Introduction

Symmetry powerfully unifies the various notions of geometry. Based on ideas of Sophus Lie, Felix Klein's 1872 Erlangen program proposed that geometry is the study of properties of a space X invariant under a group G of transformations of X. For example Euclidean geometry is the geometry of n-dimensional Euclidean space $\mathbb{R}^n$ invariant under its group of rigid motions. This is the group of transformations which transforms an object ξ into an object congruent to ξ. In Euclidean geometry one can speak of points, lines, parallelism of lines, angles between lines, distance between points, area, volume, and many other geometric concepts. All these concepts can be derived from the notion of distance, that is, from the metric structure of Euclidean geometry. Thus any distance-preserving transformation or *isometry* preserves all of these geometric entities.

Notions more primitive than that of distance are the *length* and *speed* of a smooth curve. Namely, the distance between points a, b is the infimum of the length of curves γ joining a and b. The length of γ is the integral of its speed $\|\gamma'(t)\|$. Thus Euclidean geometry admits an infinitesimal description in terms of the *Riemannian metric tensor,* which allows a measurement of the size of the velocity vector $\gamma'(t)$. In this way standard Riemannian geometry generalizes Euclidean geometry by imparting Euclidean geometry to each tangent space.

Other geometries "more general" than Euclidean geometry are obtained by removing the metric concepts, but retaining other geometric notions. *Similarity geometry* is the geometry of Euclidean space where the equivalence relation of congruence is replaced by the broader equivalence relation of similarity. It is the geometry invariant under similarity transformations. Similarity geometry does not involve distance, but rather involves angles, lines, and parallelism. *Affine geometry* arises when one speaks only of points, lines and the relation of parallelism. And when one removes the notion of parallelism and only studies lines, points and the relation of incidence between them (for example, three points being *collinear* or three lines being *concurrent*) one arrives at *projective geometry.* However in projective geometry, one must enlarge the space to *projective space,* which is the space upon which all the projective transformations are defined.

Here is a basic example illustrating the differences among the various geometries. A particle moving along a smooth path has a well-defined velocity vector field, representing its *infinitesimal displacement* at any time. This uses only the differentiable structure of $\mathbb{R}^n$. The magnitude of the velocity is the *speed,* which makes sense in Euclidean geometry. Thus "motion at unit speed" (that is, "arc-length-parametrized geodesic") is a meaningful concept there. But in affine geometry, the concept of "speed" or "arc-length" must be abandoned: yet "motion at constant speed" remains meaningful since the property of moving at constant speed along a straight line can be characterized as motion with zero acceleration. This is equivalent to the parallelism of the velocity vector field. In projective geometry this notion of "constant speed along a straight line" (or "parallel velocity") must be further weakened to the concept of "projective parameter" introduced by J. H. C. Whitehead [**346**].

Synthetic projective geometry was developed by the architect Desargues in 1636–1639 out of attempts to understand the geometry of perspective. Two hundred years later non-Euclidean (hyperbolic) geometry was developed independently — and practically simultaneously — by Bolyai in 1833 and Lobachevsky in 1826–1829. These geometries were unified in 1871 by Klein who noticed that Euclidean, affine, hyperbolic, and elliptic geometry were all "present" in projective geometry.

Later in the nineteenth century, mathematical crystallography developed, leading to the theory of *Euclidean crystallographic groups.* Answering Hilbert's eighteenth problem on the finiteness of the number of space groups in any given dimension n, Bieberbach developed a structure theory in 1911–1912. For torsion free groups, the quotient spaces identified with *flat Riemannian manifolds* of dimension n, that is, Riemannian n-manifolds having zero sectional curvature. Such Riemannian structures are locally isometric to Euclidean space E^n. In particular, every point has an open neighborhood isometric to an open subset of E^n. These local isometries define a local Euclidean geometry on the neighborhood. Furthermore on overlapping neighborhoods, the local Euclidean geometries "agree," that is, they are related by restrictions of global isometries $\mathsf{E}^n \to \mathsf{E}^n$. The neighborhoods form *coordinate patches,* the local isometries from the patches to

E^n are the *coordinate charts,* and the restrictions of isometries of E^n are the corresponding *coordinate changes.* In this way a flat Riemannian manifold is defined by a coordinate atlas for a *Euclidean structure.*

More generally, for any geometry one can define geometric structures on a manifold M modeled on the homogeneous space (G, X). A geometric atlas consists of an open covering of M by patches $U \hookrightarrow M$, together with a system of charts $U \xrightarrow{\psi} X$ such that the coordinate changes are locally restrictions of transformations of X which lie in G.

The plethora of different geometries suggests that, at least at a superficial level, no general inclusive theory of locally homogeneous geometric structures exists. Each geometry has its own features and idiosyncrasies, and special techniques particular to each geometry are used in each case. For example, a surface modeled on $\mathbb{C}\mathsf{P}^1$ has the underlying structure of a Riemann surface, and viewing a $\mathbb{C}\mathsf{P}^1$-structure as a projective structure on a Riemann surface provides a satisfying classification of $\mathbb{C}\mathsf{P}^1$-structures. Namely, as was presumably understood by Poincaré, *the deformation space of* $\mathbb{C}\mathsf{P}^1$*-structures on a closed surface* Σ *with* $\chi(\Sigma) < 0$ *identifies with a holomorphic affine bundle over the Teichmüller space of* Σ. When X is a complex manifold upon which G acts biholomorphically, holomorphic mappings provide a powerful tool in the study, a class of local mappings more flexible than "constant" maps (maps which are "locally in G") but more rigid than general smooth maps. Another example occurs when X admits a G-invariant connection, such as an invariant (pseudo-)Riemannian structure. Then the geodesic flow provides a powerful tool for the study of (G, X)-manifolds.

We emphasize the interplay between different mathematical techniques as an attractive aspect of this general subject. See [**160**] for a recent historical account of this material.

Organization of the text

The book divides into three parts. Part One describes affine and projective geometry and provides some of the main background on these extensions of Euclidean geometry. As noted by Lie and Klein, most classical geometries can be modeled in projective geometry. We introduce projective geometry as an extension of affine geometry, so we begin with a detailed discussion of affine geometry as an extension of Euclidean geometry and projective geometry as an extension of affine geometry. Part Two describes how to put the geometry of a Klein geometry (G, X) on a manifold M, and gives the basic examples and constructions. One goal is to *classify* the (G, X)-structures on a fixed topology in terms of a *deformation space* whose points correspond to equivalence classes of *marked structures,* whereby a marking is an extra piece of information which fixes the topology as the geometry of

M varies. Part Three describes recent developments in this general theory of locally homogeneous geometric structures.

Part One: Affine and projective geometry

Chapter 1 introduces affine geometry as the geometry of parallelism. Two objects are *parallel* if they are related by a *translation.* Translations form a vector space V, and act *simply transitively* on affine space. That is, for two points $p, q \in \mathsf{A}$ there is a unique translation taking p to q. In this way, points in A identify with the vector space V, but this identification depends on the (arbitrary) choice of a basepoint, or *origin* which identifies with the zero vector in V. One might say that an affine space is a vector space, where the origin is forgotten. More accurately, the special role of the zero vector is suppressed, so that all points are regarded equally.

The action by translations now allows the definition of *acceleration* of a smooth curve. A curve is a *geodesic* if its acceleration is zero, that is, if its velocity is parallel. In affine space itself, unparametrized geodesics are straight lines; a parametrized geodesic is a curve following a straight line at "constant speed." Of course, the "'speed" itself is undefined, but the notion of "constant speed" just means that the acceleration is zero.

This notion of parallelism is a special case of the notion of an *affine connection,* except the existence of *globally defined* translations effecting the notion of parallelism is a special feature to our setting — the setting of *flat connections.* Just as Euclidean geometry is affine geometry with a parallel Riemannian metric, other linear-algebraic notions enhance affine geometry with parallel tensor fields. The most notable (and best understood) are flat Lorentzian (and pseudo-Riemannian) structures.

Chapter 2 develops the geometry of projective space, viewed as the compactification of affine space. *Ideal points* arise as "where parallel lines meet." A more formal definition of an ideal point is an equivalence class of lines, where the equivalence relation is parallelism of lines. Linear families (or *pencils*) of lines form planes, and indeed the set of ideal points in a projective space form a *projective hyperplane,* that is, a projective space of one lower dimension. Projective geometry appears when the ideal points lose their special significance, just as affine geometry appears when the zero vector $\mathbf{0}$ in a vector space loses its special significance.

However, we prefer a more efficient (if less synthetic) approach to projective geometry in terms of linear algebra. Namely, the *projective space associated to a vector space* V is the space $\mathsf{P}(\mathsf{V})$ of 1-dimensional linear subspaces of V (that is, lines in V passing through $\mathbf{0}$). *Homogeneous coordinates* are introduced on projective space as follows. Since a 1-dimensional linear subspace is determined by any nonzero element, its coordinates determine a point in projective space. Furthermore the homogeneous coordinates are uniquely defined up to *projective equivalence,* that is, the equivalence relation defined by multiplication by nonzero scalars. Projectivizing linear subspaces of V produces projective subspaces of $\mathsf{P}(\mathsf{V})$, and projectivizing linear automorphisms of V yields *projective automorphisms,* or *collineations* of $\mathsf{P}(\mathsf{V})$.

The equivalence of the geometry of incidence in $\mathsf{P}(\mathsf{V})$ with the algebra of V is remarkable. Homogeneous coordinates provide the "dictionary" between projective geometry and and linear algebra. The collineation group is compactified as a projective space of "projective endomorphisms;" this will be useful for studying limits of sequences of projective transformations. These "singular projective transformations" are important in controlling developing maps of geometric structures, as developed in the second part.

Chapter 3 discusses, first from the classical viewpoint of polarities, the Cayley–Beltrami–Klein model for hyperbolic geometry. Polarities are the geometric version of nondegenerate symmetric or skew-symmetric bilinear forms on vector spaces. They provide a natural context for hyperbolic geometry, which is one of the principal examples of geometry in this study.

The Hilbert metric on a properly convex domain in projective space is introduced and is shown to be equivalent to the categorically defined Kobayashi metric [**220**, **222**]. Later this notion is extended to manifolds with projective structure.

Chapter 3 develops notions of convexity. The Cayley–Beltrami–Klein metric on hyperbolic space is a special case of the Hilbert metric on properly convex domains. These define natural metric structures on certain well-studied projective structures. An application of the Hilbert metric is Vey's semisimplicity theorem [**339**], which is later used to characterize closed hyperbolic projective manifolds as quotients of sharp convex cones. Then another metric (due to Vinberg [**340**]) is introduced, and is used to give a new proof of *Benzécri's Compactness theorem* [**46**] that the collineation group acts properly and cocompactly on the space of convex bodies in projective space — in particular the quotient is a compact (Hausdorff) manifold. This is used to characterize the boundary of convex domains which cover convex projective manifolds. Recently Benzécri's theorem has been used by Cooper, Long, and Tillmann [**100**] in their study of cusps of $\mathbb{RP}^n$-manifolds.

Part Two: Geometric manifolds

The second part globalizes these geometric notions to manifolds, introducing *locally homogeneous geometric structures* in the sense of Whitehead [**345**] and Ehresmann [**122**] in Chapter 5. We associate to every transformation group (G, X) a category of geometric structures on manifolds locally modeled on the geometry of X invariant under the group G. Because of the "rigidity" of the local coordinate changes of open sets in X which arise from transformations in G, these structures on M intimately relate to the fundamental group $\pi_1(M)$.

Chapter 5 presents three different viewpoints to study these structures. First are coordinate atlases for the pseudogroup arising from (G, X). Using the aforementioned rigidity, these are globalized in terms of a *developing map*

$$\widetilde{M} \xrightarrow{\mathsf{dev}} X,$$

defined on the universal covering space $\widetilde{M}$ of the geometric manifold M. The developing map is equivariant with respect to the holonomy homomorphism

$$\pi_1(M) \xrightarrow{h} G$$

which represents the group $\pi_1(M)$ of deck transformations of $\widetilde{M} \to M$ in G. Each of these two viewpoints represents M as a quotient: in the coordinate atlas description, M is the quotient of the disjoint union

$$\mathcal{U} := \coprod_{\alpha \in A} U_\alpha$$

of the coordinate patches U_α; in the second description, M is represented as the quotient of $\widetilde{M}$ by the action of the group $\pi_1(M)$. While a map defined on a connected space $\widetilde{M}$ may seem more tractable than a map defined on the disjoint union $\mathcal{U}$, the space $\widetilde{M}$ can still be quite large.

The third viewpoint replaces $\widetilde{M}$ with M and replaces the developing map by a section of a bundle defined over M. The bundle is a *flat bundle,* (that is, has *discrete structure group* in the sense of Steenrod [**317**]). The corresponding *developing section* is characterized by transversality with respect to the foliation arising from the flat structure. This replaces the coordinate charts (respectively the developing map) being local diffeomorphisms into X.

Chapter 6 discusses examples of geometric structures from these three points of view. Although the main interest in these notes is structures modeled on affine and projective geometry, we describe other interesting structures.

These structures interrelate: Geometries may "contain" or "refine" other geometries. For example, affine geometry *contains* Euclidean geometry — abandon the metric notions but retain the notion of *parallelism.* This corresponds to the inclusion of the Euclidean isometry group (consisting of transformations $x \longmapsto Ax + b$, where A is orthogonal) as a subgroup of the affine automorphism group (consisting of transformations $x \longmapsto Ax + b$

where A is only assumed to be linear). Other examples include the projective and conformal models for non-Euclidean geometry. In these examples, the model space of the refined geometry is an open subset of the larger model space, and the transformations in the refined geometry are restrictions of transformations in the larger geometry.

This *hierarchy of geometries* plays a crucial role in the theory. This is simply the geometric interpretation of the inclusion relations between closed subgroups of Lie groups. This *algebraicization* of geometries in the 19^{th} century by Lie and Klein satisfactorily organized the proliferation of classical geometries. This viewpoint is the cornerstone in our construction and classification of geometric structures. The classification of geometric manifolds often shows that a manifold modeled on one geometry may actually have a *stronger* geometry. For example, Fried's theorem [**135**] (see 11.4) asserts a closed manifold M with a similarity structure is either Euclidean or a manifold modeled on $\mathbb{R}^n \backslash \{\mathbf{0}\} \cong \mathbb{S}^{n-1} \times \mathbb{R}$ with its invariant (product) Riemannian metric. In particular M admits a finite covering space which is either a flat torus (the Euclidean case) or a *Hopf manifold,* a cyclic quotient of $\mathbb{R}^n \backslash \{\mathbf{0}\}$.

Chapter 7 deals with the general classification of (G, X)-structures from the point of view of developing sections. The main result is an important observation due to Thurston [**323**] that the *deformation space* of marked (G, X)-structures on a fixed topology Σ is itself "locally modeled" on the quotient of the space $\mathsf{Hom}\big(\pi_1(\Sigma), G)\big)$ by the group $\mathsf{Inn}(G)$ of inner automorphisms of G. The description of $\mathbb{R}\mathsf{P}^1$-manifolds is described in this framework. The deformation space, however, is a non-Hausdorff 1-manifold, while the subspace consisting of closed affine 1-manifolds identifies with $[0, \infty)$. For affine structures on $\mathbb{T}^2$, the deformation space is not even a (non-Hausdorff) manifold.

Chapter 8 deals with the important notion of *completeness,* for taming the developing map. In general, the developing map may be quite pathological — even for closed (G, X)-manifolds — but under various hypotheses, can be proved to be a covering space onto its image. However, the main techniques borrow from Riemannian geometry, and involves *geodesic completeness* of the Levi–Civita connection (the Hopf–Rinow theorem). A complete affine manifold M is a quotient $\Gamma\backslash\mathsf{A}$, where A is an affine space and $\Gamma < \mathsf{Aff}(\mathsf{A})$ is a discrete subgroup acting properly on A. Equivalently, a developing map $\widetilde{M} \to \mathsf{A}$ is a homeomorphism (an affine isomorphism) of the universal covering space $\widetilde{M}$ onto A.

