^0\mathcal{O}(K-dK) = 0 \quad \text{and}$$

(8.8.27) $$\mathbf{h}^0\mathcal{O}(dK) = \deg(dK) + 1 - g = (2d-1)(g-1).$$

In our case $g = 3$, and $\mathbf{h}^0\mathcal{O}(dK) = 4d-2$, when $d > 1$.

The number of monomials of degree d in $n+1$ variables is $\binom{n+d}{n}$. Here $n = 2$, so that number is $\binom{d+2}{2}$.

We assemble this information into a table:

d	0	1	2	3	4	5
monomials of deg d	1	3	6	10	15	21
$\mathbf{h}^0\mathcal{O}(dK)$	1	3	6	10	14	18

Let $(\alpha_0, \alpha_1, \alpha_2)$ be a basis of $H^0(\mathcal{O}(K))$. The monomials of degree d in $\alpha_0, \alpha_1, \alpha_2$ are global sections of $\mathcal{O}(dK)$. The table shows that there is at least one nonzero homogeneous polynomial $f(x_0, x_1, x_2)$ of degree 4, such that $f(\alpha) = 0$, and Y lies in the zero locus of that polynomial. When Y isn't hyperelliptic, there won't be any linear relation among monomials of degree less than 4. So Y is a quartic curve, and f is, up to scalar factor, the only homogeneous quartic that vanishes on Y. The monomials of degree 4 in α span a space of dimension 14, and therefore they span $H^0(\mathcal{O}(4K))$.

The table also shows that there are (at least) three independent polynomials of degree 5 that vanish on Y. They are $x_0 f, x_1 f, x_2 f$.

Curves of genus 4. When Y is a smooth projective curve of genus 4 that isn't hyperelliptic, the canonical map embeds Y as a curve of degree 6 in $\mathbb{P}^3$. Let's leave the analysis of this case as an exercise.

Curves of genus 5. With genus 5, things become more complicated. Let Y be a smooth projective curves of genus 5 that isn't hyperelliptic. The canonical map embeds Y as a curve of degree 8 in $\mathbb{P}^4$. We make a computation analogous to what was done for genus 3. For $d > 1$, the dimension of the space of global sections of $\mathcal{O}(dK)$ is

$$\mathbf{h}^0\mathcal{O}(dK) = 8d - 4$$

and the number of monomials of degree d in 5 variables is $\binom{d+4}{4}$. We form a table:

d	0	1	2	3
monomials of deg d	1	5	15	35
$\mathbf{h}^0\mathcal{O}(dK)$	1	5	12	20

This table predicts that there are at least three independent homogeneous quadratic polynomials q_1, q_2, q_3 that vanish on the curve Y. Let Q_i be the quadric $\{q_i = 0\}$. Then Y will be contained in the zero locus $Z = Q_1 \cap Q_2 \cap Q_3$.

One can expect the intersection of the three quadrics in $\mathbb{P}^4$ to have dimension 1. If so, a version of Bézout's Theorem asserts that the degree of the intersection will be $2 \cdot 2 \cdot 2 = 8$. Let's skip the proof, which is similar to the proof of the usual Bézout Theorem. Then the intersection Z will have the same degree as Y, so $Y = Z$. In this case, Y is called a *complete intersection* of the three quadrics. However, it is possible that the intersection Z has dimension 2.

A curve Y that can be represented as a three-sheeted covering of $\mathbb{P}^1$ is called a *trigonal curve*.

8.8.28. Proposition. *A trigonal curve of genus* 5 *isn't isomorphic to an intersection of three quadrics in* $\mathbb{P}^4$.

