
Index

- K_k , Discrete Single-Scale Polynomial
Average, 142, 236, 264
- L^2 -Multiplier Theory, 68
- $L^2(\mathbb{R})$ - $\ell^2(\mathbb{Z})$ Transference, 156
- L^∞ -Inverse Theorem, \mathbb{R} , Proof, 471
- L^∞ -Inverse Theorem, $\mathbb{Z}/p\mathbb{Z}$, 458
- L^∞ -Inverse Theorem, $\mathbb{Z}/p\mathbb{Z}$, Proof, 460
- L^∞ -Inverse Theorem, Degree Lowering,
 \mathbb{R} , 477
- L^∞ -Inverse Theorem, Degree Lowering,
 $\mathbb{Z}/p\mathbb{Z}$, 462
- L^p -Spaces, 23
- TT^* Approach: Maximally Modulated
Singular Integrals, 369
- TT^* Approach: the Dyadic
Hardy-Littlewood Maximal
Function, 367
- U^2 -Inverse Theorem, Cyclic Setting,
511
- U^2 -Inverse Theorem, Euclidean Setting,
468
- U^s Norms are Norms, 512
- V_k , Fourier Transform of Single-Scale
Monomial Curve, 152, 236, 264
- $\Gamma(\mathbf{m})$, Maximal Multiplier Bound on
 $L^2(\mathbb{R}^D)$, 163
- lcm Bound, 160
- $\{\mathcal{R}_s : s \geq 1\}$ Partition of \mathbb{Q} , 150
- r -Variation, 91

- Analytic Approximation, Discrete
Modulated Singular Integral, 423,
424, 426

- Analytic Approximation, Discrete
Monomially-Modulated Singular
Integral, 382–384, 386
- Analytic Approximation, Discrete
Polynomial Multiplier, 153
- Analytic Approximation, Discrete
Spherical Multiplier, 312, 351
- Arithmetic Condensation of
Singularities Lemma, 524
- Arithmetic Maximal Function, Discrete
Maximally Modulated Singular
Integral, 429
- Arithmetic Maximal Function, Discrete
Maximally Monomially-Modulated
Singular Integral, 388
- Arithmetic Maximal Theorem, Discrete
Maximally Monomially-Modulated
Singular Integral, 390
- Arithmetic Sobolev Estimate, 446, 455
- Arithmetic Sobolev Estimate, \mathbb{R} , 456
- Arithmetic Sobolev Estimate, $\mathbb{Z}/p\mathbb{Z}$,
456
- Arithmetic Sobolev Model Case: \mathbb{R} , 466
- Arithmetic Sobolev Model Case: $\mathbb{Z}/p\mathbb{Z}$,
457

- Big- O Notation, 18
- Bilinear Averages, Continuous Setting,
444
- Bilinear Pinned Incidence along a
Parabola, 442
- Bilinear Pinned Incidence Estimate, 451

