Preface

Spectral theory can be viewed as a generalization of linear algebra with a focus on linear operators on infinite-dimensional spaces. However, it is a branch of mathematical analysis that has its roots in the Fourier decomposition of a periodic function into sines and cosines. Those sines and cosines are solutions of the boundary value problem $-f'' = \lambda f$, $f(0) = f(2\pi)$, $f'(0) = f'(2\pi)$. In modern language, they are eigenvectors of a differential operator (second derivative on an interval with periodic boundary conditions), acting on a suitable space of functions (which is an infinite-dimensional vector space). Modern spectral theory studies classes of recurrence and differential operators which are motivated by mathematical physics, orthogonal polynomials, partial differential equations, and integrable systems.

This text is intended as a first course in spectral theory, with a focus on the general theory of self-adjoint operators on separable Hilbert spaces and the direct spectral theory of Jacobi matrices and one-dimensional Schrödinger operators. It has been written as a textbook for three adjacent purposes:

(a) an undergraduate course on bounded self-adjoint operators,

(b) a first course for graduate students interested in the spectral theory of bounded and unbounded self-adjoint operators,

(c) a topics course on continuum one-dimensional Schrödinger operators.

The intended audience for this text includes beginning graduate students and advanced undergraduates, so the text is written with minimal prerequisites. It is assumed that the reader knows linear algebra and basic analysis, including basic complex analysis. In an effort to keep the text accessible, we avoid unnecessary abstractions and get by without topology.

Measure theory is not assumed as a prerequisite; the required background in measure theory is developed in Chapter 1, including some specialized results needed for our purposes (e.g., a criterion for a subalgebra of bounded Borel functions to be the entire algebra, used for the proof of uniqueness of the Borel functional calculus for self-adjoint operators).

Chapter 2 introduces Banach spaces; these are vector spaces equipped with a norm (a suitable notion of length of vectors) which are complete. This chapter is a nonstandard introduction to functional analysis shaped by a spectral theorist's needs: it includes a discussion of Banach space valued integrals, Banach space valued analytic functions, and important examples of Banach spaces, without going deep into abstract Banach space theory.

Chapter 3 introduces Hilbert spaces, which are a special case of Banach space equipped with an inner product (an abstract version of a dot product). The chapter includes infinite direct sums of Hilbert spaces (needed for the multiplication operator form of the spectral theorem) and tensor products.

Chapter 4 describes the general structure and properties of bounded linear operators on Hilbert spaces; this provides the basic language for the remainder of the text.

Chapter 5 begins the study of bounded self-adjoint operators. Self-adjoint operators can be viewed as an infinite-dimensional generalization of Hermitian matrices, and this chapter can be viewed as a generalization of diagonalizability of Hermitian matrices. Spectral measures are introduced and two central results are proved; namely, the spectral theorem and the Borel functional calculus. The spectral theorem is presented in multiplication operator form, which we find more useful and intuitive (we introduce and use spectral projections later in this text, but we do not use integration with respect to projection-valued measures or the historically more common approach via resolution of the identity). The Borel functional calculus is constructed using the spectral theorem.

Chapter 6 presents several measure decompositions (continuous/pure point, absolutely continuous/singular, and decompositions with respect to Hausdorff measures) and pointwise descriptions of these decompositions. This is part of the standard vocabulary of spectral theory, where continuity properties of spectral measures are of great importance.

One of the goals of this text is to present the spectral theory of self-adjoint operators from the ground up as a correspondence of three main objects: self-adjoint operators, their spectral measures (which are measures on $\mathbb{R}$), and Herglotz functions (which are complex-analytic functions from the upper half-plane to itself). Accordingly, Herglotz functions are introduced in Chapter 7. Through an integral representation, they are related to measures on $\mathbb{R}$, and this chapter studies this correspondence. This may seem like a detour from spectral theory, but the truth is quite the opposite: although Chapter 7 doesn't mention operators, we will see that it contains the hard parts of proofs of important spectral theoretic results.

In Chapter 8, we study unbounded self-adjoint operators, culminating in their spectral theorem and Borel functional calculus. The presentation is independent from the bounded case, although the bounded case serves as a strong motivation. Many techniques from the bounded case have suitable analogues or restatements in the unbounded case, but there are technical complications. This chapter includes the study of symplectic forms over the complex field of scalars and a description of self-adjoint extensions of a symmetric operator.

Chapter 9 can be read as a continuation of Chapter 5 or of Chapter 8. It describes many general consequences of the spectral theorem and the Borel functional calculus, such as spectral type, spectral multiplicity, etc., which are part of the basic language of spectral theory. It contains a study of Stone's theorem and its applications to constructing *diagonalizations* of differential operators; for instance, we provide a self-contained introduction to the Fourier transform on $L^2(\mathbb{R})$ through the problem of *diagonalizing* the derivative $-i\frac{d}{dx}$ viewed as an unbounded self-adjoint operator on $\mathbb{R}$. This approach is constructive and based on Stone's theorem, and it serves as a warm-up for eigenfunction expansions of Schrödinger operators.

Chapter 10 discusses bounded and unbounded Jacobi matrices, which are a well-studied class of self-adjoint operators corresponding to a second-order recurrence relation on $\ell^2(\mathbb{N})$ and $\ell^2(\mathbb{Z})$. While they can be viewed as an extended example for general spectral theory, their connections to orthogonal polynomials and mathematical physics make them a classical subject of their own; we present some of their general properties and techniques for their study. We emphasize the correspondence with Weyl m-functions and use Weyl disks as a robust way of deriving approximation results, such as Carmona's theorem. The chapter includes subordinacy theory, eigenfunction expansions for full-line Jacobi matrices, and the Weyl M-matrix approach. Finally, we present the direct spectral theory of periodic Jacobi matrices, using the Marchenko–Ostrovski map as a central object.

Chapter 11 is dedicated to one-dimensional Schrödinger operators $-\frac{d^2}{dx^2}+V$, considered on a finite or infinite interval, with locally integrable potentials V. The chapter starts with self-adjointness and the limit point-limit circle alternative. Eigenfunction expansions are introduced constructively, using Stone's theorem. Weyl disks are used to derive various approximation results, including Carmona's formula and continuity of m-functions under L^1_{loc} perturbations of the potential. We also prove the local Borg–Marchenko theorem, asymptotic behavior of the m-functions, and Schnol's theorem. We conclude this chapter with the direct spectral theory of periodic Schrödinger operators studied via the Marchenko–Ostrovski map.

The book can of course be read cover to cover, but various selections of the material are possible. For instance, beyond the introductory chapters, we suggest the following.

- A course on bounded self-adjoint operators can contain Chapters 4 and 5 and Section 10.1. It can continue, time permitting, with Sections 6.1–6.2 and Sections 9.1–9.6.
- A course on unbounded self-adjoint operators can contain Chapter 4, Sections 7.1–7.5, Chapter 8, and a selection of topics from Chapters 9, 10, 11.
- A course on Jacobi or Schrödinger operators can be based on the corresponding Chapter 10 or 11. It requires Chapter 5 or Chapter 8 as a prerequisite; it is also heavily reliant on Chapters 6, 7, 9, which can be studied in preparation or in parallel with Chapter 10 or 11.

Many analytical tools are developed in Chapters 6 and 7, applied in Chapter 9 to self-adjoint operators, then refined in more specialized settings in Chapters 10 and 11. They can be studied by taking cross-sections of different chapters.

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