This requires relating geometric structures to *connections,* since all of the locally homogeneous geometric structures discussed in this book can be approached through this general concept. However, we do *not* discuss the general notion of *Cartan connections,* but rather refer to the excellent introduction to this subject by R. Sharpe [**305**]. Some aspects of the general theory of affine connections have been relegated to Appendix B.

Chapter 8 introduces some of the basic examples in our theory. Bieberbach's theorems [**53, 54**] successfully describe the structure and classification of Euclidean structures on closed manifolds:[1] *Every closed Euclidean manifold M is a biquotient $\Lambda\backslash\mathbb{R}^{/}\Phi$ where $\Lambda < \mathbb{R}^n$ is a lattice and Φ is a finite group of automorphisms of Λ.* In other words M is finitely covered by *flat torus,* such as $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$.

One wonders if a similar picture holds when M is only assumed to be *affine,* and this is the *Auslander–Milnor question,* (sometimes called the "Auslander conjecture"): whether the fundamental group $\pi_1(M)$ is virtually polycyclic. In that case, M is finitely covered by a *solvmanifold* $\Gamma\backslash G$ where G is a solvable Lie group and $\Gamma < G$ is a lattice. Here G has a left-invariant complete affine structure, meaning that it acts simply transitively and affinely on affine space. It plays the role of the group of translations for Euclidean manifolds.

This question is open for closed manifolds, but Margulis [**253**] found proper affine actions of the 2-generator free group $\mathbb{F}_2$ on A^3, and their quotients, called *Margulis spacetimes,* are discussed in §15.4.

The first 3-dimensional examples are described, including $\mathsf{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$, the Heisenberg nilmanifold $\mathsf{Heis}_{\mathbb{Z}}\backslash\mathsf{Heis}_{\mathbb{R}}$, and hyperbolic torus bundles $\mathsf{Sol}_{\mathbb{Z}}\backslash\mathsf{Sol}_{\mathbb{R}}$. They represent three of the eight *Thurston geometries* in dimension 3.

We classify complete affine structures on the 2-torus $\mathbb{T}^2$ (originally due to Kuiper [**234**]). The Hopf manifolds introduced in §6.2 are fundamental examples of incomplete structures. That affine structures on compact manifolds are generally incomplete is one dramatic difference between affine geometry and traditional Riemannian geometry.

The successful classification of affine (and projective) strucutres on $\mathbb{T}^2$ began with Kuiper [**234**] in the convex case. It was completed by Nagano–Yagi [**279**] and Arrowsmith–Furness [**141**]; Baues [**33**] provides an excellent exposition. They provide many basic examples, some of which generalize to higher dimensions. The classification of affine (and projective) 2-manifolds is somewhat messy but provides a paradigm for the problems discussed in this book. The classification is revisited several times to motivate some of the general theory, including deformation spaces and affine Lie groups.

[1]See Charlap [**84**] for a good exposition of this theory.

Part Three: Affine and projective structures

Chapter 9 begins the classification of affine structures on surfaces. We prove Benzécri's theorem [**45**] that a closed surface Σ admits an affine structure if and only if its Euler characteristic vanishes. We discuss the famous conjecture of Chern that the Euler characteristic of a closed affine manifold vanishes, giving the proof of Kostant–Sullivan [**225**] in the complete case.

Chapter 10 offers a detailed study of left-invariant affine structures on Lie groups. We will call a Lie group with a left-invariant affine structure an *affine Lie group.* These provide many examples; in particular all the nonradiant affine structures on $\mathbb{T}^2$ are *invariant* affine structures on the *Lie group* $\mathbb{T}^2$. For these structures the holonomy homomorphism and the developing map blend together in an intriguing way.[2] Covariant differentiation of left-invariant vector fields lead to well-studied nonassociative algebras called *algèbres symétriques à gauche* or (*left-symmetric* algebras). Such algebras have the property that their *associators* are s c in the left two variables. Commutator defines the structure of an underlying Lie algebra. Associative algebras correspond to *bi-invariant affine structures,* so the "group objects" in the category of affine manifolds correspond naturally to associative algebras. These structures were introduced by Ernest Vinberg [**340**] in his study of homogeneous convex cones in affine space, and further developed by Jean-Louis Koszul and his school. We take a decidedly geometric approach to these ubiquitous mathematical structures. For example, many closed affine surfaces are affine Lie groups.

Chapter 11 describes the question (apparently first raised by L. Markus [**254**]) of whether, for a closed orientable affine manifold, completeness is equivalent to *parallel volume.* The existence of a parallel volume form is equivalent to unimodularity of the linear holonomy group, that is, whether the holonomy preserves volume. An "infinitesimal analog" of this question for left-invariant affine structures on Lie groups is the conceptual and suggestive result that completeness is equivalent to parallelism of *right-invariant* vector fields, (Exercise 10.3.9 in §10.3.5.)

This tantalizing question has led to much research, subsuming various questions which we discuss. Carrière's proof that compact flat Lorentzian manifolds are complete [**78**] is a special case, and Smillie's nonexistence theorem is another special case, discussed in §11.3. Section 11.2 treats the case when the affine holonomy group Γ is nilpotent. Another example is Fried's sharp classification of closed similarity manifolds [**135**] (proved independently by a much different argument by Vaisman–Reischer [**331**]).

[2] Perhaps this provides a conceptual basis for the unexpected relation between the 1-dimensional property of geodesic completeness and the top-dimensional property of volume-preserving holonomy.

Chapter 12 expounds the notions of “hyperbolicity” of Vey [**337**] and Kobayashi [**222**]. *Hyperbolic affine manifolds* are quotients of properly convex cones. A closed hyperbolic manifold is a radiant suspension of an $\mathbb{R}\mathsf{P}^n$-manifold, which itself is a quotient of a divisible domain. In particular we describe how a *completely incomplete* closed affine manifold must be affine hyperbolic in this sense. (That is, we tame the developing map of an affine structure with *no* two-ended complete geodesics.) This striking result is similar to the tameness where *all* geodesics are complete — complete manifolds are also quotients. The key ingredient is the *infinitesimal Kobayashi pseudo-metric,* which measures the (in)completeness of a geodoesic with given velocity.

Chapter 13 summarizes some aspects of the now blossoming subject of $\mathbb{R}\mathsf{P}^2$-structures on surfaces, in terms of the explicit coordinates and deformations which extend some of the classic geometric constructions on the deformation space of hyperbolic structures on closed surfaces. We describe the analog of Fenchel–Nielsen coordinates and other coordinate systems, briefly mentioning a more analytic approach due independently to Loftin and Labourie. Then we describe the grafting construction, and the first examples, due to Smillie and Sullivan–Thurston, of a projective structure on $\mathbb{T}^2$ with pathological developing map.

Chapter 14 describes the classic subject of $\mathbb{C}\mathsf{P}^1$-manifolds, which traditionally identify with *projective structures on Riemann surfaces.* Using the Schwarzian derivative, these structures are classified by the points of a holomorphic affine bundle over the Teichmüller space of Σ. This parametrization (presumably known to Poincaré), is remarkable in that is completely *formal,* using standard facts from the theory of Riemann surfaces. One knows precisely the deformation space without any knowledge of the developing map (besides it being a local biholomorphism). This is notable because the developing maps can be pathological; indeed the first examples of pathological developing maps were $\mathbb{C}\mathsf{P}^1$-manifolds on hyperbolic surfaces. The theory of projective structures on Riemann surfaces is a suggestive paradigm for a successful classifaction of highly nontrivial geometric structures.

Chapter 15 surveys known results, and the many open questions, in dimension three. This complements Thurston’s book [**324**] and expository articles of Scott [**302**] and Bonahon [**56**], which deal with geometrization and the relations to 3-manifold topology. In particular we describe the classification, due to Serge Dupont [**119, 120**], of projective structures on hyperbolic torus bundles

Chapter 1

Affine geometry

This section introduces the geometry of affine spaces. After a rigorous definition of affine spaces and affine maps, we discuss how linear algebraic constructions define geometric structures on affine spaces. Affine geometry is then transplanted to manifolds. The section concludes with a discussion of affine subspaces, vector fields, volume, and the notion of center of gravity.

1.1. Euclidean space

We begin with a short summary of Euclidean geometry in terms of its underlying space and its group of isometries.

Euclidean geometry can be described in many different ways. Here is one simple approach. Denote by E^n the set of points in the vector space $\mathbb{R}^n$ (that is, ordered n-tuples of real numbers) with the distance function

$$\begin{aligned} \mathsf{E}^n \times \mathsf{E}^n &\xrightarrow{\mathsf{d}} \mathbb{R} \\ (p,q) &\longmapsto \|p-q\|. \end{aligned}$$

Exercise 1.1.1. Let $(\mathsf{E}^n, \mathsf{d}) \xrightarrow{g} (\mathsf{E}^n, \mathsf{d})$ be an isometry. Then

$$g(p) \;=\; \mathbf{A}p + \mathbf{b}$$

for an orthogonal matrix $\mathbf{A} \in \mathsf{O}(n)$ and a vector $\mathbf{b} \in \mathbb{R}^n$. (*Hint:* First show that translations and orthogonal linear transformations are isometries. Then it suffices to prove that an isometry which fixes the origin and all tangent vectors at the origin is the identity. This can be done by characterizing straight-line paths — that is, *geodesics* — as curves which locally minimize length. Compare Exercise 1.3.4.)

That is, the isometry g of Euclidean n-space E^n is a composition of the *linear isometry* defined by $\mathbf{A}$ and the *translation*

$$p \xmapsto{\tau_{\mathbf{b}}} p + \mathbf{b}$$

by $\mathbf{b}$.

Two objects X, Y are *parallel* if they are related by the action of a translation, in which case we write $X \parallel Y$.

Exercise 1.1.2. Show that translations form a normal subgroup $\mathsf{Trans}(\mathsf{E}^n)$ isomorphic to $\mathbb{R}^n$ and

$$\mathsf{Isom}(\mathsf{E}^n) \;=\; \mathsf{Trans}(\mathsf{E}^n) \rtimes \mathsf{O}(n).$$

Deduce that $\mathsf{Isom}(\mathsf{E}^n)$ preserves the relation of parallelism.

Another feature of Euclidean geometry is the notion of *angle:*

Exercise 1.1.3. Every isometry of E^n preserves angles. (Hint: use the fact that angles can be defined in terms of inner products on $\mathbb{R}^n$.)

That is, every isometry is angle-preserving, or *conformal.* The equivalence relation of *similarity* is generated by the group $\mathsf{Sim}(\mathsf{E}^n)$ of conformal transformations of E^n.

An element of $\mathsf{Sim}(\mathsf{E}^n)$ which is not an isometry is the *homothety* given by scalar multiplication $p \longmapsto \lambda p$, where $\lambda \in \mathbb{R}^\times$ and $\lambda \neq \pm 1$. (See §1.6.2 for the general definition of homotheties.) Denote the group of scalar multiplications by $\lambda > 0$ by $\mathbb{R}^+$.

Exercise 1.1.4. The group $\mathsf{Sim}(\mathsf{E}^n)$ is generated by $\mathsf{Isom}(\mathsf{E}^n)$ and $\mathbb{R}^+$. Indeed,

$$\mathsf{Sim}(\mathsf{E}^n) \;=\; \mathsf{Sim}_0(\mathsf{E}^n) \ltimes \mathsf{Trans}(\mathsf{E}^n)$$

where

$$\mathsf{Sim}_0(\mathsf{E}^n) := \mathbb{R}^+ \times \mathsf{O}(n)$$

is the group of *linear similarities* of E^n. Explicitly a transformation g of E^n lies in $\mathsf{Sim}(\mathsf{E}^n)$ if it has the form

$$p \longmapsto \lambda \mathbf{A}(p) + \mathbf{b}$$

where $\mathbf{A} \in \mathsf{O}(n)$ and $\lambda \in \mathbb{R}^+$.

Yet another feature of Euclidean geometry is *volume:*

Exercise 1.1.5. Show that an orientation-preserving isometry of Euclidean space is volume-preserving. Show that an orientation-preserving similarity transformation preserves volume if and only if it is an isometry.

Compare §1.4.3 for further discussion of volume in affine geometry.

1.2. Affine space

What geometric properties of E^n do not involve the metric notions of distance, angle and volume? For example, the notion of *straight line* is invariant under translations and more general linear maps which are not Euclidean isometries. It enjoys a metric characterization as a curve which is locally length-minimizing — that is, every subpath is "the shortest path joining its endpoint." However *geodesics* are more fundamentally characterized as curves of zero acceleration. However the definition of acceleration requires comparing the velocity vectors at *different* points along the curve. This is achieved by the *parallel transport* of the velocity along the curve, and hence involves the notion of *parallelism.* (This is the notion of an *affine connection,* which is way to "connect" the infinitesimal displacements at different locations.)

Here is our first definition of an *affine transformation:*

Definition 1.2.1. An *affine transformation* of $\mathbb{R}^n$ is a mapping of the form

$$\begin{aligned} \mathbb{R}^n &\xrightarrow{g} \mathbb{R}^n \\ p &\longmapsto \mathbf{A}p + \mathbf{b} \end{aligned}$$

where $\mathbf{A} \in \mathsf{GL}(n, \mathbb{R})$ is an $n \times n$ invertible matrix and $\mathbf{b} \in \mathbb{R}^n$ is a vector. $\mathbf{A}$ is called the *linear part* of g, and denoted $\mathsf{L}(g)$ and $\mathbf{b}$ is called the *translational part* of g, and denoted $\mathbf{u}(g)$.

Thus an affine transformation g is:

- a translation if and only if $\mathsf{L}(g) = \mathbb{I}$;
- a Euclidean isometry if and only if $\mathsf{L}(g) \in \mathsf{O}(n)$;
- a Euclidean similarity (conformal transformation) if and only if $\mathsf{L}(g) \in \mathsf{Sim}_0(\mathbb{R}^n) = \mathbb{R}^+ \times \mathsf{O}(n)$;
- a volume-preserving (ore *special*) affine transformation if and only if $\mathsf{L}(g) \in \mathsf{SL}(n, \mathbb{R})$.

1.2.1. The geometry of parallelism. Here is a more formal definition of an affine space. Although less intuitive, it embodies the idea that affine geometry is the geometry of *parallelism.*

Recall that subsets $X, Y \subset \mathsf{E}^n$ are *parallel* (written $X \parallel Y$) if and only if $\tau_{\mathbf{v}}(X) = Y$ for some vector $\mathbf{v} \in \mathbb{R}^n$. (Here $\tau_{\mathbf{v}} \in \mathsf{Trans}(\mathsf{E}^n)$ denotes the translation $p \longmapsto p + \mathbf{v}$.) Affine geometry is the geometry arising from the simply transitive action of the vector space of translations (isomorphic to $\mathbb{R}^n$).

Recall that an action of a group G on a space X is *simply transitive* if and only if for some (and then necessarily every) $x \in X$, the evaluation map

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto g \cdot x \end{aligned}$$

is bijective: that is, for all $x, y \in X$, a unique $g \in G$ takes x to y. Equivalently, the action is both:

- *Transitive:* There is only one orbit, and
- *Free:* No nontrivial element fixes a point.

For further general discussion about group actions, see §A.3.

Definition 1.2.2. Let G be a group. A *G-torsor* is a space X with a simply transitive G-action.

Thus a G-torsor is like the group G, except that the special role of its identity element is "forgotten." Thus all the points are regarded as equivalent. What is remembered is the algebraic structure of the transformations in G which transport uniquely between the points.