- Birkhoff's Pointwise Ergodic Theorem, 82
 Bounding the Long Variation,
 Birkhoff's Theorem, 107
 Bounding the Short Variation,
 Birkhoff's Theorem, Physical Approach, 111
 Bounding the Short Variation,
 Birkhoff's Theorem, Spectral Approach, 118
 Bourgain's Double-Logarithmic Lemma, 247
 Bourgain's Maximal Polynomial Ergodic Theorem, L^2 -Formulation, 137
 Bourgain's Multi-frequency Maximal Function, 158
 Bourgain's Multi-Frequency Maximal Function, Large Scales, 173
 Bourgain's Multi-Frequency Maximal Function, Small Scales, 176
 Bourgain's Multi-frequency Maximal Theory, 168
 Bourgain's Multi-frequency Maximal Theory, Exploiting Periodicity, 173
 Bourgain's Polynomial Ergodic Theorem, $L^p(X)$ -Extension, 249
 Bourgain, J., xxi, xxii, 3–6, 32, 90, 91, 94, 106, 120, 125, 137, 153, 156, 163, 168, 173, 183, 227, 235, 237, 238, 246, 247, 249, 257, 301, 302, 306, 343, 347, 366, 367, 375, 376, 394, 439, 442, 445, 489–491, 493, 495, 496
 Burkholder-Davis Martingale Inequalities, Euclidean Case, 97
 Calderón's Transference Principle, 86
 Calderón-Zygmund Decomposition, 201
 Calderón-Zygmund Norm, 327, 363, 404
 Calderón-Zygmund Theory,
 $\ell^2(\mathbb{Z})$ -Approach, 208
 Calderón-Zygmund Theory, Singular Integrals, 198
 Calderón-Zygmund Theory, Singular Integrals, Proof, 202
 Calderón-Zygmund Theory, the Approach, 197
 Carleson's Theorem, 363
 Cauchy-Schwarz Inequality, 19
 Chernoff's Inequality, 190
 Chernoff/Entropy Argument, 194
 Coefficient Norm Bound for Exponential Sums, 414
 Compactly-Supported, Indefinitely Differentiable Functions, 59
 Continuous Gowers Norms, 467
 Continuous Improving Inequalities, Spherical Averages, 340
 Continuous Operators on Infinite Dimensional Vector Spaces, 47
 Continuous Stein-Wainger Operator, 364
 Controlling a Maximal Function on L^2 , 161
 de la Vallée Poussin Kernel, 195
 Density Argument, Pointwise Convergence, 44
 Density Increment Argument, 220, 500
 Density of Schwartz Functions, 60
 Discrete Analogue of a Theorem of Stein-Wainger, 403
 Discrete Fractional Integrals along Paraboloids, 358
 Discrete Fractional Integrals along Polynomial Orbits, 358
 Discrete Improving Inequalities and Fractional Integration, 354
 Discrete Improving Inequalities: Paraboloids, 347
 Discrete Improving Inequalities: Polynomial Curves, 347
 Discrete Improving Inequalities: Quadratic Curves, 342
 Discrete Improving Inequalities: Spherical Averages, 349
 Discrete Integral Operator Norm, 378
 Discrete Maximal Function, Polynomial Averages, 137
 Discrete Polynomial Coefficient Norm, 413
 Discrete Spherical Averages, 305
 Discrete Spherical Maximal Function, 305
 Discrete Stein-Wainger Decomposition, 418
 Discrete Stein-Wainger Operator, 417
 Discrete Sublevel Estimates, 415
 Discretized Sobolev Embedding, 165
 Distributional Equivalence of (Dyadic) Hardy-Littlewood Maximal Functions, 41

- Dual Formulation of Type of an Operator, 26
 Dual Function, \mathbb{R} , 472
 Dual Function, $\mathbb{Z}/p\mathbb{Z}$, 459
 Duoandikoetxea-Rubio de Francia Bootstrapping, 325
 Dyadic Conditional Expectation Operators on $[0, 1]$, 73
 Dyadic Differencing Operators, 95
 Dyadic Grid, Half-Open Cubes, 40
 Dyadic Hardy-Littlewood Maximal Function, 40
 Dyadic Hardy-Littlewood Maximal Inequality, 41
 Dyadic Maximal Function and Dyadic Square Function are Comparable, 99
 Dyadic Maximal Function Controls Dyadic Square Function, 99
 Dyadic Square Function, 96
 Dyadic Square Function Controls Dyadic Maximal Function, 100

 Energy Increment, 228
 Equidistribution of Polynomial Sequence, 507
 Euclidean Fourier Transform, 56
 Euclidean Lacunary Maximal Functions, 322
 Euclidean Polynomial Coefficient Norm, 371, 405, 518
 Euclidean Spherical Maximal Function, 301
 Euclidean Spherical Maximal Function, Proof, 305
 Exponential Sums and Discrete Sublevel Estimates, 412

 Farey Dissection, 309, 343
 Fejér Kernel, 53
 Finite Fourier Analysis on Cyclic Groups, 501
 Fourier Inversion, Euclidean Space, 64
 Fourier Transform of Continuous Single-Scale Modulated Singular Integral, 423
 Fourier Transform of Continuous Single-Scale Monomially-Modulated Singular Integral, 382
 Fourier Transform of Spherical Measure and its Derivative, 302

 Fourier Transform on $\ell^2(\mathbb{Z})$, 50
 Fourier Transform on $\ell^2(\mathbb{Z}^D)$, 55
 Fourier Transform on Schwartz Functions, 61
 Fractional Integration in Euclidean Space, 355
 Furstenberg-Bergelson-Leibman Conjecture, 442
 Furstenberg-Weiss Averages, 441

