Foreword

This book grew out of a series of lecture notes used in graduate topics classes at Caltech over the 2017−2019 academic years, aimed at familiarizing (harmonic) analysts with the circle of ideas used to study classical Euclidean-analytic operators on the integers; the title reflects the over-arching impact of Jean Bourgain and Eli Stein on the field.

Modern discrete harmonic analysis developed out of Bourgain's work on pointwise ergodic theorems along polynomial orbits: for instance, the initial focus of his efforts in this direction were the pointwise convergence of the averages,

$$\frac{1}{N} \sum_{n \leq N} \tau^{n^2} f := \frac{1}{N} \sum_{n \leq N} f(\tau^{n^2} \cdot)$$

where $f \in L^2(X, \mu)$ is a square-integrable function on some σ-finite measure space, (X, μ), and $\tau : X \to X$ is a *measure-preserving transformation*: $\mu(E) = \mu(\tau^{-1}E)$ for any measurable $E \subset X$. Although the issue of pointwise convergence is qualitative, Bourgain's insight was to quantify the rate at which convergence occurred – and then to use an abstract transference argument to deduce these quantitative estimates from a single "universal" measure preserving system.

The moral of this transference argument is that *it is the acting group that dictates the field of engagement:* our attention narrows to the integers with counting measure and the shift $(\mathbb{Z}, |\cdot|, \tau : x \mapsto x - 1)$.

In particular, Bourgain was after quantitative estimates on the oscillation of the averaging operators

$$\frac{1}{N}\sum_{n\leq N} f(x-n^2), \tag{0.1}$$

applied to $\ell^2(\mathbb{Z})$-functions. A natural perspective on (0.1) is as a convolution of f and

$$K_N(x) := \frac{1}{N}\sum_{n\leq N}\delta_{n^2}(x)$$

where δ_m denotes the point-mass at $m \in \mathbb{Z}$; as this problem is ℓ^2-based, the Fourier transform method is naturally employed, and the key to the analysis is an understanding of the exponential sums

$$\frac{1}{N}\sum_{n\leq N} e^{-2\pi i\beta n^2},$$

which is accomplished via the *circle method* from analytic number theory; the interplay between the "soft" analytic issue of pointwise convergence and "hard" analytic estimates on the integers/Euclidean space via analytic-number-theoretic means is characteristic of the field of discrete harmonic analysis, and is personally very appealing.

Somewhat less motivated by ergodic considerations, Eli Stein, his student and long-time collaborator Stephen Wainger, and their collaborators, began an investigation into the related issue of symmetry of ℓ^2 functions with respect to polynomial orbits; in fullest generality, this line of inquiry remained open until 2020, when it was closed by Stein and (different) collaborators [**137**]. Along the way, in collaboration with Wainger and another one of his students, Akos Magyar, Stein initiated a study of oscillation of discrete *spherical* averaging operators,

$$\frac{1}{N(r)}\sum_{|m|=r} f(x-m) = \Big(\frac{1}{N(r)}\sum_{|m|=r}\delta_m\Big) * f(x), \tag{0.2}$$

where

$$N(r) := |\{m \in \mathbb{Z}^D : |m| = r\}|$$

is bounded above and below by constant multiples of r^{D-2} in the appropriate dimension. The Euclidean variants of these operators were introduced and studied by Stein, with the theory ultimately being fully developed by Bourgain.

Indeed, a main focus of this book is the bilateral nature of the relationship between the work of Bourgain and Stein. For instance, our approach to the study of oscillation of (0.2) is through the lens of Bourgain [**92**] –

while our understanding of the oscillation of (0.1) is ultimately completed through that of Stein and collaborators, [**133**], [**134**], [**137**].

The purpose of this book is to explore these connections, both in the context of (0.1) and (0.2), and beyond. Doing so, however, requires ideas and techniques not just from ergodic theory, analytic number theory, and harmonic analysis – but also abstract Hilbert-space orthogonality methods, martingale techniques from probability theory, chaining arguments from Banach-space geometry, additive combinatorics, and ultimately some p-adic analysis. Although many of the ideas are elegant and natural, significant technical complications provide a fairly high barrier to entry to the field. To facilitate this transition, I have included a wide range of exercises – of varying degrees of difficulty – with the aim of helping the reader get their "hands dirty," and develop the technique necessary to appreciate the beauty of the field.

In my efforts to get my own hands dirty, I have been fortunate enough to have benefited from the support and guidance of many of my friends and collaborators. My hope is that this book will inspire interest and collaboration in a field that I love.