Now we give the formal definition of a an affine space:

Definition 1.2.3. An *affine space* is a V-torsor A, where V is a vector space. We call V the *vector space underlying* A, and denote it by $\mathsf{Trans}(\mathsf{A})$, the elements of which are the *translations* of A.

This abstract approach provides the usual coordinates for an affine space, which identifies A with V. (In turn, choosing a basis of V identifies V with k^n.) Namely, choose a basepoint $p_0 \in \mathsf{A}$ which will correspond to the *origin* which identifies with the *zero vector* $\mathbf{0} \in \mathsf{V} \cong \mathsf{k}^n$. Any other point $p \in \mathsf{A}$ relates to p_0 by a unique translation $\tau \in \mathsf{Trans}(\mathsf{A})$ satisfying $p = \tau(p_0)$. (This translation exists because the action is transitive, and is unique because the action is free.) Identifying the transformations $\mathsf{Trans}(\mathsf{A})$ with vectors $\mathbf{v} \in \mathsf{V}$ in the usual coordinates, the vector $\mathbf{v}$ corresponding to τ is just $\mathbf{v} = p - p_0$, and τ is the mapping

$$\begin{aligned} \mathsf{A} &\xrightarrow{\tau} \mathsf{A} \\ p &\longmapsto p + \mathbf{v}. \end{aligned}$$

1.2.2. Affine transformations. Here is the second (more abstract) definition of the notion of a transformation being *affine.* The key point is that the arbitrary choices of basepoints enables "choosing origins" and identifies the affine spaces with their underlying vector spaces.

Affine maps are maps between affine spaces A, A' which are *compatible* with these simply transitive actions of vector spaces. In other words, they preserve the structures on A, A' as *torsors.* Denote the underlying vector spaces

$$\mathsf{V} \longleftrightarrow \mathsf{Trans}(\mathsf{A})$$
$$\mathsf{V}' \longleftrightarrow \mathsf{Trans}(\mathsf{A}')$$

respectively Then a continuous map

$$\mathsf{A} \xrightarrow{f} \mathsf{A}'$$

is *affine* if for each $\tau \in \mathsf{Trans}(\mathsf{A})$, there exists a translation $\tau' \in \mathsf{Trans}(\mathsf{A}')$ such that the diagram

$$\begin{array}{ccc} \mathsf{A} & \xrightarrow{f} & \mathsf{A}' \\ {\scriptstyle\tau}\big\downarrow & & \big\downarrow{\scriptstyle\tau'} \\ \mathsf{A} & \xrightarrow{f} & \mathsf{A}' \end{array}$$

commutes, that is, $f \circ \tau = \tau' \circ f$. That is, for each vector $\mathbf{v} \in \mathsf{V}$, there exists a vector $\mathbf{v}' \in \mathsf{V}'$ such that f conjugates translation by $\mathbf{v}$ in A to translation by $\mathbf{v}'$ in A'.

Exercise 1.2.4. Suppose that f is affine as above, conjugating τ to τ'.

- Show that τ' is unique, and therefore f defines a map $\mathsf{V} \longrightarrow \mathsf{V}'$.
- Show that this map is a linear map of vector spaces.

This linear map, which we denote A, is the *linear part* of f, denoted $\mathsf{L}(f)$. Denoting the space of all affine maps $\mathsf{A} \longrightarrow \mathsf{A}'$ by $\mathsf{aff}(\mathsf{A}, \mathsf{A}')$ and the space of all linear maps $\mathsf{V} \longrightarrow \mathsf{V}'$ by $\mathsf{Hom}(\mathsf{V}, \mathsf{V}')$, linear part defines a map

$$\mathsf{aff}(\mathsf{A}, \mathsf{A}') \xrightarrow{\mathsf{L}} \mathsf{Hom}(\mathsf{V}, \mathsf{V}')$$

When $\mathsf{A} = \mathsf{A}'$, the set of affine *endomorphisms* of an affine space A will be denoted by $\mathsf{aff}(\mathsf{A})$ and the group of affine *automorphisms* of A will be denoted $\mathsf{Aff}(\mathsf{A})$. In particular if $f \in \mathsf{aff}(\mathsf{A})$, then $\mathsf{L}(f) \in \mathsf{End}(\mathsf{V})$. Moreover

$$\mathsf{L}(f) \in \mathsf{Aut}(\mathsf{V}) \cong \mathsf{GL}(n, \mathsf{k})$$

if and only if $f \in \mathsf{Aff}(\mathsf{A})$.

The notion of *translational part* involves choosing basepoints $p_0 \in \mathsf{A}$ and $p_0' \in \mathsf{A}'$, respectively. In particular, choice of a basepoint $p_0 \in \mathsf{A}$ identifies $\mathsf{GL}(V)$ as the subgroup of $\mathsf{Aff}(\mathsf{A})$ fixing p_0. More generally, $\mathsf{Hom}(\mathsf{V}, \mathsf{V}')$ identifies with the subspace of $\mathsf{aff}(\mathsf{A}, \mathsf{A}')$ comprising affine maps $\mathsf{A} \to \mathsf{A}'$ taking p_0 to p_0'.

Using the identifications

$$\mathsf{V} \longleftrightarrow \mathsf{A}$$
$$\mathsf{V}' \longleftrightarrow \mathsf{A}'$$

defined by the respective basepoints, the *translational part* of $f \in \mathsf{aff}(\mathsf{A}, \mathsf{A}')$ is simply the vector $\mathbf{b}' \in \mathsf{V}'$ corresponding to the translation $\tau_f \in \mathsf{Trans}(\mathsf{A}')$ taking $p_0' \in \mathsf{A}'$ to $f(p_0) \in \mathsf{A}'$. Then $(\tau_f)^{-1} \circ f$ maps p_0 to p_0', and is a *linear* map $\mathsf{V} \to \mathsf{V}'$ in the above sense.

Exercise 1.2.5. An arbitrary point $p \in \mathsf{A}$ corresponds to a vector

$$\mathbf{x} \;=\; p - p_0 \in \mathsf{V},$$

that is, translation

$$\tau = \; \tau_{\mathbf{x}} \in \mathsf{Trans}(\mathsf{A})$$

by $\mathbf{x}$ takes p_0 to p. Show that under the above identifications, f corresponds to the map

$$p \;\leftrightarrow \mathbf{x} \quad \longmapsto \quad \mathbf{A}\mathbf{x} + \mathbf{b}' \leftrightarrow f(p).$$

The following suggestive facts are due to Vey [**337**], and foreshadow the discussion of affine Lie groups in Chapter 10.

Exercise 1.2.6. The space $\mathsf{aff}(\mathsf{A}, \mathsf{A}')$ of affine maps $\mathsf{A} \to \mathsf{A}'$ itself is an affine space, with underlying vector space $\mathsf{Hom}(\mathsf{V}, \mathsf{V}')$.

- The Cartesian product $\mathsf{Aff}(\mathsf{A}) \times \mathsf{Aff}(\mathsf{A}')$ acts by composition on $\mathsf{aff}(\mathsf{A}, \mathsf{A}')$, preserving the affine structure.
- More generally, if $\Omega \subset \mathsf{A}$ is a subdomain, show that *affine automorphism group* $\mathsf{Aff}(\Omega)$ is a Lie group with a bi-invariant affine structure (as defined in Chapter 10). Identify the Lie algebra of $\mathsf{Aff}(\mathsf{A})$ with $\mathsf{aff}(\mathsf{A})$.

§1.6 identifies $\mathsf{aff}(\mathsf{A})$ with the Lie algebra of *affine vector fields* on A.

Affine geometry is the study of affine spaces and affine maps between them. If $U \subset \mathsf{A}$ is an open subset, then a map $U \xrightarrow{f} \mathsf{A}'$ is *locally affine* if for each connected component U_i of U, there exists an affine map $f_i \in \mathsf{aff}(\mathsf{A}, \mathsf{A}')$ such that the restrictions of f and f_i to U_i are identical. Note that two affine maps which agree on a nonempty open set are identical.

1.3. The connection on affine space

Now we discuss the structure of an affine space A as a smooth manifold. To analyze the differentiable structure on A, we consider smooth paths in A and their velocity vector fields, which live in the tangent bundle TA. From this we "connect" the tangent spaces to define covariant differentiation enabling us to define acceleration as the covariant derivative of the velocity. Geodesics are curves of zero acceleration.

1.3.1. The tangent bundle of an affine space. Let $\gamma(t)$ denote a *smooth curve* in A; that is, in coordinates

$$\gamma(t) = \big(x^1(t), \dots, x^n(t)\big)$$

where $x^j(t)$ are smooth functions of the time parameter, which ranges in an interval $[t_0, t_1] \subset \mathbb{R}$. The vector $\gamma(t) - \gamma(t_0)$ corresponds to the unique translation taking $\gamma(t_0)$ to $\gamma(t)$, and lies in the vector space V underlying A. It represents the displacement of the curve γ as it goes from t_0 to t. Define its *velocity vector* $\gamma'(t) \in \mathsf{V}$ as the derivative of this path in the vector space V of translations. It represents the *infinitesimal* displacement of $\gamma(t)$ as t varies.

The set of tangent vectors is a vector space, denoted $\mathsf{T}_p\mathsf{A}$, and naturally identifies with V as follows. If $\mathbf{v} \in \mathsf{V}$ is a vector, then the path $\gamma_{(p,\mathbf{v})}(t)$ defined by:

$$t \mapsto p + t\mathbf{v} = \tau_{t\mathbf{v}}(p) \tag{1.1}$$

is a smooth path with $\gamma(0) = p$ and velocity vector $\gamma'(0) = \mathbf{v}$. Conversely, the above discussion of infinitesimal displacement implies that every smooth path through $p = \gamma(0)$ with velocity $\gamma'(0) = \mathbf{v}$ is tangent to the curve (1.1) as above.

The tangent spaces *linearize* smooth manifolds as follows. Let M, M' be smooth manifolds and

$$M \xrightarrow{f} M'$$

a continuous map. Then f is *differentiable at* $p \in M$ if every infinitesimal displacement $\mathbf{v} \in \mathsf{T}_pM$ maps to an infinitesimal displacement $\mathsf{D}_pf(\mathbf{v}) \in \mathsf{T}_qM'$, where $q = f(p)$. That is, if γ is a smooth curve with $\gamma(0) = q$ and $\gamma'(0) = \mathbf{v}$, then we require that $f \circ \gamma$ is a smooth curve through q at $t = 0$; then we call the new velocity $(f \circ \gamma)'(0)$ the value of the *derivative*

$$\begin{aligned} \mathsf{T}_pM &\xrightarrow{\mathsf{D}_pf} \mathsf{T}_qM' \\ \mathbf{v} &\longmapsto (f \circ \gamma)'(0) \end{aligned}$$

1.3.2. Parallel transport. On an affine space A, all the tangent spaces identify with each other. Namely, if $x, y \in \mathsf{A}$, let $\tau \in \mathsf{Trans}(\mathsf{A})$ be the unique translation taking x to y. (τ corresponds to the vector $y - x$.) The differential $(\mathsf{D}\tau)_x$ maps $\mathsf{T}_x\mathsf{A}$ isomorphically to $\mathsf{T}_y\mathsf{A}$) and we denote this by:

$$\mathsf{T}_x\mathsf{A} \xrightarrow{\mathbb{P}_{x,y}} \mathsf{T}_y\mathsf{A}$$

We call this map *parallel transport* from x to y.

Exercise 1.3.1. Another construction involves the linear structure of $\mathsf{V} \longleftrightarrow \mathsf{Trans}(\mathsf{A})$. Namely, the action of V by translations identifies the vector space V with $\mathsf{T}_x\mathsf{A}$. Denoting this isomorphism by $\mathsf{V} \xrightarrow{\alpha_x} \mathsf{T}_x\mathsf{A}$, show that

$$\mathbb{P}_{x,y} = \alpha_y \circ (\alpha_x)^{-1}.$$

A vector field $\xi \in \mathsf{Vec}(\mathsf{A})$ is *parallel* if it is invariant under parallel transport. That is, $\mathbb{P}_{x,y}(\xi_x) = \xi_y$ for any $x, y \in \mathsf{A}$. This just means that ξ is a "constant vector field," defined by a constant map $\mathsf{A} \xrightarrow{\mathbf{v}} \mathsf{V}$: as a differential operator

$$\mathsf{C}^\infty(\mathsf{A}) \xrightarrow{\xi} \mathsf{C}^\infty(\mathsf{A})$$
$$f \longmapsto v^i(x)\frac{\partial f}{\partial x^i}$$

where $\mathbf{v}(x)$ is constant. Thus V identifies with the space of parallel vector fields on A, and is based by the coordinate vector fields

$$\frac{\partial}{\partial x^i} \in \mathsf{Vec}(A),$$

which we abbreviate simply by ∂_i.

Exercise 1.3.2. Show that $\xi \in \mathsf{Vec}(\mathsf{A})$ is parallel if and only if it generates a one-parameter group of translations.

Similarly, the dual vector space V^* identifies with parallel 1-forms as follows. A 1-form (covector field) on A corresponds to a constant map $\mathsf{A} \longrightarrow \mathsf{V}^*$. The basis of parallel covector fields dual to the coordinate basis $\{\partial_1, \dots, \partial_n\}$ of parallel vector fields is denoted $\{dx^1, \dots, dx^n\}$ (as usual).

Exercise 1.3.3. Show that a parallel 1-form is exact, and hence closed.

1.3.3. Acceleration and geodesics. The velocity vector field $\gamma'(t)$ of a smooth curve $\gamma(t)$ is an example of a *vector field along the curve* $\gamma(t)$*:* For each t, the tangent vector $\gamma'(t) \in \mathsf{T}_{\gamma(t)}\mathsf{A}$. Differentiating the velocity vector field raises a significant difficulty: since the values of the vector field live in different vector spaces, we need a way to compare, or to *connect* them. The natural way is use the simply transitive action of the group V of translations of A. That is, suppose that $\gamma(t)$ is a smooth path, and $\xi(t)$ is a vector field along $\gamma(t)$. Let τ_s^t denote the translation taking $\gamma(t+s)$ to $\gamma(t)$, that is, in coordinates:

$$\mathsf{A} \xrightarrow{\tau_s^t} \mathsf{A}$$
$$p \longmapsto p + \big(\gamma(t) - \gamma(t+s)\big)$$

Its differential

$$\mathsf{T}_{\gamma(t+s)}\mathsf{A} \xrightarrow{\mathsf{D}\tau_s^t} \mathsf{T}_{\gamma(t)}\mathsf{A}$$

then maps $\xi(t+s)$ into $\mathsf{T}_{\gamma(t)}\mathsf{A}$ and the *covariant derivative* $\frac{\mathsf{D}}{dt}\xi(t)$ is the derivative of this smooth path in the *fixed* vector space $\mathsf{T}_{\gamma(t)}\mathsf{A}$:

$$\frac{\mathsf{D}}{dt}\xi(t) := \frac{d}{ds}\bigg|_{s=0} (\mathsf{D}\tau_s^t)\big(\xi(t+s)\big)$$

$$= \lim_{s\to 0} \frac{(\mathsf{D}\tau_s^t)\big(\xi(t+s)\big) - \xi(t)}{s}$$

In this way, define the *acceleration* as the covariant derivative of the velocity:

$$\gamma''(t) := \frac{\mathsf{D}}{dt}\gamma'(t)$$

A curve with zero acceleration is called a *geodesic.*

Exercise 1.3.4. Given a point p and a tangent vector $\mathbf{v} \in \mathsf{T}_p\mathsf{A}$, show that the unique *geodesic* $\gamma(t)$ with

$$\big(\gamma(0), \gamma'(0)\big) = (p, \mathbf{v})$$

is given by (1.1).