 Gauss Sums, Discrete Modulated Singular Integral, 423
 Gauss Sums, Discrete Monomially-Modulated Singular Integral, 382
 Gauss Sums, Discrete Polynomial Averaging Operators, 151, 236, 264
 Gauss Sums, Discrete Spherical Averages, 310, 351
 Gaussian Large-Deviation Inequality for ± 1 i.i.d. Random Variables, 188
 Gaussians, 59
 Generalized Rademacher-Menshov Inequality, 167
 Generic Chaining, 166
 Good Major Arc Collections, 274
 Good- λ Inequalities, 98
 Gowers Norms Control s -APs, 510
 Gowers Norms on $\mathbb{Z}/p\mathbb{Z}$, 504
 Greedy Jump-Counting Function, 102

 Hölder's Inequality, 20
 Haar Functions, 95
 Hahn-Banach-Type Separation Lemma, \mathbb{R} , 483
 Hahn-Banach-Type Separation Lemma, $\mathbb{Z}/p\mathbb{Z}$, 465
 Hardy-Littlewood Maximal Function, 33
 Hardy-Littlewood Maximal Theorem, 36
 Hardy-Littlewood Maximal Theorem, Proof, 39
 Hardy-Littlewood-Sobolev Inequality, 355
 Hausdorff-Young Inequality, 65
 High- $\ell_H^p(\mathbb{Z})$ Rubio de Francia-Type Orthogonality Estimate, 277
 High-Dimensional Oscillatory Integrals with Polynomial Phases, Classical Theory, 527

- High-Dimensional Oscillatory Integrals with Polynomial Phases, No Differential Structure, 529
- High-Dimensional Oscillatory Integrals, Classical Theory, 528
- High-Low Decomposition of Bourgain's Polynomial Maximal Operator, 249
- High-Low Decomposition of Discrete Single-Scale Quadratic Average, 344
- High-Low Decomposition of Discrete Spherical Average, 354
- High-Low Decomposition of Discrete Spherical Maximal Function, 316
- High-Low Decomposition of Euclidean Spherical Maximal Function, 305
- High-Low Method, 32
- Higher Order Uniformity Norm, 509
- Higher-dimensional Haar Functions, 126
- Hitting Times of Bernoulli Random Variables, 192
- Hopf-Dunford-Schwartz maximal theorem for semigroups, 84
- Hua's Theorem, 151
- Improving Argument, \mathbb{R} , 481
- Improving Argument, $\mathbb{Z}/p\mathbb{Z}$, 464
- Independent Events, 186
- Independent Random Variables, 186
- Interpolation of Functions, 24
- Introduction to 4-APs, 505
- Ionescu's Dual Sets, 283
- Ionescu, A., xxv, 6, 32, 235, 257–262, 264, 265, 273, 275, 277, 306, 315, 387, 428, 445, 446, 493, 494
- Ionescu-Wainger Construction, 258
- Ionescu-Wainger Height, 259
- Ionescu-Wainger Projection, 260
- Ionescu-Wainger Theorem after T. Tao, 263, 275
- Ionescu-Wainger Theory: Proof Overview, 275
- Jump-Counting Function, 102
- Khinchine's Inequality, 191
- Kloosterman Refinement, 352
- Kloosterman Sums, 351
- Krause-Mirek-Tao Theorem, 442
- Krause-Roos Maximally Monomially-Modulated Singular Integral, 366
- Lépingle's Inequality, 101
- Lépingle's Inequality, Proof, 126
- Lacey, M., xxv, 173, 328, 342, 349, 366, 490
- Lacunary Discrete Spherical Maximal Function, 319, 329
- Lacunary Discrete Spherical Maximal Function, Low- $\ell^p(\mathbb{Z}^D)$ Counter-Example, 333
- Lacunary Discrete Spherical Maximal Theorem, 329
- Lacunary Euclidean Spherical Maximal Function, 319
- Lacunary Trigonometric Polynomials, 321
- Lebesgue Differentiation Theorem, 45
- List of Symbols: Asymptotic Notation, xi
- List of Symbols: Fourier Multipliers, Discrete Setting, xvii
- List of Symbols: Functions and Operators, Discrete Setting, xvi
- List of Symbols: Functions and Operators, Euclidean Setting, xiv
- List of Symbols: Miscellaneous Symbols, xii
- Littlewood-Paley Square Functions, 115
- Long vs. Short Variation, 106
- Magyar-Stein-Wainger Discrete Spherical Maximal Theorem, 306
- Magyar-Stein-Wainger Discrete Spherical Maximal Theorem, Proof, 316
- Magyar-Stein-Wainger Transference Lemma, 239
- Major Arc Approximation, Discrete Modulated Singular Integrals, 424
- Major Arc Approximation, Discrete Monomially-Modulated Singular Integrals, 383
- Major Arc Approximation, Discrete Polynomial Averaging Operators, 152
- Major Arc Sets, $\{X_j\}$, General Dimensions, 424
- Major Arc Systems, 284
- Major Arcs, Discrete Modulated Singular Integrals, 424
- Major Arcs, Discrete Monomially-Modulated Singular Integrals, 384

