Preface

The main subject of the book are group-theoretic aspects of topological dynamics with a focus on the study of asymptotic properties of groups of dynamical origin and on using group theory in symbolic dynamics.

One of the most basic ways to construct a group is to describe a set of symmetries of a structure either by taking all its symmetries, or by considering a group generated by a collection of symmetries. It is natural to study groups defined in this way by using the underlying symmetric structure. For example, groups of isometries of a metric space are effectively studied using the geometry of the space, fundamental groups of topological spaces are closely related the universal coverings, groups of matrices are studied via their action on the vector spaces. In a similar way, many interesting and important groups are defined and studied using dynamics of their action on topological spaces.

Traditionally, dynamical systems are defined as actions of subgroups or subsemigroups of the group of real numbers, representing time evolution of a system. We are mainly interested in more complicated groups. In fact, most of our groups are rather "exotic" from the point of view of more traditional group theory. They are either defined as groups generated by a given set of homeomorphisms of a topological space, or as groups naturally associated with classical dynamical systems (as their full groups or iterated monodromy groups, for example). Classical group-theoretic methods do not work well in these situations. For example, such groups usually can not be defined by a finite set of relations, and can not be built from simpler groups using classical group theoretic constructions. On the other hand, dynamical approaches are fruitful: one can study topological properties of the orbits of the action, the action on open sets, e.t.c., and derive properties of groups from the obtained information.

Very often properties of the orbits are easy to understand (in particular, when they come from well studied classical dynamical systems), so they are the main tool in the study of properties of the group. For example, while geometry of the Cayley graphs of the group may be very complicated, the geometry of the graphs of the action on the orbits may be very simple.

This dynamical approach to group theory is especially effective for understanding asymptotic properties of the groups. For instance, all currently known examples of groups of intermediate growth (i.e., such groups that the number of elements that can be written as a product of generators of length at most n grows faster than any polynomial but slower than any exponential function of n) are built and studied using their action on topological spaces and dynamical properties of their orbits. Similarly, all known examples of non-elementary amenable groups are also defined and studied using topological dynamics. One of the main purposes of this book is to serve as an introduction to this approach to asymptotic group theory.

The interplay between group theory and topological dynamics also goes in the opposite direction. We will see how a locally expanding covering map $f : \mathcal{X} \longrightarrow \mathcal{X}$ of a compact metric space can be completely encoded in an algebraic structure consisting of a group, called the *iterated monodromy group*, and a "self-similarity" on it (which can be described in different equivalent ways: as a virtual endomorphism of the group, as a homomorphism from the group to the wreath product of the group with a symmetric group, or as a pair of commuting actions on a set). One can reconstruct the dynamical system $f \circlearrowright \mathcal{X}$ from the iterated monodromy group, and also use the algebraic data to study topological properties of the space and the map. For example, one can construct an approximation of the space and of the map by a piecewise linear map between polyhedra starting from the algebraic invariant. Conversely, the properties of the iterated monodromy groups can be deduced from the properties of the dynamical system. In most cases the iterated monodromy groups are also exotic groups from the point of view of classical group theory. For example, some of them are of intermediate growth, non-elementary amenable. Most of them are not finitely presented.

Iterated monodromy groups and self-similarity of groups was the subject of the previous book [**Nek05**]. The current book contains some new applications of iterated monodromy groups (e.g., iterated monodromy groups of maps of several variables, using iterated monodromy groups to study semi-conjugacies of dynamical systems, some new algebraic facts about iterated monodromy groups, e.t.c.), it is less focused on the algebraic properties of self-similar groups, and puts the theory of iterated monodromy groups into a wider framework of groups associated with dynamical systems. We have also included many examples of applications of iterated monodromy groups as exercises.

As it was mentioned above, one of the main objects of study in topological dynamics are topological properties of orbits. It became clear in recent decades that orbits by themselves, even when we "forget" the action, carry a lot of interesting information and have rich algebraic structure (even in the Borel setting, as in the theory of Borel equivalence relations [**KM04**]). One way of formalizing this approach is via the notion of a *groupoid*, i.e., a small category of isomorphisms. Here the objects of the category are points of the space and two points are isomorphic if they belong to one orbit. It is natural in the context of topological dynamics to enrich this groupoid by adding some topological information, for example by looking at the germs of the action. Due to their nature as a bridge between topological dynamics and algebra, topological groupoids play an essential role in the study of group theoretical aspects of topological dynamics.

A part of the material is based on two graduate courses taught by the author at Texas A&M University. Other parts where taught by the author in several mini-courses. It is aimed at graduate students and researchers interested in group theory and topological dynamics.

The first chapter is an introduction to classical examples and basic notions of topological dynamics. It serves as a "crash course" in topological dynamics for group theorists and also develops some tools and techniques that will be important in later chapters (for example ordered Bratteli diagrams, subshifts, substitutional systems). There is also a section containing basic facts in holomorphic dynamics that will be needed later for the theory of iterated monodromy groups.

The second chapter studies some general aspects of groups acting on topological spaces. In particular, we develop basic techniques of orbital graphs, give a proof of a simplified version of M. Rubin's reconstruction theorem, and give basic definitions of theory of automata (transducers) used to define homeomorphisms of symbolic Cantor sets. A separate section is devoted to groups acting on rooted trees and related automata theory.

Chapter 3 is an introduction to the theory of topological groupoids. We use groupoids for two reasons: to encode the dynamical information about the action and to consider dynamical systems on orbispaces (generalizations of orbifolds). The former are used, for example, to define *topological full groups* and to study their properties. The latter are important for the theory of iterated monodromy groups (for example in the case of sub-hyperbolic complex rational functions). So far, existing textbooks on groupoid theory were focused on other aspects of groupoid theory: foliations, C^*-algebras, or homotopy theory. We hope that our book will fill the gap and introduce topological groupoids to current and future specialists in groups and dynamics.

Iterated monodromy groups and their applications are studied in Chapter 4. The main focus of the chapter is algebraic theory and symbolic dynamics of expanding covering maps.

Chapter 5 is devoted to topological full groups. We describe a method of constructing simple finitely generated groups associated with dynamical systems and how properties of the groupoid of germs of the action are related with the properties of the group. One of its sections describes examples of finitely generated infinite torsion groups (*groups of Burnside type*) constructed from dynamical systems. All known amenable groups of Burnside type are constructed in this way.

The last chapter is in some sense the culmination of the book. We use a large part of the developed techniques to construct groups of intermediate growth and non-elementary amenable groups.

Exercises are listed at the end of each chapter. They are roughly divided into three categories marked by asterisks at their numbers: problems for which one just has to apply the definitions in a more or less straightforward way (without asterisk), problems for which one has to come up with a nontrivial idea (one asterisk), and mini-research projects (two asterisks).

The author wants to thank Laurent Bartholdi, Collin Bleak, Rostislav Grigorchuk, Nicolas Matte Bon, Constantine Medynets, Josiah Owens, Dylan Thurston, and the anonymous referees for suggestions improving the text of the book. The work of the author was partially supported by NSF grant DMS 1709480.