In other words, geodesics in A are parametrized curves which are Euclidean straight lines traveling at *constant* speed. However, in affine geometry the *speed* itself in not defined, but "motion along a straight line at constant speed" is affinely invariant (zero acceleration).

This leads to the following important definition:

Definition 1.3.5. Let $p \in \mathsf{A}$ and $\mathbf{v} \in \mathsf{T}_p(\mathsf{A}) \cong \mathsf{V}$. Then the *exponential mapping* is defined by:

$$\mathsf{T}_p\mathsf{A} \xrightarrow{\mathsf{Exp}_p} \mathsf{A}$$
$$\mathbf{v} \longmapsto p + \mathbf{v}.$$

Thus the unique geodesic with initial position and velocity $(p, \mathbf{v})$ equals

$$t \longmapsto \mathsf{Exp}_p(t\mathbf{v}) = p + t\mathbf{v}.$$

1.4. Parallel structures

Many important refinements of affine geometry involve structures which are *parallel.* Parallelism generalizes the notion of "constant" when the targets vary from point to point.

For example, the most familiar geometry is *Euclidean geometry,* extremely rich with metric notions such as distance, angle, area, and volume.

We have seen that *affine geometry* underlies it with the more primitive notion of *parallelism.* Euclidean geometry arises from affine geometry by introducing a Riemannian structure on A, which is *parallel.*

Parallel vector fields and 1-forms were introduced back in §1.3.2, where parallel vector fields correspond to vectors in V and parallel 1-forms (parallel covector fields) correspond to covectors in V^*. Now we consider parallel tensor fields of higher order.

1.4.1. Parallel Riemannian structures. Let B be an inner product on V and $\mathsf{O}(\mathsf{V};\mathsf{B}) \subset \mathsf{GL}(\mathsf{A})$ the corresponding orthogonal group. Then B defines a flat Riemannian metric on A and the inverse image

$$\mathsf{L}^{-1}(\mathsf{O}(\mathsf{V};\mathsf{B})) \cong \mathsf{O}(\mathsf{V};\mathsf{B}) \cdot \mathsf{Trans}(\mathsf{A})$$

is the full group of isometries, that is, the *Euclidean group.* If B is a nondegenerate indefinite form, then there is a corresponding flat pseudo-Riemannian metric on A and the inverse image $\mathsf{L}^{-1}\big(\mathsf{O}(\mathsf{V};\mathsf{B})\big)$ is the full group of isometries of this pseudo-Riemannian metric.

1.4.2. Similarity geometry. Euclidean space with a parallel *conformal structure* (that is, an infinitesimal notion of *angle*) is a model for *similarity geometry.*[1]

Exercise 1.4.1. Show that an affine automorphism g of Euclidean n-space E^n is conformal (that is, preserves angles) if and only if its linear part is the composition of an orthogonal transformation and scalar multiplication.

Such a transformation will be called a *similarity transformation* and the group of similarity transformations will be denoted $\mathsf{Sim}(\mathsf{E}^n)$. The scalar multiple is called the *scale factor* $\lambda(g) \in \mathbb{R}^\times$ and defines a homomorphism $\mathsf{Sim}(\mathsf{E}^n) \xrightarrow{\lambda} \mathbb{R}^\times$. In general, if $g \in \mathsf{Sim}(\mathsf{E}^n)$, then $\exists!\mathbf{A} \in \mathsf{O}(n)$ and $\mathbf{b} \in \mathbb{R}^n$ such that

$$x \xrightarrow{g} \lambda(g)\mathbf{A}x + \mathbf{b}.$$

Furthermore $\lambda(g)\mathbf{A} = \mathsf{L}(g)$ identifies with $\mathsf{D}g$.

1.4.3. Parallel tensor fields. Any tangent vector $\mathbf{v}_p \in \mathsf{T}_p\mathsf{A}$ extends uniquely to a vector field on A invariant under the group of translations. As we saw in §1.4.1, Euclidean structures are defined by extending an inner product from a single tangent space to all of E.

Dual to parallel vector fields are *parallel* 1*-forms.* Every tangent *covector* $\omega_p \in \mathsf{T}^*_p\mathsf{A}$ extends uniquely to a translation-invariant 1-form.

[1] According to Fried [**135**] Euclidean geometry is affine geometry with a parallel *ruler* and similarity geometry is affine geometry with a parallel *protractor.*

Exercise 1.4.2. Prove that a parallel 1-form is closed. Express a parallel 1-form in local coordinates.

If $n = \mathsf{dim}(\mathsf{A})$, then an exterior n-form ω must be $f(x)\,dx^1 \wedge \cdots \wedge dx^n$ in local coordinates, where $f \in C^\infty(\mathsf{A})$ is a smooth function. Then ω is parallel if and only if $f(x)$ is constant.

Exercise 1.4.3. Prove that an affine transformation $g \in \mathsf{Aff}(\mathsf{A})$ preserves a parallel volume form if and only if $\det \mathsf{L}(g) = 1$.

Parallel volume forms are discussed more extensively in §1.7

1.4.4. Complex affine geometry. We have been working entirely over $\mathbb{R}$, but it is clear one may study affine geometry over any field. If $\mathsf{k} \supset \mathbb{R}$ is a field extension, then every k-vector space is a vector space over $\mathbb{R}$ and thus every k-affine space is an $\mathbb{R}$-affine space. In this way we obtain more refined geometric structures on affine spaces by considering affine maps whose linear parts are linear over k.

Exercise 1.4.4. Relate 1-dimensional complex affine geometry to 2-dimensional similarity geometry with a fixed orientation.

This structure is another case of a parallel structure on an affine space, as follows. Recall a complex vector space has an underlying structure as a real vector space V. The difference is a notion of scalar multiplication by $\sqrt{-1}$, which is given by a linear map

$$\mathsf{V} \xrightarrow{J} \mathsf{V}$$

such that $J \circ J = -\mathbb{I}$. Such an automorphism is called a *complex structure* on V, and “turns V into” a *complex vector space.*

If M is a manifold, an endomorphism field (that is, a $(1,1)$-tensor field) J where, for each $p \in M$, the value J_p is a complex structure on the tangent space T_pM is called an *almost complex structure.* Necessarily $\mathsf{dim}(M)$ is even.

Recall that a *complex manifold* is a manifold with an atlas of coordinate charts where coordinate changes are biholomorphic. (Such an atlas is called a *holomorphic atlas.*) Every complex manifold admits an almost complex structure, but not every almost complex structure arises from a holomorphic atlas, except in dimension two.

Exercise 1.4.5. Prove that a complex affine space is the same as an affine space with a parallel almost complex structure.

1.5. Affine subspaces

Suppose that $\mathsf{A}_1 \overset{\iota}{\hookrightarrow} \mathsf{A}$ is an injective affine map; then we say that $\iota(\mathsf{A}_1)$ (or with slight abuse, ι itself) is an *affine subspace.* If A_1 is an affine subspace then for each $x \in \mathsf{A}_1$ there exists a linear subspace $\mathsf{V}_1 \subset \mathsf{Trans}(\mathsf{A})$ such that A_1 is the orbit of x under V_1. That is, "an affine subspace in a vector space is just a coset (or translate) of a linear subspace $\mathsf{A}_1 = x + \mathsf{V}_1$." An affine subspace of dimension 0 is thus a point and an affine subspace of dimension 1 is a line. The next exercise describes how the quotient of an affine space by an affine subspace has the natural structure of a *vector space.*

Exercise 1.5.1. Let $\mathsf{A}_1 \subset \mathsf{A}$ is an affine subspace; then the cosets $p + \mathsf{A}_1$ are parallel affine subspaces defining equivalence classes for an equivalence relation on A. Let A/A_1 be the corresponding quotient space. Define a vector space structure on A/A_1 so that every affine transformation of A which preserves A_1 induces a linear transformation of A/A_1.

The next exercise describes a natural *affine parameter* along an affine line.

Exercise 1.5.2. Show that if ℓ, ℓ' are (affine) lines and

$$(x, y) \in \ell \times \ell,\ x \neq y$$
$$(x', y') \in \ell' \times \ell',\ x' \neq y'$$

are pairs of distinct points. Then there is a unique affine map $\ell \xrightarrow{f} \ell'$ such that

$$f(x) = x',$$
$$f(y) = y'.$$

If $x, y, z \in \ell$ (with $x \neq y$), then define $[x, y, z]$ to be the image of z under the unique affine map $\ell \xrightarrow{f} \mathbb{R}$ with $f(x) = 0$ and $f(y) = 1$. Show that if $\ell = \mathbb{R}$, then $[x, y, z]$ is given by the formula

$$[x, y, z] = \frac{z - x}{y - x}.$$

1.6. Affine vector fields

A vector field X on A is said to be *affine* if it generates a one-parameter group of affine transformations. Affine vector fields include parallel vector fields and radiant vector fields. Parallel vector fields generate one-parameter groups of translations, and *radiant* vector fields generate one-parameter groups of homotheties. Covariant differentiation provides general criteria characterizing affine vector fields.

1.6.1. Translations and parallel vector fields. Now we discuss the important case of "constant" vector fields, which play the role of *infinitesimal translations.*

Definition 1.6.1. A vector field X on A is *parallel* if, for every $p, q \in \mathsf{A}$, the values $X_p \in \mathsf{T}_p\mathsf{A}$ and $X_q \in \mathsf{T}_q\mathsf{A}$ are parallel.

Since translation $\tau_{\mathbf{v}}$ by $\mathbf{v} = q - p$ is the unique translation taking p to q, this simply means that the differential $(\mathsf{D}\tau_{\mathbf{v}})_p$ maps X_p to X_q.

Exercise 1.6.2. Let $X \in \mathsf{Vec}(\mathsf{A})$ be a vector field on an affine space A. The following conditions are equivalent:

- X is parallel
- The coefficients of X (in affine coordinates) are constant.
- $\nabla_Y X = 0$ for all $Y \in \mathsf{Vec}(\mathsf{A})$.
- The covariant differential $\nabla X = 0$.
- Show how to identify $\mathsf{Vec}(\mathsf{A})$ with $\mathsf{Map}(\mathsf{A}, \mathsf{V})$ so that X corresponds to an affine map f with linear part $\mathsf{L}(f) = 0$.

The vector space V identifies with the space of parallel vector fields on A, which is an abelian subalgebra of the Lie algebra $\mathsf{Vec}(\mathsf{A})$.

1.6.2. Homotheties and radiant vector fields. Scalar multiplication determine another important class of affine vector fields: the *radiant vector fields,* which are *infinitesimal homotheties.*

Definition 1.6.3. An affine transformation $\phi \in \mathsf{Aff}(\mathsf{A})$ is a *homothety* if it is conjugate by a translation to scalar multiplication $\mathbf{v} \longmapsto \lambda\mathbf{v}$, for some scalar $\lambda \in \mathbb{R}^\times$ with $\lambda \neq \pm 1$. An affine vector field is *radiant* if it generates a one-parameter group of homotheties.

Observe that a homothety fixes a unique point $p \in \mathsf{A}$, which we often take to be the origin. The only zero of the corresponding radiant vector field is p. We denote by Rad_p the unique radiate vector field vanishing at p, which, in coordinates, equals:

$$\mathsf{Rad}_p := \sum_{i=1}^{n} (x^i - p^i) \frac{\partial}{\partial x^i}. \tag{1.2}$$

Radiant vector fields are also called *Euler vector fields,* due to their role in Euler's theorem on homogeneous functions: Recall that a function $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$ is *homogeneous of degree m* if and only if

$$f(\lambda x) = \lambda^m f(x)$$

for all $\lambda \in \mathbb{R}^+$.

Exercise 1.6.4. Prove *Euler's theorem* on homogeneous functions: Suppose that Rad is the radiant vector field vanishing at the origin $\mathbf{0}$. Then f is *homogeneous of degree* m if and only if $\mathsf{Rad} f = mf$.

Radiant vector fields play an important role, since important examples of affine manifolds admit radiant vector fields. Furthermore radiant vector fields provide a link between n-dimensional affine manifolds and $(n-1)$-dimensional projective manifolds, through the *radiant suspension* construction discussed in §6.5.3.

Exercise 1.6.5. Let $X \in \mathsf{Vec}(\mathsf{A})$ be a vector field. Then the following conditions are equivalent:

- X is radiant;
- $\nabla_Y X = X$, for all $Y \in \mathsf{Vec}(\mathsf{A})$
- $\nabla X = \mathbb{I}_{\mathsf{A}}$ where $\mathbb{I}_{\mathsf{A}} \in \mathcal{A}^1(\mathsf{A}, \mathsf{TA})$ is the TA-valued 1-form on A (solder form) defined in § B.1 corresponding to the identity endomorphism field (defined above);

In particular, Rad generates the one-parameter group of homotheties

$$p \longmapsto b + e^t(p-b)$$

fixing b. Thus a radiant vector field is a special kind of affine vector field. Furthermore Rad generates the center of the isotropy group of $\mathsf{Aff}(\mathsf{A})$ at b, which is conjugate (by translation by b) to $\mathsf{GL}(\mathsf{V})$.

Exercise 1.6.6. Show that the radiant vector fields on A form an affine subspace of $\mathsf{Vec}(\mathsf{A})$ isomorphic to A.

- Show that the sum of a parallel vector field and a radiant vector field and a parallel vector is radiant.
- Define a vector field X to be ϵ*-radiant* if it equals ϵR for a radiant vector filed. Show that X is ϵ-radiant if and only if $\nabla_Y X = \epsilon Y$, $\forall Y \in \mathsf{Vec}(\mathsf{A})$, or, equivalently, $\nabla X = \epsilon X$.
- Let Y be a parallel vector field. Find a family X_ϵ of vector fields such that X_ϵ is ϵ-radiant and
$$\lim_{\epsilon \to 0} X_\epsilon = Y.$$

1.6.3. Affineness criteria. Affine vector fields can be characterized in terms of the covariant differential

$$\mathcal{T}^p(M; \mathsf{T}M) \xrightarrow{\nabla} \mathcal{T}^{p+1}(M; \mathsf{T}M)$$

where $\mathcal{T}^p(M; \mathsf{T}M)$ denotes the space of $\mathsf{T}M$-valued covariant p-tensor fields on M (the tensor fields of type $(1,p)$). Thus $\mathcal{T}^0(M; \mathsf{T}M) = \mathsf{Vec}(M)$ and

$\mathcal{T}^1(M;\mathsf{T}M)$ comprises endomorphism fields, (alternatively $\mathsf{T}M$-valued 1-forms).

Exercise 1.6.7. X is affine if and only if it satisfies any of the following equivalent conditions:

- For all $Y, Z \in \mathsf{Vec}(\mathsf{A})$,
$$\nabla_Y \nabla_Z X = \nabla_{(\nabla_Y Z)} X.$$
- $\nabla\nabla X = 0$.
- The coefficients of X are affine functions, that is,
$$X = \sum_{i,j=1}^{n} ({a^i}_j x^j + b^i)\frac{\partial}{\partial x^i}$$
for constants ${a^i}_j, b^i \in \mathbb{R}$.

Write
$$\mathsf{L}(X) = \sum_{i,j=1}^{n} {a^i}_j x^j \frac{\partial}{\partial x^i}$$
for the linear part (which corresponds to the matrix $({a^i}_j) \in \mathsf{gl}(\mathbb{R}^n)$) and
$$X(0) = \sum_{i=1}^{n} b^i \frac{\partial}{\partial x^i}$$
for the translational part (the translational part of an affine vector field is a parallel vector field). We denote the affine transformation corresponding to $X \in \mathsf{Vec}(\mathsf{A})$ by $\hat{X}$. Thus $\hat{X} = \left[\mathsf{L}(X) \;\middle|\; X(0)\right]$.