- Major Arcs, Discrete Polynomial Averaging Operators, 150
- Major-Arc Sets, $\{X_j\}$, One Dimension, 375, 377
- Marcinkiewicz Interpolation Theorem, 28
- Marcinkiewicz Interpolation Theorem, Proof, 30
- Marcinkiewicz-Zygmund Theorem, 281
- Martingales, 94
- Maximal Estimates Imply Vector-Valued Estimates, 110
- Maximal Fourier Multipliers on $L^2(\mathbb{R}^D)$, 163
- Maximally Monomially-Modulated Singular Integrals are $L^p(\mathbb{R})$ -Bounded, 373
- Minkowski's Inequality, 21
- Minor Arc Estimate for Discrete Maximally Modulated Singular Integral, 420
- Minor Arc Estimate for Discrete Maximally Modulated Singular Integral, Proof, 421
- Minor Arc Estimate for Discrete Monomially-Modulated Singular Integral, 377
- Minor Arc Estimate for Discrete Monomially-Modulated Singular Integral, Proof, 380
- Mirek, M., xxv, 6, 7, 126, 176, 257, 258, 262, 320, 365, 439, 442, 493, 497
- Mirek: Remembering my Collaboration with Stein and Bourgain, 493
- Multi-Dimensional Exponential Sums with Polynomial Phases, 534
- Multi-frequency Maximal Theory for Modulated Singular Integral, 428, 432
- Multi-frequency Maximal Theory for Monomially-Modulated Singular Integral, 393
- Multi-frequency Maximal Theory for Monomially-Modulated Singular Integral, Intermediate Frequency Contribution, 396
- Multi-frequency Maximal Theory for Monomially-Modulated Singular Integral, Low Frequency Contribution, 394
- Multi-frequency Maximal Theory for Monomially-Modulated Singular Integral, Proof of Low Frequency Contribution, 399
- Multi-Frequency Variation Operator, Large Scales, 268
- Multi-Frequency Variation Operator, Small Scales, 269
- Multi-index Notation, 406
- Multiplicative Differencing Operator, 457, 499
- Nonces, 284
- Notation for Discrete Modulated Singular Integral, 405
- Number-Theoretic Approximation of the Discrete Modulated Singular Integral, 423, 425
- Number-Theoretic Approximation of the Discrete Monomially-Modulated Singular Integral, 382–385, 424
- Number-Theoretic Approximation of the Discrete Polynomial Averaging Operator, 150, 152, 153
- Number-Theoretic Approximation of the Discrete Spherical Averaging Operator, 308–312, 314
- Numerical Rademacher-Menshov Inequality, 176
- One-Dimensional Exponential Sums with Polynomial Phases, 522
- One-Dimensional Oscillatory Integrals with Polynomial Phases, No Differential Structure, 518
- One-Dimensional Oscillatory Integrals, Classical Theory, 516
- One-Sided Hardy-Littlewood Maximal Theorem, Discrete Formulation, 88
- Open Problems, 489
- Oscillation Inequality, 179
- Oscillatory Integrals with Polynomial Phases, 405
- Oscillatory Modulated Singular Integrals, 417
- Plancherel's Theorem, Euclidean Space, 64
- Plancherel's Theorem, Integer Formulation, 50