Proof. Given a trigonal curve Y of genus 5, we inspect the morphism of degree 3 to the projective line X. We choose a fibre of that morphism, say the fibre $\{q_1, q_2, q_3\}$ over a point p of X, and we adjust coordinates in X so that p is the point at infinity. With coordinates (x_0, x_1) on X, the rational function $u = x_1/x_0$ on X has poles $D = q_1 + q_2 + q_3$ on Y. So $H^0(Y, \mathcal{O}_Y(D))$ contains 1 and u, and therefore $\mathbf{h}^0\mathcal{O}_Y(D) \geq 2$. By Riemann-Roch, $\chi(\mathcal{O}_Y(D)) = 3+1-g = -1$. So $\mathbf{h}^1\mathcal{O}_Y(D) \geq 3$, and therefore $\mathbf{h}^0\mathcal{O}_Y(K-D) \geq 3$. There are three independent global sections of $\mathcal{O}_Y(K)$ that vanish on D, say $\alpha_0, \alpha_1, \alpha_2$. We extend this set to a basis $(\alpha_0, \dots, \alpha_4)$ of $\mathcal{O}_Y(K)$, and we embed Y into $\mathbb{P}^4$ by that basis. With coordinates $x_0, \dots, x_4$ in $\mathbb{P}^4$, the three hyperplanes $H_i : \{x_i = 0\},\ i = 0, 1, 2$, contain the points q_1, q_2, q_3. The intersection of those hyperplanes is a line L in $\mathbb{P}^4$ that contains the three points.

We go back to the quadrics Q_1, Q_2, Q_3 in $\mathbb{P}^4$ that contain Y. Since the quadrics contain Y, they contain D. A quadric intersects a line in at most two points unless it contains that line. Therefore each of the quadrics Q_i contains L. So $Q_1 \cap Q_2 \cap Q_3$ contains L. Since the point p over which the fibre of the morphism $Y \to X$ is q_1, q_2, q_3 was arbitrary, $Q_1 \cap Q_2 \cap Q_3$ contains a family of lines L parametrized by X, a ruled surface. It doesn't have dimension 1. □

This is the only exceptional case. A curve of genus 5 is hyperelliptic or trigonal, or else it is a complete intersection of three quadrics in $\mathbb{P}^4$, but we omit the proof. We've done enough.

8.9. What Is Next

These remarks are meant for someone who has become reasonably comfortable with the material in these notes and wishes to continue.

First, spend some time, not too much time, learning about varieties over arbitrary ground fields and schemes. I suggest reading in the books by Hartshorne or Cutkosky that are listed in the references.

If you wish to continue with algebraic geometry, I recommend the book *Complex Algebraic Surfaces* by Beauville. Algebraic surfaces have an interesting intrinsic geometry, formed by curves and their intersections. They were classified by the mathematicians Castelnuovo and Enriques, who spent much of their careers studying them. Enriques wrote, with tongue in cheek: *mentre le curve algebriche sono create da Dio, le superficie invece sono opera del Demonio.*

If you are interested in arithmetic, I recommend reading one of Serre's books, such as *Algebraic Groups and Class Fields*. Indeed, one can't go wrong reading any of Serre's books. Even when the book doesn't cover what you think you want, it will be so clear as to be useful.

In either case, a long-range goal should be to understand Mumford's work on geometric invariant theory.

Exercises

8.1. Let Y be a smooth projective curve of genus $p_a = 1$. Use Version 1 of the Riemann-Roch Theorem to determine the dimensions of the $\mathcal{O}$-modules $\mathcal{O}(rp)$.

8.2. Let C be a smooth projective curve. Use Version 1 of Riemann-Roch to prove the following:
  (i) There are positive divisors D such that $\mathbf{h}^1\mathcal{O}(D) = 0$.
  (ii) Let D be a positive divisor such that $\mathbf{h}^0\mathcal{O}(D) \geq 2$ and $\mathbf{h}^1\mathcal{O}(D) = 0$. If p is a generic point of C, then $\mathbf{h}^1\mathcal{O}(D-p) = 0$.
  (iii) If $d \geq p_a(C)$, there is a positive divisor D of degree $d \geq p_a$, such that $\mathbf{h}^1\mathcal{O}(D) = 0$.

8.3. Prove that a projective curve Y such that $\mathbf{h}^1(\mathcal{O}_Y) = 0$, smooth or not, is isomorphic to the projective line $\mathbb{P}^1$.

8.4. Let C be a smooth projective curve of genus 2. It follows from Exercise 8.2 that there is a positive divisor D of degree 4 such that $\mathbf{h}^0\mathcal{O}(D) = 3$. A basis of global sections of $\mathcal{O}(D)$ defines a morphism $C \to \mathbb{P}^2$.
  (i) Prove that the image of C is either a plane curve of degree 4 with a double point or it is a conic.
  (ii) Prove that C can be represented as a double cover of $\mathbb{P}^1$.