Exercise 1.6.8. Under this correspondence, covariant derivative corresponds to composition of affine maps (matrix multiplication):
$$\nabla_B A \quad \longleftrightarrow \hat{A}\hat{B}$$

The Lie bracket of two affine vector fields is given by:

- $\mathsf{L}([X,Y]) = [\mathsf{L}(X), \mathsf{L}(Y)] = \mathsf{L}(X)\mathsf{L}(Y) - \mathsf{L}(X)\mathsf{L}(Y)$ (matrix multiplication)
- $[X,Y](0) = \mathsf{L}(X)Y(0) - \mathsf{L}(Y)X(0)$.

In this way the space $\mathsf{aff}(\mathsf{A}) = \mathsf{aff}(\mathsf{A},\mathsf{A})$ of affine endomorphisms of A is a Lie algebra.

Let M be an affine manifold. A vector field $\xi \in \mathsf{Vec}(M)$ is *affine* if in local coordinates ξ appears as a vector field in $\mathsf{aff}(\mathsf{A})$. We denote the space of affine vector fields on an affine manifold M by $\mathsf{aff}(M)$.

Exercise 1.6.9. Let M be an affine manifold.

(1) Show that $\mathsf{aff}(M)$ is a subalgebra of the Lie algebra $\mathsf{Vec}(M)$.

(2) Show that the identity component of the affine automorphism group $\mathsf{Aut}(M)$ has Lie algebra $\mathsf{aff}(M)$.

(3) If ∇ is the flat affine connection corresponding to the affine structure on M, show that a vector field $\xi \in \mathsf{Vec}(M)$ is affine if and only if, for all $v \in \mathsf{Vec}(M)$,

$$\nabla_\xi v = [\xi, v].$$

1.7. Volume in affine geometry

Although an affine automorphism of an affine space A need not preserve a natural measure on A, Euclidean volume nonetheless does behave rather well with respect to affine maps. The Euclidean volume form ω can almost be characterized affinely by its parallelism: it is invariant under all translations. Moreover two $\mathsf{Trans}(\mathsf{A})$-invariant volume forms differ by a scalar multiple but there is no natural way to normalize. Such a volume form will be called a *parallel volume form.* If $g \in \mathsf{Aff}(\mathsf{A})$, then the distortion of volume is:

$$g^*\omega = \det \mathsf{L}(g) \cdot \omega.$$

Compare Exercise 1.4.3.) Thus although there is no canonically normalized volume or measure there is a natural affinely invariant line of measures on an affine space. The subgroup $\mathsf{SAff}(\mathsf{A})$ of $\mathsf{Aff}(\mathsf{A})$ consisting of volume-preserving affine transformations is the inverse image $\mathsf{L}^{-1}(\mathsf{SL}(\mathsf{V}))$, sometimes called the *special affine group* of A. Here $\mathsf{SL}(\mathsf{V})$ denotes the *special linear group*

$$\mathsf{Ker}\big(\mathsf{GL}(\mathsf{V}) \xrightarrow{\det} \mathbb{R}^\times\big) \;=\; \{g \in \mathsf{GL}(\mathsf{V}) \mid \det(g) = 1\}.$$

1.7.1. Centers of gravity.

Given a finite subset $F \subset \mathsf{A}$ of an affine space, its *center of gravity* or *centroid* $\bar{F} \in \mathsf{A}$ is point associated with F in an affinely invariant way: that is, given an affine map $\mathsf{A} \xrightarrow{\phi} \mathsf{A}'$ we have

$$\overline{\phi(F)} = \phi(\bar{F}).$$

This operation can be generalized as follows.

Theorem 1.7.1. Let μ be a compactly supported probability measure on an affine space A. Then there exists a unique point $\bar{x} \in \mathsf{A}$ (the *centroid* of μ) such that for all affine maps $\mathsf{A} \xrightarrow{f} \mathbb{R}$,

$$f(\bar{x}) = \int_{\mathsf{A}} f\, d\mu \qquad (1.3)$$

Proof. Let $(x^1, \ldots, x^n)$ be an affine coordinate system on A. Let $\bar{x} \in \mathsf{A}$ be the points with coordinates $(\bar{x}^1, \ldots, \bar{x}^n)$ given by

$$\bar{x}^i = \int_{\mathsf{A}} x^i \, d\mu.$$

This uniquely determines $\bar{x} \in \mathsf{A}$; we must show that (1.3) is satisfied for all affine functions. Suppose $\mathsf{A} \xrightarrow{f} \mathbb{R}$ is an affine function. Then there exist $a_1, \ldots, a_n, b$ such that

$$f = a_1 x^1 + \cdots + a_n x^n + b$$

and thus

$$\begin{aligned} f(\bar{x}) = a_1 \int_{\mathsf{A}} x^1 \, d\mu + \cdots + a_n \int_{\mathsf{A}} x^n \, d\mu \\ + b \int_{\mathsf{A}} d\mu \; = \; \int_{\mathsf{A}} f \, d\mu \end{aligned}$$

as claimed. □

We denote the centroid of μ by $\mathsf{centroid}(\mu)$.

Now let $C \subset \mathsf{A}$ be a *convex body*, that is, a convex open subset having compact closure. Then C determines a probability measure μ_C on A by

$$\mu_C(X) = \frac{\int_{X \cap C} \omega}{\int_C \omega}$$

where ω is any parallel volume form on A.

Proposition 1.7.2. Let $C \subset \mathsf{A}$ be a convex body. Then the centroid $\bar{C}$ of C lies in C.

Proof. C is the intersection of halfspaces, that is, C consists of all $x \in \mathsf{A}$ such that $f(x) > 0$ for all affine maps

$$\mathsf{A} \xrightarrow{f} \mathbb{R}$$

such that $f|_C > 0$.

If f is such an affine map, then clearly $f(\bar{C}) > 0$. Therefore $\bar{C} \in C$. □

1.7.2. Divergence. If $\xi \in \mathsf{Vec}(M)$, then the infinitesimal distortion of volume is the *divergence* of ξ, defined as the function $\mathsf{div}(\xi)$ such that

$$\mathcal{L}_\xi(\omega) = \mathsf{div}(\xi)\omega$$

where ω is (any) parallel volume form and $\mathcal{L}_\xi$ denotes Lie differentiation with respect to ξ. If, in coordinates $\xi = \xi^i \partial_i$, then

$$\mathsf{div}(\xi) = \partial_i \xi^i$$

(the usual formula).

Exercise 1.7.3. The Lie algebra of the special affine group $\mathsf{SAff}(\mathsf{A})$ consists of affine vector fields of divergence zero.

1.8. Linearizing affine geometry

Associated to every affine space A is an embedding $\mathcal{A}'$ of A as an *affine hyperplane* in a vector space W as follows.

Exercise 1.8.1. Let A be an affine space over a field k with underlying vector space $\mathsf{V} := \mathsf{Trans}(\mathsf{A})$. Let $\mathsf{W} := \mathsf{V} \oplus \mathsf{k}$ and let $\mathsf{W} \xrightarrow{\psi} \mathsf{k}$ denote linear projection onto the second summand.

- For each $s \in \mathsf{k}$, the group V acts simply transitively on the affine hyperplane $\psi^{-1}(s)$.
  - A identifies with $\psi^{-1}(1)$.
  - V identifies with $\mathsf{Ker}(\psi) = \psi^{-1}(0)$.
- Define a bijective correspondence between n-dimensional affine spaces A and pairs (W, ψ) where W is an $n+1$-dimensional vector space and $\psi \in \mathsf{W}^*$ is a nonzero covector, where A corresponds to $\psi^{-1}(1)$.
- Identify the affine group $\mathsf{Aff}(\mathsf{A})$ with the subgroup of $\mathsf{GL}(\mathsf{W})$ preserving this hyperplane, as well as the stabilizer of ψ.
- If $\left[\mathbf{A} \,\middle|\, \mathbf{b}\right]$ represents an affine transformation with linear part $\mathbf{A}$ and translational part $\mathbf{b}$, show that the corresponding linear transformation of W is represented by the block matrix

$$\begin{bmatrix} \mathbf{A} & \mathbf{b} \\ \mathbf{0} & 1 \end{bmatrix}$$

  where $\mathbf{0}$ is the row vector representing the zero map $\mathsf{V} \to \mathbb{R}$.

In coordinates, $\mathcal{A}'(\mathbf{v}) = \begin{bmatrix} v^1 \\ \vdots \\ v^n \\ 1 \end{bmatrix} \in \mathsf{W}$ and $\psi = \begin{bmatrix} 0 & \dots & 0 & 1 \end{bmatrix} \in \mathsf{W}^*$. The affine transformation has linear part $\mathbf{A} = \begin{bmatrix} A^1{}_1 & \dots & A^1{}_n \\ \vdots & & \vdots \\ A^n{}_1 & \dots & A^n{}_n \end{bmatrix} \in \mathsf{GL}(\mathsf{V})$ and translational part $\mathbf{b} = \begin{bmatrix} b^1 \\ \vdots \\ b^n \end{bmatrix} \in \mathsf{V}$:

$$\mathbf{v} \xmapsto{\left[\mathbf{A} \,\middle|\, \mathbf{b}\right]} \begin{bmatrix} A^1{}_1 & \dots & A^1{}_n \\ \vdots & A^i{}_j & \vdots \\ A^n{}_1 & \dots & A^n{}_n \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^j \\ \vdots \\ v^n \end{bmatrix} + \begin{bmatrix} b^1 \\ \vdots \\ b^i \\ \vdots \\ b^n \end{bmatrix} = \begin{bmatrix} A^1{}_j v^i + b^1 \\ \vdots \\ A^i{}_j v^j + b^i \\ \vdots \\ A^n{}_j v^j + b^n \end{bmatrix}$$

$$\mathcal{A}'(\mathbf{v}) \longmapsto \begin{bmatrix} A^1{}_1 & \dots & A^1{}_n & b^1 \\ \vdots & & \vdots & \\ A^n{}_1 & \dots & A^n{}_n & b^n \\ 0 & \dots & 0 & 1 \end{bmatrix} \begin{bmatrix} v^1 \\ \vdots \\ v^n \\ 1 \end{bmatrix} = \begin{bmatrix} A^1{}_i v^i + b^1 \\ \vdots \\ A^n{}_i v^i + b^n \\ 1 \end{bmatrix}.$$

Chapter 2

Projective geometry

Projective geometry arose historically out of the efforts of Renaissance artists to understand perspective. Imagine a painter looking at a 2-dimensional canvas with one eye closed. The painter's open eye plays the role of the origin in the 3-dimensional vector space W and the canvas plays the role of an affine hyperplane $\mathsf{A} \subset \mathsf{W}$ as in §1.8. As the canvas tilts, the geometry seen by the painter changes. Parallel lines no longer appear parallel (like the railroad tracks described below) and distance and angle are distorted. But lines stay lines and the basic relations of collinearity and concurrence are unchanged. The change in perspective given by "tilting" the canvas is determined by a linear transformation of W, since a point on A is determined completely by the 1-dimensional linear subspace of V containing it. (One must solve systems of linear equations to write down the effect of such transformation.) Projective geometry is the study of points, lines and the incidence relations between them.

Projective space *closes off* affine space — that is, it *compactifies* affine space by adding *ideal points* at infinity. To develop an intuitive feel for projective geometry, consider how points in A^n may go to infinity:

The easiest way to go to infinity in A^n is by following geodesics, since they have zero acceleration. Furthermore two geodesics approach the same ideal point if they are parallel. A good model is railroad tracks running parallel to each other — they meet ideally at the horizon. We thus force parallel lines to intersect by attaching ideal points where the extended parallel lines are to intersect.

The passage between the geometry of P and the algebra of V is a "dictionary" between linear algebra and projective geometry. Linear maps and linear subspaces correspond geometrically to projective maps and projective subspaces: inclusions, intersections, and linear spans correspond to incidence relations in projective geometry. Thus projective geometry lets us visually understand linear algebra and linear algebra enables to prove theorems in geometry by calculation.

A good general reference for projective geometry (especially in dimension two) is Coxeter [**103**], as well as Berger [**48, 49**] and Coxeter [**102**]. More classical treatments are Busemann–Kelly [**73**], Semple–Kneebone [**304**] and Veblen–Young [**333, 334**].

2.1. Ideal points

Parallelism of lines in A is an equivalence relation. Define an *ideal point* of A as an equivalence class. The *ideal set* of an affine space A is the space $\mathsf{P}_\infty(\mathsf{A})$ of ideal points, with the quotient topology. If $l, l' \subset \mathsf{A}$ are parallel lines, then the point in P_∞ corresponding to their equivalence class is defined as their intersection. So two lines are parallel if and only if they intersect at infinity.

Projective space is defined as the union $\mathsf{P} := \mathsf{A} \cup \mathsf{P}_\infty(\mathsf{A})$, with a suitable topology. The natural structure on P is perhaps most easily seen in terms of an alternate, maybe more familiar, description. Embed A as an affine hyperplane in a vector space $\mathsf{W} \cong \mathsf{V} \oplus \mathsf{k}$ as in §1.8, where $\mathsf{V} = \mathsf{Trans}(\mathsf{A})$ is the vector space underlying A. For example, if

$$p = \begin{bmatrix} p^1 \\ \vdots \\ p^n \end{bmatrix} \in \mathsf{A}^n,$$

then its embedding in $\mathsf{W} = \mathsf{k}^{n+1}$ is the nonzero vector

$$\mathcal{A}'(p) = \begin{bmatrix} p^1 \\ \vdots \\ p^n \\ 1 \end{bmatrix} \in \mathsf{W}.$$

Furthermore the line $\mathsf{k}\mathcal{A}'(p)$ spans meets the affine hyperplane $\mathsf{A} \leftrightarrow \mathsf{k}^n \times \{1\}$ in a single point. Think of A as the "canvas" or *viewing hyperplane,* and the line $\mathsf{k}\mathcal{A}'(p)$ as the line of sight as the point p is viewed from the origin $\mathbf{0} \in \mathsf{W}$ (the eye of the painter). The nonzero elements of this line $\mathsf{k}^\times\mathcal{A}'(p)$ form the *projective equivalence class* $[\mathcal{A}'(p)]$.

Now suppose that p travels to infinity along an affine geodesic $\ell \subset \mathsf{A}$:

$$p(t) := p + t\mathbf{v},$$

where $t \in \mathsf{k}$ and $\mathbf{v} \in \mathsf{V}$ is a nonzero vector. Denote the corresponding path of vectors in W by

$$\mathbf{w}(t) := \mathcal{A}'\big(p(t)\big) \in \mathsf{W}.$$

Although $\lim_{t\to\infty} \mathbf{w}(t)$ does not exist, the corresponding lines $\mathsf{k} \cdot \mathbf{w}(t)$ converge to the line $\mathsf{k} \cdot \mathbf{v}$ corresponding to $\mathbf{v} \in \mathsf{W}$. This limiting line defines the *ideal point* of the affine line ℓ:

$$\lim_{t\to\infty} [\mathbf{w}(t)] = [\mathbf{v}]$$

This motivates the following fundamental definition:

Definition 2.1.1. Let W denote a vector space over k. The *projective space associated to* W is the space $\mathsf{P}(\mathsf{W})$ of projective equivalence classes $[\mathbf{w}]$ of nonzero vectors $\mathbf{w} \in \mathsf{W}$, with the quotient topology.

Thus a point in $\mathsf{P}(\mathsf{W})$ (a "projective point") corresponds to a *line* (that is, a 1-dimensional linear subspace) in W. If $\mathbf{w} = \begin{bmatrix} w^1 \\ \vdots \\ w^{n+1} \end{bmatrix} \in \mathsf{W}$, the corresponding projective point is

$$p := [\mathbf{w}] = \left[\!\!\left[\begin{matrix} w^1 \\ \vdots \\ w^{n+1} \end{matrix} \right]\!\!\right] \in \mathsf{P}(\mathsf{W})$$

and $w^1, \dots, w^{n+1}$ are the *homogeneous coordinates* of p. Since linear transformations of W preserve lines, $\mathsf{GL}(\mathsf{W}) = \mathsf{Aut}(\mathsf{W})$ acts on $\mathsf{P}(\mathsf{W})$; the induced transformations are the *projective transformations* or *collineations* of $\mathsf{P}(\mathsf{W})$.