- Pointwise Convergence of Multilinear Polynomial Averages, Ergodic Theory, 439
- Pointwise Convergence of Polynomial Ergodic Averages on $L^2(X)$, 179
- Pointwise Convergence of Random Ergodic Averages on L^1 , 196, 197
- Pointwise Convergence of Random Ergodic Averages on L^p , $p > 1$, 193
- Poisson Summation, 66
- Poisson Summation, an Application, 334
- Pre-Ionsecu-Wainger Projection, 156
- Probabilistic Decoupling, 290
- Probability Space $(\mathbb{Z}/2\mathbb{Z})^\infty$, 183
- Probability Theory, 183
- Rademacher-Menshov Inequality, 176
- Ramanujan Sums, 331
- Real-Variable Fejér Kernel, 239
- Reduction to Lacunary Times, Pointwise Convergence, 178
- Reduction to Sub-lacunary Times, 263
- Restriction Estimate via Hardy-Littlewood-Sobolev, 355
- Reveiw of Stein-Wainger, 407
- Riemann-Lebesgue Lemma, 56
- Roth's Theorem, 499
- Roth's Theorem, Proof, 502
- Rough Square Functions and Long Variation of Truncated (Rough) Singular Integrals, 323
- Rubio de Francia Square Function Inequality, 278
- Sarközy's Theorem, 220
- Sarközy's Theorem, Pinned Extension, 226
- Schwartz Functions, 57
- Shifted Square Functions, 132
- Sketch of Bilinear Pointwise Ergodic Theorem, 443
- Sketch of Discrete Maximally Monomially-Modulated Singular Integral Argument, 375
- Sobolev Embedding Lemma, 162
- Square Function Argument, 68
- Statistics of Coin Flips, 187
- Stein's Maximal Principle, 76
- Stein's Purely Quadratic Carleson Operator, 365
- Stein, E., xxi, xxii, 5, 6, 68, 76, 84, 126, 238, 239, 257, 258, 262, 279, 282, 284, 301, 302, 319, 320, 354, 357, 358, 364, 365, 376, 395, 405, 407, 435, 489, 491, 493, 494, 496
- Stein-Wainger Theorem on Maximally Modulated Singular Integrals, 407
- Stein-Wainger Theorem, Oscillatory Estimate, 409
- Stein-Wainger Theorem, Proof Summary, 412
- Stirling's Formula, 298
- Strong Law of Large Numbers, 207
- Sublevel Estimates for Polynomials Over the p -adics, 449
- Sublevel Estimates for Polynomials, 405
- Summary of Contents, 8
- Sunflowers, 286
- Super-positions of Ramanujan Sums, 331
- Tao, T., xxv, 7, 29, 257, 263, 273, 284, 290, 320, 439–442, 497
- Type II Superorthogonal Hypersystems, 284
- Type II Superorthogonal Systems, 284
- Type of an Operator, 26
- Uniqueness of Fourier Coefficients, Euclidean Space, 64
- Uniqueness of Fourier Coefficients, Functions on the Torus, 53
- Universally L^p -Good Sets, 82
- Upper Banach Density, 226, 440
- van der Corput Oscillatory Integral Estimates, 516
- Van der Corput's Lemma and Consequence, 469
- Variational Approach to Birkhoff's Theorem, 106
- Variational Polynomial Ergodic Theorem, 263
- Variational Polynomial Ergodic Theorem, Proof, 265
- Vitali-Type Covering Lemma, 38
- Wainger, S., xxii, xxv, 5, 6, 32, 235, 238, 239, 257–262, 264, 265, 273, 275, 277, 279, 282, 354, 357, 358, 364, 387, 395, 405, 407, 428, 435, 445, 446, 493, 494

- Wedding-Cake Characterization of
 L^p -Norms, 23
- Weighted Summation by Parts, 196
- Weyl Sum Estimates, 541
- Weyl's Lemma, 153, 543
- Worked Example, Calderón-Zygmund
 Theory, 199
- Worked Example, Dyadic r -Variations,
 91
- Worked Example, Dyadic
 Hardy-Littlewood Maximal
 Function, 40
- Worked Example, Dyadic Square
 Function, 96
- Worked Example, Single-Scale Bilinear
 Average, 443
- Worked Example, Spherical Maximal
 Function, 301
- Worked Examples, Averages vs.
 Expectations, 74
- Worked Examples, Hardy-Littlewood
 Maximal Function, 36
- Worked Examples, Single-Scale Discrete
 Polynomial Radon Transform, 142
- Young's Inequality, 31
- Young's Scalar Inequality, 18