8.5. Let D be a divisor on a smooth projective curve Y, and suppose $\mathbf{h}^0\mathcal{O}_X(D) > 1$. When Y is mapped to projective space using a basis for $H^0(\mathcal{O}_Y(D))$, what is the inverse image in Y of a hyperplane?

8.6. (i) Prove that every projective curve of degree 2 is a plane conic.
  (ii) Classify projective curves of degree 3.

8.7. For the branched covering $Y \to X$, where $X = \operatorname{Spec}\mathbb{C}[x]$ and Y is the locus $y^e = x$ in $\operatorname{Spec}\mathbb{C}[x,y]$, compute $\boldsymbol{\tau}(dy)$ and $\boldsymbol{\tau}(dy/y)$.

8.8. Let D be a divisor of degree d on a smooth projective curve Y, such that $\mathbf{h}^0\mathcal{O}(D) = k > 0$. Prove that $\mathbf{h}^0\mathcal{O}(D) \leq d+1$ and that if $\mathbf{h}^0\mathcal{O}(D) = d+1$, then X is isomorphic to $\mathbb{P}^1$.

8.9. Prove that every nonempty open subset of a smooth affine curve is affine.

8.10. Let Y be a smooth projective curve of genus 2. Determine the possible dimensions of $H^q(Y, \mathcal{O}(D))$, when D is an effective divisor of degree n.

8.11. The two standard affine open sets $U^0 = \operatorname{Spec} R_0$ and $U^1 = \operatorname{Spec} R_1$ with $R_0 = \mathbb{C}[u]$ with $u = x_1/x_0$ and $R_1 = \mathbb{C}[u^{-1}]$ cover $\mathbb{P}^1$. The intersection U^{01} is the spectrum of the Laurent polynomial ring $R_{01} = \mathbb{C}[u, u^{-1}]$. The units of R_{01} are the monomials cu^k, where k can be any integer.

(i) Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be an invertible R_{01}-matrix. Prove that there is an invertible R_0-matrix Q and that there is an invertible R_1-matrix P, such that $Q^{-1}AP$ is diagonal.

(ii) Use (i) to prove the Birkhoff-Grothendieck Theorem for torsion-free $\mathcal{O}_X$-modules of rank 2.

8.12. On $\mathbb{P}^1$, when is $\mathcal{O}(m) \oplus \mathcal{O}(n)$ isomorphic to $\mathcal{O}(r) \oplus \mathcal{O}(s)$?

8.13. Let Y be an elliptic curve.

(i) Prove that, with the law of composition $\oplus$ defined in Subsection 8.8.14, Y is an abelian group.

(ii) Let p be a point of Y. Describe the sum $p \oplus p \cdots \oplus p$ of k copies of p.

(iii) Determine the number of points of order 2 on Y.

(iv) Suppose that Y is a plane curve and that the origin is a flex point. Show that the other flexes of Y are the points of order 3, and determine the number of points of Y of order 3.

8.14. How many real flex points can a real cubic curve have?

8.15. Prove that a finite $\mathcal{O}$-module on a smooth curve is a direct sum of a torsion module and a locally free module.

8.16. (i) Let $B = A[x]$ be the ring of polynomials in one variable with coefficients in a finite-type domain A. Describe the module Ω_B in terms of Ω_A.

(ii) Let A_s be a localization of A. Describe the module Ω_{A_s} in terms of Ω_A.

8.17. Let K be the function field of a variety X of dimension d. Prove that Ω_K is a K-module of dimension d.

8.18. Let $Y \to X$ be a branched covering, and let p be a point of X whose inverse image in Y consists of one point q. Prove a local analytic version of the main theorem on the trace map for differentials by computation.

8.19. Let $Y = \operatorname{Spec} B$ be a smooth affine curve, and let y be an element of B. At what points is dy a local generator for Ω_Y?

8.20. Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth affine curves, $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$, and let $B \xrightarrow{\delta} {}_A(B, \Omega_A)$ be the composition of the derivation $B \xrightarrow{d} \Omega_B$ with the trace map $\Omega_B \approx {}_B(B, \Omega_B) \xrightarrow{\tau\otimes} {}_A(B, \Omega_A)$. Prove that δ is a derivation from B to the B-module ${}_A(B, \Omega_A)$.