Exercise 2.1.2. The action of $\mathsf{GL}(\mathsf{W})$ on $\mathsf{P}(\mathsf{W})$ is *not* effective. Its kernel consists of the group $\mathsf{k}^\times$ of nonzero scalings, which forms the center of $\mathsf{GL}(\mathsf{W})$. The *projective group* or *collineation group* is the quotient

$$\mathsf{PGL}(\mathsf{W}) := \mathsf{GL}(\mathsf{W})/\mathsf{k}^\times$$

which does act effectively.

2.2. Projective subspaces

Returning to the projective geometry of the line ℓ, note that the affine line $\mathbf{w}(t)$ in W lies in the linear 2-plane $\mathsf{span}(p, \mathbf{v}) \subset \mathsf{W}$. The 1-dimensional linear subspaces contained in this linear 2-plane is a *projective line.*

Definition 2.2.1. Let $\mathsf{P} = \mathsf{P}(\mathsf{W})$ be a projective space, and let d be a nonnegative integer. A d-dimensional *projective subspace* S of P is the collection of all projective equivalence classes $[\mathbf{v}]$ of nonzero vectors $\mathbf{v}$ lying in a fixed

$d+1$-dimensional linear subspace $\mathsf{S} \subset \mathsf{W}$. We write $S = \mathsf{P}(\mathsf{S})$ and call S the *projectivization* of S.

Thus a *projective line* is the projectivization of a linear 2-plane in W and a *projective hyperplane* is the projectivization a linear hyperplane in W.

A linear embedding $S_1 \hookrightarrow S_2 \subset \mathsf{W}$ of linear subspaces induces an embedding of projective subspaces $\mathsf{P}(\mathsf{S}_1) \hookrightarrow \mathsf{P}(\mathsf{S}_2)$ and we say that the subspaces $\mathsf{P}(\mathsf{S}_1)$ and $\mathsf{P}(\mathsf{S}_2)$ are *incident.* Clearly projective transformations preserve the relation of incidence. Conversely, *an incidence-preserving transformation of projective space is a projective transformation.* This is a deep theorem, sometimes called the *fundamental theorem of projective geometry.*

Exercise 2.2.2. Show that the set of ideal points is a projective hyperplane.

Suppose that $S_1, S_2 \subset \mathsf{P}$ are projective subspaces. Write $S_i = \mathsf{P}(\mathsf{S}_i)$ for respective linear subspaces $\mathsf{S}_i \subset \mathsf{W}$. Their sum $\mathsf{S}_1 + \mathsf{S}_2$ is a vector space and denote its projectivization as the projective subspace

$$\mathsf{span}(S_1, S_2) \ := \ \mathsf{P}(\mathsf{S}_1 + \mathsf{S}_2).$$

Exercise 2.2.3. If $S_1 \cap S_2 = \emptyset$, then

$$\mathsf{dim}\big(\mathsf{span}(S_1, S_2)\big) = \mathsf{dim}(S_1) + \mathsf{dim}(S_2) + 1.$$

If S_1 and S_2 are points, then $\mathsf{span}(S_1, S_2)$ is a line, and we use the more familiar notation $\overleftrightarrow{S_1 S_2}$. If S_i are projective subspaces and $S_1 \cap S_2 \neq \emptyset$, althen $S_1 \cap S_2$ is a projective subspace and

$$\mathsf{dim}\big(\mathsf{span}(S_1, S_2)\big) + \mathsf{dim}(S_1 \cap S_2) = \mathsf{dim}(S_1) + \mathsf{dim}(S_2).$$

Evidently $\mathsf{span}(S_1, S_2)\big)$ is the smallest projective subspace containing S_1 and S_2.

2.2.1. Affine patches. Ideal points are only special when projective space is the completion of affine space; by changing the viewing hyperplane, one gets different notions of "ideal." Indeed, every projective point has neighborhoods which are affine subspaces.

Let P be d-dimensional projective space and $H \subset \mathsf{P}$ be a projective hyperplane Then the complement $\mathsf{P} \setminus H$ is an *affine patch* and has the structure as a d-dimensional affine space with underlying vector space V via an *affine chart*

$$\mathsf{V} \xrightarrow[\approx]{\mathcal{A}} \mathsf{P} \setminus H,$$

defined as follows. Write $\mathsf{P} = \mathsf{P}(\mathsf{W})$. Choose a covector $\psi \in \mathsf{W}^*$ such that $H = \mathsf{P}(\mathsf{V})$ where $\mathsf{V} := \mathsf{Ker}(\psi)$. Choose a vector $\mathbf{w}_0 \in \mathsf{W}$ with $\psi(\mathbf{w}_0) = 1$ to define an origin in the affine patch. Then

$$\begin{aligned} \mathsf{V} &\xrightarrow{\mathcal{A}^{(\psi,\mathbf{w}_0)}} \mathsf{P} \setminus H \\ \mathbf{v} &\longmapsto [\mathbf{w}_0 + \mathbf{v}] \end{aligned}$$

defines an affine chart on $\mathsf{P} \setminus H$. Compare §1.8.

Writing

$$p = [\mathbf{X}] = \begin{bmatrix} X^1 \\ \vdots \\ X^{d+1} \end{bmatrix} \in \mathsf{P},$$

some homogeneous coordinate X^i of the nonzero vector $\mathbf{X}$ is nonzero. Let $\psi \in \mathsf{V}^*$ denote the covector corresponding to the homogeneous coordinate X^i. Then corresponding affine patch is defined by $X^i \neq 0$ and has a chart

$$\mathsf{k}^d \xrightarrow{\mathcal{A}^i} \mathsf{P}$$

$$\begin{bmatrix} v^1 \\ \vdots \\ v^d \end{bmatrix} \longmapsto \begin{bmatrix} v^1 \\ \vdots \\ v^{i-1} \\ 1 \\ v^i \\ \vdots \\ v^d \end{bmatrix}$$

and $p \in \mathsf{A}^{(i)} := \mathcal{A}^{(i)}(\mathsf{V})$. These $d+1$ *coordinate affine patches* define a covering by contractible open sets.

Exercise 2.2.4. Suppose that $1 \leq i \neq j \leq d+1$.

(1) Express the intersection $\mathsf{A}^{(i)} \cap \mathsf{A}^{(j)}$ in terms of the charts $\mathcal{A}^{(i)}, \mathcal{A}^{(j)}$.

(2) Compute the change of coordinates

$$\left(\mathcal{A}^{(i)}\right)^{-1}\left(\mathsf{A}^{(i)} \cap \mathsf{A}^{(j)}\right) \xrightarrow{\left(\mathcal{A}^{(j)}\right)^{-1} \circ \mathcal{A}^{(i)}} \left(\mathcal{A}^{(j)}\right)^{-1}\left(\mathsf{A}^{(i)} \cap \mathsf{A}^{(j)}\right).$$

(3) Let $\mathsf{k} = \mathbb{R}$ and let $a, b \in \mathbb{R}$ with $a < b$. Suppose that

$$(a, b) \xrightarrow{\gamma} \mathsf{V}$$

is a curve such that $\mathcal{A}^{(i)}\big(\gamma(t)\big) \in \mathsf{A}^{(i)}$ and

$$\mathcal{A}^{(j)}\big(\gamma(t)\big) \in \mathsf{A}^{(j)}$$

for $a < t < b$. Suppose that $\mathcal{A}^{(i)} \circ \gamma$ is a geodesic in $\mathsf{A}^{(i)}$. Show there exists a *reparametrization,* that is, a diffeomorphism

$$(a,b) \xrightarrow[\approx]{\tau} \tau\big((a,b)\big) \subset \mathbb{R}$$

such that the composition $\mathcal{A}^{(j)} \circ \gamma \circ \tau$ is a geodesic in $\mathsf{A}^{(j)}$.

In general, the topology of projective space is complicated. Since it arises from a *quotient* and not a *subset* construction, it is more sophisticated than a subset. Indeed, projective space generally does not arise as a *hypersurface* in Euclidean space. Although P can be covered by $d+1$ contractible open sets, it cannot be covered by fewer contractible open sets. For either $\mathsf{k} = \mathbb{R}$ or $\mathbb{C}$, projective space $\mathsf{P}^d(\mathsf{k})$ is a compact smooth manifold. We summarize some basic facts about the topology.

Exercise 2.2.5. Suppose $\mathsf{k} = \mathbb{R}$. Exhibit $\mathsf{P}^d(\mathbb{R})$ as a quotient of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ by the antipodal map.[1] Show that $\mathsf{P}^1(\mathbb{R}) \approx \mathbb{S}^1$ and for $d > 1$ the fundamental group $\mathsf{P}^d(\mathbb{R})$ has order two. Show that $\mathsf{P}^d(\mathbb{R})$ is orientable if and only if d is odd.

Exercise 2.2.6. Suppose that $\mathsf{k} = \mathbb{C}$. Exhibit $\mathsf{P}^d(\mathbb{C})$ as a quotient of the unit sphere $\mathbb{S}^{2d-1} \subset \mathbb{C}^{d+1}$ by the group $\mathbb{T}$ of unit complex numbers. Show that $\mathsf{P}^1(\mathbb{C}) \approx \mathbb{S}^2$, and $\mathsf{P}^d(\mathbb{C})$ is simply connected and orientable for all $d \geq 1$.

Exercise 2.2.7. Find a *natural* $\mathbb{S}^1$-fibration $\mathsf{P}^{2d+1}(\mathbb{R}) \longrightarrow \mathsf{P}^d(\mathbb{C})$.

2.3. Projective mappings

Linear mappings $\mathsf{V} \xrightarrow{\phi} \mathsf{W}$ between vector spaces define mappings between the corresponding projective spaces. However, if ϕ is not injective, the corresponding projective map is not defined on all of $\mathsf{P}(\mathsf{V})$. We begin discussing with projective maps defined by injective linear maps, particularly emphasizing *automorphisms,* classically known as *collineations.* Collineations arise from linear automorphisms of the vector space W.

2.3.1. Embeddings and Collineations. A projective subspace $S \subset \mathsf{P}(\mathsf{W})$ determines a projective map, determined by the linear inclusion $\mathsf{S} \hookrightarrow \mathsf{W}$, where S is the linear subspace of W projectivizing to S.

Exercise 2.3.1. Show that an injective projective map $\mathsf{P}(\mathsf{S}) \longrightarrow \mathsf{P}(\mathsf{W})$ is determined by an injective linear map $\mathsf{S} \longrightarrow \mathsf{W}$ unique up to left-composition with scalar multiplication on S and right-composition with scalar multiplication on W.

[1] This sphere can be described intrinsically in a projectively-invariant way as the *sphere of directions.* Compare §6.2.1.

A linear automorphism of a vector space W induces an invertible transformation $\mathsf{P}(\mathsf{W}) \xrightarrow{\phi} \mathsf{P}(\mathsf{W})$. We call such a transformation a *collineation* or a *homography.* Evidently a collineation preserves projective subspaces, and the relations between them. An *involution* is a collineation of order two, that is, $\phi = \phi^{-1}$.

Exercise 2.3.2. Show that the projective automorphisms of P form a group $\mathsf{Aut}(\mathsf{P})$ which has the following description. If

$$\mathsf{P} \xrightarrow{f} \mathsf{P}$$

is a projective automorphism, some linear isomorphism

$$\mathsf{V} \xrightarrow{\tilde{f}} \mathsf{V}$$

induces f. Indeed,

$$1 \longrightarrow \mathbb{R}^\times \longrightarrow \mathsf{GL}(\mathsf{V}) \longrightarrow \mathsf{Aut}(\mathsf{P}) \longrightarrow 1$$

is a short exact sequence, where $\mathbb{R}^\times \longrightarrow \mathsf{GL}(\mathsf{V})$ is the inclusion of the group of multiplications by nonzero scalars. This quotient, *the projective general linear group*

$$\mathsf{PGL}(\mathsf{V}) \ := \ \mathsf{PGL}(\mathsf{V}) \ := \mathsf{GL}(\mathsf{V})/\mathbb{R}^\times \cong \mathsf{Aut}(\mathsf{P}^n),$$

is also denoted $\mathsf{PGL}(n+1,\mathbb{R})$ If n is even, then

$$\mathsf{PGL}(n+1,\mathbb{R}) \cong \mathsf{SL}(n+1;\mathbb{R}).$$

If n is odd, then $\mathsf{PGL}(n+1,\mathbb{R})$ has two connected components, and its identity component is doubly covered by $\mathsf{SL}(n+1;\mathbb{R})$ and is isomorphic to $\mathsf{SL}(n+1;\mathbb{R})/\{\pm\mathbb{I}\}$.

Exercise 2.3.3. Let $\mathbb{R}\mathsf{P}^n$ be a real projective space of dimension n, and let ϕ be an involution of $\mathbb{R}\mathsf{P}^n$.

- Suppose n is even. Then $\mathsf{Fix}(\phi)$ is the union of two disjoint projective subspaces of dimensions d_1, d_2 where $d_1 + d_2 = n - 1$.
- Suppose $n = 2m + 1$ is odd. Then either:
  - $\mathsf{Fix}(\phi) \neq \emptyset$ and equals the union of two disjoint projective subspaces of dimensions d_1, d_2 where $d_1 + d_2 = n - 1$, or
  - $\mathsf{Fix}(\phi) = \emptyset$, and ϕ leaves invariant an $\mathbb{S}^1$-fibration

$$\mathbb{R}\mathsf{P}^n \longrightarrow \mathbb{C}\mathsf{P}^m.$$

For example, an involution ϕ of $\mathbb{R}\mathsf{P}^2$ has an isolated fixed point p and a disjoint fixed line ℓ. In an affine patch containing p, the involution ϕ looks like a symmetry in p, preserving the local orientation. In contrast, ϕ looks like a reflection in ℓ in an affine patch containing ℓ, reversing the local orientation. Since $\mathbb{R}\mathsf{P}^2$ is nonorientable no *global* orientation exists to either preserve or reverse.

Here is an explicit example. The involution defined in homogeneous coordinates by:

$$\mathbb{R}\mathsf{P}^2 \xrightarrow{\iota} \mathbb{R}\mathsf{P}^2$$
$$\begin{bmatrix}\!\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}\!\end{bmatrix} \longmapsto \begin{bmatrix}\!\begin{bmatrix} -X \\ -Y \\ Z \end{bmatrix}\!\end{bmatrix}$$

fixes the point

$$p = \begin{bmatrix}\!\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\!\end{bmatrix}$$

and the projective line defined by $Z = 0$, that is, $\ell = \begin{bmatrix}\!\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}\!\end{bmatrix}$. In the affine chart $\mathcal{A}^3$, the isolated fixed point has coordinates $(0,0)$ and ι appears as the *symmetry*

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \longmapsto \begin{bmatrix} -v^1 \\ -v^2 \end{bmatrix}.$$

and in the affine chart $\mathcal{A}^1$, the fixed line has coordinates $(*,0)$ and ι appears as the reflection

$$\begin{bmatrix} v^1 \\ v^2 \end{bmatrix} \longmapsto \begin{bmatrix} -v^1 \\ v^2 \end{bmatrix}$$

fixing the vertical axis. That a single reflection can appear simultaneously as a symmetry in a point and reflection in a line indicates the topological complexity of P^2: A reflection in a line *reverses* a local orientation about a point on the line, and a symmetry in a point *preserves* a local orientation about the point.

