Preface

This book explores geometric structures on manifolds locally modeled on a classical geometry.

This subject mediates between *topology* and *geometry,* where a fixed topology is given local coordinate systems in the geometry of a homogeneous space of a Lie group. A familiar example puts Euclidean geometry on a manifold; such a *Euclidean structure* is nothing more than a Riemannian metric of zero curvature. In this sense, the topology of the 2-dimensional sphere $\mathbb{S}^2$ is incompatible with the geometry of Euclidean space: *There is no metrically accurate atlas of the world.* In contrast, however, the topology of the 2-dimensional torus $\mathbb{T}^2$ *does* support Euclidean geometry. Indeed, the classification of Euclidean structures on the torus is part of a rich and central area of mathematics (elliptic curves, modular forms). Indeed, Euclidean structures on $\mathbb{T}^2$ are classified by the action of the modular group on the Poincaré upper halfplane.

Topology and geometry communicate via *group theory. Topology* contributes its group, — the fundamental group — and *Geometry* contributes the group of symmetries of the given geometry. Thus our approach starts from the Klein–Lie algebraicization of geometry via Lie groups and homogeneous spaces, and quickly evolves into studying representations of discrete groups in Lie groups.

This book surveys the theory, with a special emphasis on affine and projective geometry. Many important geometries (for example, hyperbolic geometry) have projective models, and these projective models unify the diverse geometries.

This work is based on examples. I have tried to present examples as a way to suggest the general theory. Because of the dramatic growth of this subject in the last decades, I tried to collect many facets of this subject and present them from a single viewpoint. Since Ehresmann's 1936 initiation of this subject, there have been many "success stories" in the classification of geometric structures on a given topology. I have to tried to present some of these in this book.

Despite the profound interrelations between different geometries, each geometry enjoys special features. A developing map only goes so far, and heavier machinery is often required, drawing on techniques special to the particular geometry. Learning new techniques and adapting to different areas of mathematics has been an exciting part of this journey.

Furthermore some of the material — which I feel should be better known — is unpublished, untranslated, or aimed at a different readership. The literature suffers from many errors (including some of my own) which I have tried to correct and clarify. However, I am certain many errors still persist, and I take full responsibility.

This book is suitable for a graduate textbook and contains many exercises. Some exercises are routine and others are more difficult. Many are used in other parts of the text. Others are meant to introduce ideas and examples before a subsequent detailed discussion.

To preserve the expository flow, several developments have been put in appendices. I have tried to illustrate geometric ideas with pictures and algebraic ideas with tables.

I have tried to keep the prerequisites fairly minimal. Material from beginning graduate courses in topology, differential geometry, and algebra are assumed, although some of the material which is crucial or less standard is summarized. The relationship between Lie groups and Lie algebras is heavily used, but little of the general structure theory/representation theory is assumed.

I began this area of research working with Dennis Sullivan and Bill Thurston at Princeton University in 1976. Their influence is evident throughout this work. Thurston formulated his *geometrization* of 3-manifolds in the context of geometric structures modeled on 3-dimensional Riemannian homogeneous spaces. Since then the study of more general (but not necessarily Riemannian) locally homogeneous geometric structures has become a very active field with interactions to other areas of mathematics and physics.

Several important topics have been omitted or only briefly mentioned. Flat structures on Riemann surfaces — namely, singular Euclidean structures modeled on translations — are barely mentioned despite their fundamental role in modern Teichmüller theory. Their strata are fascinating and mysterious examples of incomplete complex affine manifolds. Nor are holomorphic affine and projective structures on complex manifolds. The algebraic theory of character varieties and representations of fundamental groups, is not really developed thoroughly. In particular the theory of surface group representations into Lie groups of higher rank, sometimes called *higher Teichmüller theory*, is not extensively discussed, despite its remarkable recent activity. Integral affine structures (important in mirror symmetry) are not discussed. Other very natural topics in this subject have not been discussed in detail, for reasons of space: These include the convex decomposition theorem of Suhyoung Choi, completeness results of Carrière and Klingler for constant curvature Lorentzian manifolds, and affine structures with diagonal holonomy as developed by Smilie and Benoist.

I welcome suggestions, comments, and feedback of (almost) all sorts. Through the AMS bookstore, I plan to maintain a website of errata, comments, graphics, and interactive software in connection with this book.

I hope you enjoy this journey as much as I have!

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