8.21. Can a trigonal curve be hyperelliptic?

8.22. Let $K = \mathbb{C}(t)$ be the function field of $Y = \mathbb{P}^1$, and let F be a subfield of K that properly contains $\mathbb{C}$.
- (i) Prove that $[K : F] < \infty$, that F is the function field of a smooth projective curve X, and that Y is a branched covering of X.
- (ii) Prove *Lüroth's Theorem*: X is isomorphic to $\mathbb{P}^1$.

8.23. Let F_1, F_1 be subfields of $K = \mathbb{C}(t)$ that contain $\mathbb{C}$ and such that $[K : F_i] = 2$. Let $L = F_1 \cap F_2$. What can be said about $[K : L]$?

8.24.
- (i) Let Y be a plane curve of degree 5 with a node as its only singularity. Show that the projection of the plane to $\mathbb{P}^1$ with the double point as center of projection represents Y as a trigonal curve of genus 5.
- (ii) The canonical embedding of a trigonal genus 5 curve Y will have three collinear points p_1, p_2, p_3. Let $D = p_1 + p_2 + p_3$. Show that $\mathbf{h}^0\mathcal{O}(K-D) = 3$, that $\mathcal{O}(K-D)$ has no base point. Show that a basis of $H^0\mathcal{O}(K-D)$ maps Y to a curve of degree 5 in $\mathbb{P}^2$ with a double point.

8.25. *The basepoint-free trick.* Let D be an effective divisor on a smooth projective curve Y. Suppose that $\mathcal{O}(D)$ has no base point and that $\mathbf{h}^1\mathcal{O}(D) = 0$. Choose global sections α, β of $\mathcal{O}(D)$ with no common zeros. Prove the following:
- (i) The sections α, β generate the $\mathcal{O}$-module $\mathcal{O}(D)$, and there is an exact sequence

$$0 \to \mathcal{O}(-D) \xrightarrow{(-\beta,\alpha)} \mathcal{O}^2 \xrightarrow{(\alpha,\beta)^t} \mathcal{O}(D) \to 0.$$

- (ii) The tensor product of this sequence with $\mathcal{O}(kD)$ is an exact sequence

$$0 \to \mathcal{O}((k-1)D) \xrightarrow{(-\beta,\alpha)} \mathcal{O}(kD)^2 \xrightarrow{(\alpha,\beta)^t} \mathcal{O}((k+1)D) \to 0.$$

- (iii) If $H^1\mathcal{O}((k-1)D) = 0$, every global section of $\mathcal{O}((k+1)D)$ can be obtained as a combination $\alpha u + \beta v$ with $u, v \in H^0\mathcal{O}(kD)$.

8.26. Let Y be a smooth projective curve of genus g, and let d be an integer. Prove the following:

(i) If $d < g-1$, then for every divisor D of degree d on Y, $\mathbf{h}^1\mathcal{O}(D) > 0$.

(ii) If $d \le 2g-2$, there exist divisors D of degree d on Y such that $\mathbf{h}^1\mathcal{O}(D) > 0$.

(iii) If $d \ge g-1$, there exist divisors D of degree d on Y such that $\mathbf{h}^1\mathcal{O}(D) = 0$.

(iv) If $d > 2g-2$, then for every divisor D of degree d on Y, $\mathbf{h}^1\mathcal{O}(D) = 0$.

8.27. Let $Y \xrightarrow{\pi} X$ be a branched covering of smooth curves. Prove that the direct image of $\mathcal{O}_Y$ is isomorphic to the direct sum $\mathcal{O}_X \oplus \mathcal{M}$ for some locally free $\mathcal{O}_X$-module $\mathcal{M}$.