2.3.2. Singular projective mappings. When the linear map ϕ is not injective, then $\mathsf{P}\big(\mathsf{Ker}(\phi)\big)$ is a projective subspace, upon which the projectivization $[\phi]$ of ϕ is not defined. For that reason, we call it the *undefined set* of $[\phi]$, denoting it $\mathbb{U}([\phi])$. If V, V' are vector spaces with associated projective spaces P, P' then a linear map $\mathsf{V} \xrightarrow{\tilde{f}} \mathsf{V}'$ maps lines through 0 to lines through 0. But $\widetilde{f}$ only induces a map $\mathsf{P} \xrightarrow{f} \mathsf{P}'$ if it is injective, since $f(x)$

can only be defined if $\widetilde{f}(\ \tilde{x}\ ) \neq 0$ (where $\tilde{x}$ is a point of $\Pi^{-1}(x) \subset \mathsf{V} \setminus \{0\}$). Suppose that $\tilde{f}$ is a (not necessarily injective) linear map and let

$$\mathbb{U}(f) = \Pi\big(\mathsf{Ker}(\tilde{f})\big).$$

The resulting *projective endomorphism* of P is defined on the complement $\mathsf{P} \setminus \mathbb{U}(f)$. If $\mathbb{U}(f) \neq \emptyset$, the corresponding projective endomorphism is by definition a *singular projective transformation* of P. If f is singular, its image is a proper projective subspace, called the *range* of f and denoted $\mathcal{R}(f)$.

A projective map $\mathsf{P}_1 \xrightarrow{\iota} \mathsf{P}$ corresponds to a linear map $\mathsf{V}_1 \xrightarrow{\tilde{\iota}} \mathsf{V}$ between the corresponding vector spaces (well-defined up to scalar multiplication). Since ι is defined on all of P_1, $\tilde{\iota}$ is an injective linear map and hence corresponds to an embedding. Such an embedding (or its image) will be called a *projective subspace.* Projective subspaces of dimension k correspond to linear subspaces of dimension $k+1$. (By convention the empty set is a projective space of dimension -1.) Note that the "bad set" $\mathbb{U}(f)$ of a singular projective transformation is a projective subspace. Two projective subspaces of dimensions k, l where $k + l \geq n$ intersect in a projective subspace of dimension at least $k+l-n$. The *rank* of a projective endomorphism is defined to be the dimension of its image.

Exercise 2.3.4. Let P be a projective space of dimension n. Show that the (possibly singular) projective transformations of P form themselves a projective space of dimension $(n+1)^2 - 1$. We denote this projective space by $\mathsf{End}(\mathsf{P})$. Show that if $f \in \mathsf{End}(\mathsf{P})$, then

$$\mathsf{dim}\ \mathbb{U}(f) + \mathsf{rank}(f) = n - 1.$$

Show that $f \in \mathsf{End}(\mathsf{P})$ is nonsingular (in other words, a collineation) if and only if $\mathsf{rank}(f) = n$, that is, $\mathbb{U}(f) = \emptyset$. Equivalently, $\mathcal{R}(f) = \mathsf{P}$.

An important kind of projective endomorphism is a *projection,* also called a *perspectivity.* Let $A^k, B^l \subset \mathsf{P}^n$ be disjoint projective subspaces whose dimensions satisfy $k + l = n - 1$. Define the projection $\Pi = \Pi_{A^k,B^l}$ onto A^k from B^l

$$\mathsf{P}^n - B^l \xrightarrow{\Pi} A^k$$

as follows. For every $x \in \mathsf{P}^n - B^l$ the minimal projective subspace

$$\overleftrightarrow{xB} := \mathsf{span}(\{x\} \cup B^l)$$

containing $\{x\} \cup B^l$ is unique and has dimension $l + 1$. It intersects A^k transversely in a 0-dimensional projective subspace, that is, a unique point $\Pi_{A^k,B^l}(x)$, that is:

$$\{\Pi_{A^k,B^l}(x)\} \;=\; \overleftrightarrow{xB} \cap A^k.$$

Perspectivities are projective mappings obtained as restrictions of projections:

Exercise 2.3.5. Let $A' \subset \mathsf{P}$ be a projective subspace of dimension k disjoint from B.

- Restriction $\Pi_{A,B}|_{A'}$ is a projective isomorphism $A' \to A$.
- Express an arbitrary projective isomorphism between projective subspaces as a composition of perspectivities.

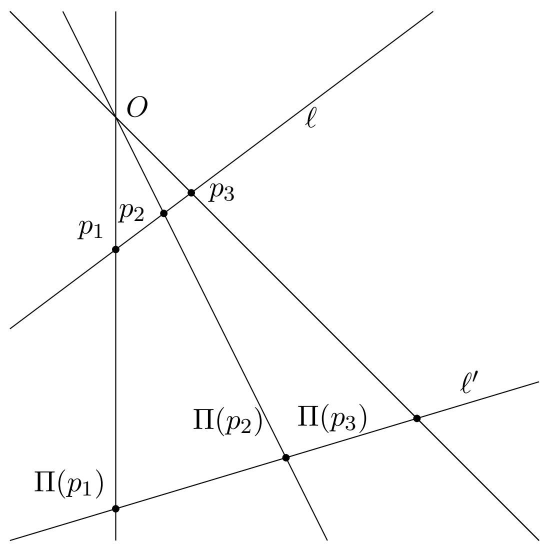

Figure 2.1. A perspectivity $\ell \xrightarrow{\Pi} \ell'$ is a projective mapping between two lines ℓ, ℓ' in the plane. Π is the restriction to ℓ of the projection $\mathsf{P}^2 \setminus \{O\} \twoheadrightarrow \ell'$. The undefined set $\mathbb{U}(\Pi)$ is traditionally called the *center* of the perspectivity.

2.3.3. Locally projective maps. Suppose that P, P' are projective spaces and $U \subset \mathsf{P}$ is an open subset. A map $U \xrightarrow{f} \mathsf{P}'$ is *locally projective* if for each component $U_i \subset U$ there exists a linear map

$$\mathsf{V}(\mathsf{P}) \xrightarrow{\tilde{f}_i} \mathsf{V}(\mathsf{P}')$$

such that the restrictions of $f \circ \Pi$ and $\Pi \circ \tilde{f}_i$ to $\Pi^{-1}U_i$ agree. Locally projective maps (and hence also locally affine maps) satisfy the *Unique Extension Property:* if $U \subset U' \subset \mathsf{P}$ are open subsets of a projective space with U nonempty and U' connected, then any two locally projective maps $f_1, f_2 : U' \longrightarrow \mathsf{P}'$ which agree on U must be identical. (Compare §5.1.1.)

Exercise 2.3.6. Let $U \subset \mathsf{P}$ be a connected open subset of a projective space of dimension greater than 1. Let $U \xrightarrow{f} \mathsf{P}$ be a local diffeomorphism. Then f is locally projective if and only if for each line $\ell \subset \mathsf{P}$, the image $f(\ell \cap U)$ is a line.

2.4. Affine patches

Let $H \subset \mathsf{P}$ be a projective hyperplane (projective subspace of codimension one). Then the complement $\mathsf{P} \setminus H$ has a natural affine geometry, that is, it is an affine space in a natural way. Indeed the group of projective automorphisms $\mathsf{P} \to \mathsf{P}$ leaving fixed each $x \in H$ and whose differential $\mathsf{T}_x\mathsf{P} \to \mathsf{T}_x\mathsf{P}$ is a volume-preserving linear automorphism is a vector group acting simply transitively on $\mathsf{A} = \mathsf{P} \setminus H$. Moreover the subgroup of $\mathsf{Aut}(\mathsf{P})$ leaving H invariant is $\mathsf{Aff}(\mathsf{A})$. In this way affine geometry *embeds* in projective geometry.

Here is how it looks in terms of matrices. Let $\mathsf{A} = \mathbb{R}^n$; then the affine subspace of

$$\mathsf{V} = \mathsf{Trans}(\mathsf{A}) \oplus \mathbb{R} = \mathbb{R}^{n+1}$$

corresponding to A is $\mathbb{R}^n \times \{1\} \subset \mathbb{R}^{n+1}$, the point of A with *affine* or *inhomogeneous* coordinates $(x^1, \ldots, x^n)$ has homogeneous coordinates $[x^1, \ldots, x^n, 1]$. Let $f \in \mathsf{Aff}(E)$ be the affine transformation with linear part $A \in \mathsf{GL}(n; \mathbb{R})$ and translational part $\mathbf{b} \in \mathbb{R}^n$, that is, $f(x) = Ax + \mathbf{b}$, is then represented by the $(n+1)$-square matrix

$$\begin{bmatrix} A & \mathbf{b} \\ 0 & 1 \end{bmatrix}$$

where $\mathbf{b}$ is a column vector and 0 denotes the $1 \times n$ zero row vector.

2.4.1. Projective vector fields.

In the affine space A^n, let

$$\mathfrak{B}(\mathsf{A}^n) \longrightarrow \mathsf{A}^n$$

denote the *bundle of bases,* more commonly known as the *affine frame bundle* over A^n: its fiber $\mathfrak{B}_p$ over a point $p \in \mathsf{A}^n$ consists of the set of bases for the tangent space $\mathsf{T}_p\mathsf{A}^n$. Using the simply transitive action of $\mathsf{k}^n = \mathsf{Trans}(\mathsf{A}^n)$ on A^n, the total space $\mathfrak{B}(\mathsf{A}^n)$ is a torsor for the affine automorphism group $\mathsf{Aff}(\mathsf{A}^n)$: an affine automorphism is determined uniquely by its action on a basepoint $p_0 \in \mathsf{A}^n$ and a basis $\beta_0 \in \mathfrak{B}_{p_0}$ of $\mathsf{T}_p\mathsf{A}^n$. Every $(p, \beta) \in \mathfrak{B}(\mathsf{A}^n)$ is the image of (p_0, β_0) under a unique element of $\mathsf{Aff}(\mathsf{A}^n)$.

Let $g \in \mathsf{Aut}(\mathsf{P}^n)$ be a projective automorphism. Fix a basepoint p_0 and a basis β_0 of $T_{p_0}\mathsf{A}^n$ such that $g(p_0)$ is not ideal, that is, $g(p_0) \in \mathsf{A}^n$. Let

$h \in \mathsf{Aff}(\mathsf{A}^n)$ be the unique *affine* automorphism taking (p_0, β_0) to

$$(p, \beta) \;=\; g(p_0, \beta_0).$$

Then $h^{-1} \circ g$ is a projective automorphism fixing p_0 acting identically on

$$\mathsf{T}_{p_0}\mathsf{P}^n = \mathsf{T}_{p_0}\mathsf{A}^n.$$

In the affine chart $\mathcal{A}^{n+1}$ where p_0 is the origin, such a projective transformation is defined in homogeneous coordinates by:

$$\left[\!\!\left[\begin{matrix} X^1 \\ \vdots \\ X^n \\ X^{n+1} \end{matrix}\right]\!\!\right] \longmapsto \left[\!\!\left[\begin{matrix} X^1 \\ \vdots \\ X^n \\ \sum_{i=1}^n \xi_i X^i + X^{n+1} \end{matrix}\right]\!\!\right]$$

for a row vector $\xi^\dagger$ for $\xi \in \mathsf{k}^n$, that is, by the block matrix

$$\begin{bmatrix} \mathbb{I}_n & \mathbf{0} \\ \xi & 1 \end{bmatrix}.$$

In affine coordinates such a transformation is given by

$$(x^1, \dots, x^n) \xrightarrow{g_\xi} \left(\frac{x^1}{1 + \sum_{i=1}^n \xi_i x^i}, \dots, \frac{x^n}{1 + \sum_{i=1}^n \xi_i x^i} \right).$$

Exercise 2.4.1. Show that this group is isomorphic to an n-dimensional vector group (that is, the additive group of an n-dimensional vector space). Show that g_ξ lies in the one-parameter group

$$t \longmapsto g_{t\xi}$$

and the corresponding vector field equals the product

$$-\Big(\sum_{i=1}^n \xi_i x^i \Big) \,\mathsf{Rad}$$

where Rad is the radiant vector field defined in §1.6.2.

Denote the Lie algebra of such vector fields by $\mathfrak{g}_1$, the Lie algebra of parallel vector fields on A^n by $\mathfrak{g}_{-1}$, and the Lie algebra of linear vector fields on A^n by $\mathfrak{g}_0$, show that, for $\lambda = 0, \pm 1$, the subalgebra $\mathfrak{g}_\lambda$ of the Lie algebra $\mathfrak{g}$ of projective vector fields equals the λ-eigenspace of $\mathsf{ad}(\mathsf{Rad})$. Furthermore $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$ where $\mathfrak{g}_\nu := 0$ if $\nu \neq 0, \pm 1$. In particular as a vector space $\mathfrak{g}$ decomposes as a direct sum

$$\mathfrak{g} \;=\; \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \;\cong\; \mathsf{k}^n \oplus \mathsf{gl}(n) \oplus \mathsf{k}^n.$$

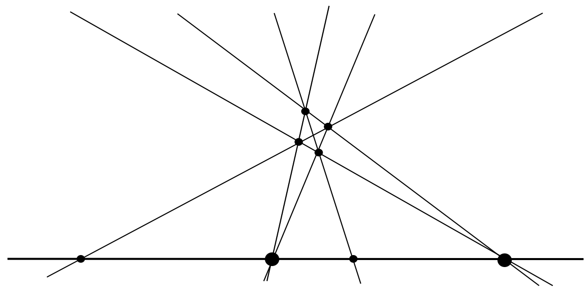

Figure 2.2. The four points on the horizontal line form a harmonic quadruple. Such a quadruple is characterized by having cross-ratio -1.

Exercise 2.4.2. Describe the corresponding group-theoretic decomposition of the projective automorphism group

$$\mathsf{Aut}(\mathsf{P}^n) := \mathsf{PGL}(n+1).$$

2.5. Classical projective geometry

This section surveys standard results in projective geometry. The *fundamental theorem of projective geometry* characterizes projective mappings. The *cross ratio* of a set of four points is the fundamental invariant in 1-dimensional projective geometry. The classical notion of a *harmonic set* is introduced, and is applied to the study of projective reflections and their products.

2.5.1. 1-dimensional reflections. Let ℓ be a projective line containing distinct points x, z. Then there exists a unique reflection[2]

$$\ell \xrightarrow{\rho_{x,z}} \ell$$

whose fixed-point set equals $\{x, z\}$. We say that a pair of points y, w are *harmonic* with respect to x, z if $\rho_{x,z}$ interchanges them. In that case one can show that x, z are harmonic with respect to y, w. Furthermore this relation is equivalent to the existence of lines p, q through x and lines r, s through z such that

$$y = \overleftrightarrow{(p \cap r)(q \cap s)} \cap \ell \tag{2.1}$$

$$z = \overleftrightarrow{(p \cap s)(q \cap r)} \cap \ell \tag{2.2}$$

Compare Figure 2.2.

This leads to a projective-geometry construction of reflection, as follows. Let $x, y, z \in \ell$ be fixed; we seek the harmonic conjugate of y with respect

[2] a *harmonic homology* in classical terminology

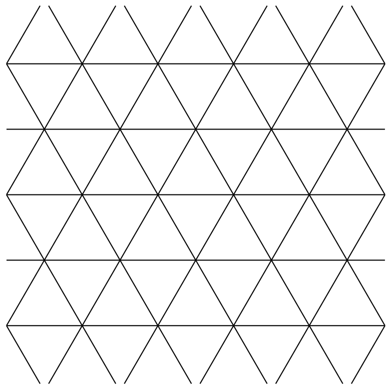

Figure 2.3. Euclidean $(3,3,3)$-triangle tessellation

to x, z, that is, the image $\rho_{x,z}(y)$. Erect arbitrary lines (in general position) p, q through x and a line r through z. Through y draw the line

$$\ell' := \overleftrightarrow{y \quad (r \cap q)}$$

through $r \cap q$; join its intersection with p with z to form a line:

$$s = \overleftrightarrow{z \quad (p \cap \ell')}.$$

Then $\rho_{x,z}(y)$ will be the intersection of ℓ with

$$\overleftrightarrow{(p \cap r)\ (s \cap q)}$$

as in Figure 2.2.