8.28. Let Y be a smooth projective curve of genus $g > 1$, and let D be an effective divisor of degree $g+1$ on Y. Suppose that $\mathbf{h}^1\mathcal{O}(D) = 0$ and $\mathbf{h}^0\mathcal{O}(D) = 2$. Let $Y \xrightarrow{\pi} X$ be the morphism to the projective line X defined by a basis $(1, f)$ of $H^0\mathcal{O}(D)$. The $\mathcal{O}_X$-module $\mathcal{O}_Y$ is isomorphic to a direct sum $\mathcal{O}_X \oplus \mathcal{M}$, where $\mathcal{M}$ is a locally free $\mathcal{O}_X$-module of rank g (Exercise 8.27).

(i) Let p be the point at infinity of X. Prove that $\mathcal{O}_Y(D)$ is isomorphic to $\mathcal{O}_Y \otimes_{\mathcal{O}_X} \mathcal{O}_X(p)$.

(ii) Determine the dimensions of cohomology of $\mathcal{M}$ and of $\mathcal{M}(p)$.

(iii) According to the Birkhoff-Grothendieck Theorem, $\mathcal{M}$ is isomorphic to a sum of twisting modules $\sum_{i=1}^{g} \mathcal{O}_X(r_i)$. Determine the twists r_i.

8.29. Let C be the plane curve defined by a homogeneous polynomial $f(x, y, z)$ of degree d.

(i) Suppose that C has a node as its only singularity. Determine the genus of its normalization $C^\#$. Do the same for a curve with a cusp.

(ii) One says that C has an *ordinary r-fold point* at $p = (0, 0, 1)$ if the homogeneous parts f_i of $f(x, y, 1)$ are zero for $i < r$ and f_r has distinct roots. Suppose that C has an ordinary triple point at p and no other singularity. Determine the genus of the normalization $C^\#$.

(iii) Do the same for an ordinary four-fold point.

8.30. Factor the polynomial $x^3y^2 - x^3z^2 + y^3z^2$.

8.31. Let f and g be irreducible, homogeneous polynomials in $\mathbb{C}[x_0, x_1, x_2, x_3]$, of degrees d and e, respectively, and suppose that g is not a scalar multiple of f. Let X be the locus of common zeros of f and g in the projective space $\mathbb{P}^3$, and let i be the inclusion $X \to \mathbb{P}$.

(i) Construct an exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-d-e) \to \mathcal{O}_{\mathbb{P}}(-d) \oplus \mathcal{O}_{\mathbb{P}}(-e) \to \mathcal{O}_{\mathbb{P}} \to i_*\mathcal{O}_X \to 0.$$

(ii) Determine the cohomology of $\mathcal{O}_X$.

(iii) Prove that X is connected, i.e., that it is not the union of two proper disjoint Zariski-closed subsets of $\mathbb{P}$.

8.32. Let

$$N = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \\ y_{31} & y_{32} \end{pmatrix}$$

be a 3×2-matrix whose entries are homogeneous polynomials of degree d in $R = \mathbb{C}[x_0, x_1, x_2]$, and let $M = (m_1, m_2, m_3)$ be the 1×3-matrix of minors:

$$\begin{aligned} m_1 &= \quad y_{21}y_{32} - y_{22}y_{31} \\ m_2 &= -y_{11}y_{32} + y_{12}y_{31} \\ m_3 &= \quad y_{11}y_{22} - y_{12}y_{21}. \end{aligned}$$

Let I be the ideal of R generated by the minors.

(i) Suppose that I is the unit ideal of R. Prove that this sequence is exact:

$$0 \leftarrow R \xleftarrow{M} R^3 \xleftarrow{N} R^2 \leftarrow 0.$$

(ii) Let $X = \mathbb{P}^2$, and suppose that the locus Y of zeros of I in X has dimension 0. Prove that this sequence is exact:

$$0 \leftarrow R/I \leftarrow R \xleftarrow{M} R^3 \xleftarrow{N} R^2 \leftarrow 0.$$

(iii) The sequence in (ii) corresponds to the following sequence, in which the terms R are replaced by twisting modules:

$$0 \leftarrow \mathcal{O}_Y \leftarrow \mathcal{O}_X \xleftarrow{M} \mathcal{O}_X(-2d)^3 \xleftarrow{N} \mathcal{O}_X(-3d)^2 \leftarrow 0.$$

Use this sequence to determine $h^0(Y, \mathcal{O}_Y)$. Check your work in an example in which y_{ij} are homogeneous linear polynomials.