2.5.2. $(3,3,3)$**-triangle tessellations.** Figure 2.5 depicts a projective tessellation of the inside of a triangle by smaller triangles. This tessellation is combinatorially equivalent to the usual Euclidean $(3,3,3)$-tessellation of the plane by equilateral triangles. Around each vertex are six triangles, and the tessellation is invariant under the group generated by Euclidean reflections in the three sides of (any) triangle. These tessellations depend on one real parameter, and can be constructed by an easy projective-geometry construction using only a straightedge.

Exercise 2.5.1. Here is the construction of the tessellation. Begin with a triangle $\triangle$ with vertices v_0, v_1, v_2. Choose a line segment s with one endpoint a vertex (say v_2) and the other endpoint on the opposite edge e_2. Choose two distinct points a, b on s. Connect these points to the other vertices v_0 and v_1.

Draw lines from p, q to the vertices v_0, v_1. These lines (together with s) cut off two triangles inside $\triangle$ with "new" vertices

$$\overleftrightarrow{v_0\ p} \cap \overleftrightarrow{v_1\ q}, \qquad \overleftrightarrow{v_0\ q} \cap \overleftrightarrow{v_1\ p}$$

which should now be joined to v_2, creating new triangles inside $\triangle$. Iterating this procedure creates the tilings depicted in Figure 2.5.

2.5.3. Rationality.

Exercise 2.5.2. Consider the projective line $\mathsf{P}^1 = \mathbb{R} \cup \{\infty\}$. Show that for every rational number $x \in \mathbb{Q}$ there exists a sequence

$$x_0, x_1, x_2, x_3, \ldots, x_n \in \mathsf{P}^1$$

such that:

- $x = x_n$;
- $\{x_0, x_1, x_2\} = \{0, 1, \infty\}$;
- For each $i \geq 3$, there is a harmonic quadruple (x_i, y_i, z_i, w_i) with

$$y_i, z_i, w_i \in \{x_0, x_1, \ldots, x_{i-1}\}.$$

If x is written in reduced form p/q then what is the smallest n for which x can be reached in this way?

Exercise 2.5.3 (Synthetic arithmetic)**.** Using the above synthetic geometry construction of harmonic quadruples, show how to add, subtract, multiply, and divide real numbers by a straightedge-and-pencil construction. In other words, draw a line l on a piece of paper and choose three points to have coordinates $0, 1, \infty$ on it. Choose arbitrary points corresponding to real numbers x, y. Using only a straightedge (not a ruler!) construct the points corresponding to

$$x + y, x - y, xy, \text{ and } x/y \text{ if } y \neq 0.$$

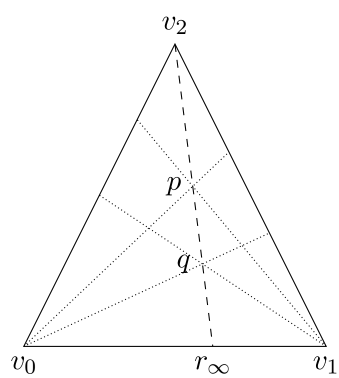

Figure 2.4. Initial configuration for non-Euclidean (3, 3, 3)-tessellation. By iterating the process of joining vertices of the original triangle to interior intersections of lines, one creates tilings like the ones depicted in Figure 2.5.

2.5.4. Fundamental theorem of projective geometry. One version of what is sometimes called the *fundamental theorem of projective geometry* is that the projective transformations (defined by linear transformations of the associated vector space) are precisely the transformations of projective space which preserve the ternary relation of collinearity (hence *collineations*). Collinearity is a special instance of the set of *incidence relations* between projective subspaces. For example, two distinct projective points p, q are incident to a unique projective line $\overleftrightarrow{p\,q}$ and (p, q, r) is a collinear triple if and only if $r \in \overleftrightarrow{p\,q}$. We do not develop this theory in detail, but refer to the texts of Berger [**49, 50**] and Coxeter [**102**] for discussion, and in particular the relation with automorphisms of the ground field k.

If $\ell \subset \mathsf{P}$ and $\ell' \subset \mathsf{P}'$ are projective lines, part of the fundamental theorem that for given triples $x, y, z \in \ell$ and $x', y', z' \in \ell'$ of distinct points there exists a unique projective map

$$\begin{aligned} \ell &\xrightarrow{f} \ell' \\ x &\longmapsto x' \\ y &\longmapsto y' \\ z &\longmapsto z' \end{aligned}$$

If $w \in \ell$ then the *cross-ratio* $[w, x, y, z]$ is defined to be the image of w under the unique collineation $l \xrightarrow{f} \mathsf{P}^1$ with

$$\begin{aligned} x &\overset{f}{\longmapsto} 1 \\ y &\overset{f}{\longmapsto} 0 \\ z &\overset{f}{\longmapsto} \infty \end{aligned}$$

If $\ell = \mathsf{P}^1$, then this linear fractional transformation is:

$$f(w) := \frac{w-y}{w-z} \bigg/ \frac{x-y}{x-z}$$

so

$$[w, x, y, z] := \frac{w-y}{w-z} \bigg/ \frac{x-y}{x-z}, \tag{2.3}$$

thus defining[3] the *cross-ratio.* The cross-ratio extends to quadruples of points on a projective line, of which at least three are distinct.

[3]The literature has several variations of the cross-ratio; the version here is used by Veblen–Young [**333**], Kneebone–Semple[**304**], Coxeter [**104**], and Ahlfors [**6**]. Other versions can be found in Goldman [**155**], Hubbard [**197**], and Ovsienko–Tabachnikov [**284**].

Exercise 2.5.4. Let σ be a permutation on four symbols. Show that there exists a linear fractional transformation Φ_σ such that

$$[x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}, x_{\sigma(4)}] = \Phi_\sigma([x_1, x_2, x_3, x_4]).$$

Determine which permutations leave the cross-ratio invariant.

The cross-ratio is a function of four variables, and therefore transforms under the action of the symmetric group $\mathfrak{S}_4$ on four symbols. The group $\mathfrak{S}_4$ is a split extension

$$\mathbb{Z}/2 \oplus \mathbb{Z}/2 \lhd \mathfrak{S}_4 \twoheadrightarrow \mathfrak{S}_3$$

where the normal subgroup $\mathbb{Z}/2 \oplus \mathbb{Z}/2$ consists of products of disjoint transpositions and a section $\mathfrak{S}_3 \hookrightarrow \mathfrak{S}_4$ corresponds to the inclusion $\{2,3,4\} \hookrightarrow \{1,2,3,4\}$.

Exercise 2.5.5. Show that the cross-ratio is invariant under the normal subgroup $\mathbb{Z}/2 \oplus \mathbb{Z}/2$, and transforms under $\mathfrak{S}_3 = \mathsf{Aut}\{2,3,4\}$ by the rules, where $z = [z_1, z_2, z_3, z_4]$:

$$\begin{aligned}
[z_1, z_2, z_4, z_3] &= 1/z \\
[z_1, z_3, z_2, z_4] &= 1 - z \\
[z_1, z_3, z_4, z_2] &= 1 - 1/z \\
[z_1, z_4, z_2, z_3] &= 1/(1 - z) \\
[z_1, z_4, z_3, z_2] &= z/(z - 1)
\end{aligned}$$

Using the isomorphism $\mathsf{Aut}(\mathbb{R}\mathsf{P}^1) \cong \mathsf{PGL}(2, \mathbb{R})$, this corresponds to an embedding $\mathfrak{S}_3 \hookrightarrow \mathsf{PGL}(2, \mathbb{Z})$ taking

$$(34) \longmapsto \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(23) \longmapsto \pm \begin{bmatrix} -1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(234) \longmapsto \pm \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}$$

$$(243) \longmapsto \pm \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$$

$$(24) \longmapsto \pm \begin{bmatrix} -1 & 0 \\ -1 & +1 \end{bmatrix}$$

which induces an isomorphism $\mathfrak{S}_3 \cong \mathsf{PGL}(2, \mathbb{Z}/2)$.

A pair $\{w, x\}$ is harmonic with respect to the pair $\{y, z\}$ (in which case we say that (x, y, w, z) is a *harmonic quadruple*) if and only if the cross-ratio $[x, y, w, z] = -1$.

Exercise 2.5.6. When $z = \infty$, the expression for the cross-ratio simplifies:

$$[x, y, w, \infty] \;:=\; \frac{x - w}{y - w}$$

and defines a fundamental *affine* invariant.

- (x, y, w, ∞) is a harmonic quadruple if and only if y is the midpoint of $\overline{xw}$.
- Suppose $(x, y, w), (x', y', w')$ are ordered triples of distinct points of A^1. Show that

  $$[x, y, w, \infty] = [x', y', w', \infty]$$

  if and only if $\exists g \in \mathsf{Aff}(\mathsf{A}^1)$ such that

  $$\begin{aligned} x' &= g(x) \\ y' &= g(y) \\ w' &= g(w). \end{aligned}$$

The process of extending a triple of points on P^1 to a harmonic quadruple is equivalent to applying a projective reflection in one pair to the remaining element. This process is called *harmonic subdivision.* Iterated harmonic subdivisions produce a countable dense subsets of P^1 corresponding to the rational numbers $\mathbb{Q} \subset \mathbb{R}$. Such a subset is called a *harmonic net* (Coxeter [**103**], §3.5) or a *net of rationality* in Veblen–Young [**333**],p.84). Compare also Busemann–Kelly [**73**], §I.6. These ideas provide an approach to the fundamental theorem:

Exercise 2.5.7. Let $\mathsf{P}^1 \xrightarrow{f} \mathsf{P}^1$ be a homeomorphism. Show that the following conditions are equivalent:

- f is projective;
- f preserves harmonic quadruples
- f preserves cross-ratios, that is, for all quadruples (x, y, w, z), the cross-ratios satisfy

  $$[f(x), f(y), f(w), f(z)] = [x, y, w, z].$$

Determine the weakest hypothesis on f to obtain these conditions.

2.5.5. Distance via cross-ratios. For later use in §12.1, here are some explicit formulas for cross-ratios on 1-dimensional spaces $\mathbb{R}^+$ and the interval $\mathbf{I} = [-1, 1]$. Their infinitesimal forms yield the Poincaré metrics on $\mathbb{R}^+$ and $\mathsf{int}(\mathbf{I})$.

Exercise 2.5.8. (Parameters on the positive ray and unit intervals)

$$[0, e^a, e^b, \infty] \;=\; [-1, \tanh(a/2), \tanh(b/2), 1] \;=\; e^{b-a}$$

2.5.6. Products of reflections. If ϕ, ψ are collineations, each of which fix a point $O \in \mathsf{P}^2$, their composition $\phi\psi = \phi \circ \psi$ fixes O. In particular its derivative

$$\mathsf{D}(\phi\psi)_O = (\mathsf{D}\phi)_O \circ (\mathsf{D}\psi)_O$$

acts linearly on the tangent space $\mathsf{T}_O\mathsf{P}$. We consider the case where ϕ, ψ are reflections in P^2. As in Exercise 2.3.3, a reflection ϕ is completely determined by its set $\mathsf{Fix}(\phi)$ which consists of a point p_ϕ and a line ℓ_ϕ such that $p_\phi \notin \ell_\phi$. Define

$$O := \ell_\phi \cap \ell_\psi$$

and P^*_O the projective line whose points are the lines incident to O.

Exercise 2.5.9. Let ρ denote the cross-ratio of the four lines

$$\ell_\phi,\; \overleftrightarrow{Op_\phi},\; \overleftrightarrow{Op_\psi},\; \ell_\psi$$

as elements of P^*_O.

- The linear automorphism

$$\mathsf{T}_O\mathsf{P}^2 \xrightarrow{\mathsf{D}(\phi\psi)_O} \mathsf{T}_O\mathsf{P}^2$$

  of the tangent space $\mathsf{T}_O\mathsf{P}^2$ leaves invariant a positive definite inner product g_O on $\mathsf{T}_O\mathsf{P}^2$.
- Furthermore $\mathsf{D}(\phi\psi)_O$ represents a rotation of angle θ in the tangent space $\mathsf{T}_O(\mathsf{P})$ with respect to g_O if and only if

$$\rho = \frac{1}{2}(1 + \cos\theta)$$

  for $0 < \theta < \pi$.
- $\mathsf{D}(\phi\psi)_O$ represents a rotation of angle π (that is, an involution) if and only if $p_\phi \in \ell_\psi$ and $p_\psi \in \ell_\phi$.[4]

[4] In other words, the four lines ℓ_ϕ, $\overleftrightarrow{Op_\phi}$, $\overleftrightarrow{Op_\psi}$, ℓ_ψ form a harmonic quadruple.

2.6. Asymptotics of projective transformations

We shall be interested in the singular projective transformations since they occur as limits of nonsingular projective transformations. The collineation group $\mathsf{Aut}(\mathsf{P})$ of $\mathsf{P} = \mathsf{P}^n$ is a large noncompact group which is naturally embedded in the projective space $\mathsf{End}(\mathsf{P})$ as an open dense subset as in Exercise 2.3.4. Thus understanding precisely what it means for a sequence of collineations to converge to a (possibly singular) projective transformation is crucial.

This leads to an algebraic description of the classical notion of limit sets of discrete groups isometries of hyperbolic space (see §2.6.2). These ideas were used by Kuiper [**232**, **234**, **235**] and later Benzéctri [**46**] who credits them to Myrberg [**278**].

In general collineations do not preserve a metric on P, although P admits many metrics; for example the Fubini–Study metric discussed in §3.2.2. Being compact, two such metrics determine the same *uniform structure* on P, enabling the definition of uniform convergence of mappings defined on subsets of P.

A *singular projective transformation* of P is a projective map f defined on the complement of a projective subspace $\mathbb{U}(f) \subset \mathsf{P}$, called the *undefined subspace* of f and taking values in a projective subspace $\mathcal{R}(f) \subset \mathsf{P}$, called the *range* of f. Furthermore

$$\dim \mathsf{P} \;=\; \dim \mathbb{U}(f) \;+\; \dim \mathcal{R}(f) \;+\; 1.$$

Proposition 2.6.1. Let $g_m \in \mathsf{Aut}(\mathsf{P})$ be a sequence of collineations of P and let $g_\infty \in \mathsf{End}(\mathsf{P})$. Then the sequence g_m converges to g_∞ in $\mathsf{End}(\mathsf{P})$ if and only if the restrictions $g_m|_K$ converge uniformly to $g_\infty|_K$ for all compact sets $K \subset \mathsf{P} - \mathbb{U}(g_\infty)$.

Convergence in $\mathsf{End}(\mathsf{P})$ may be described as follows: Let $\mathsf{P} = \mathsf{P}(\mathsf{V})$ where $\mathsf{V} \cong \mathbb{R}^{n+1}$ is a vector space. Then $\mathsf{End}(\mathsf{P})$ is the projective space associated to the vector space $\mathsf{End}(\mathsf{V})$ of $(n+1)$-square matrices. If $a = (a^i_j) \in \mathsf{End}(\mathsf{V})$ is such a matrix, let

$$\|a\| \;=\; \sqrt{\mathsf{Tr}(aa^\dagger)} \;=\; \sqrt{\sum_{i,j=1}^{n+1} |a^i_j|^2}$$

denote its Euclidean norm; projective endomorphisms then correspond to matrices a with $\|a\| = 1$, uniquely determined up to the antipodal map $a \mapsto -a$. The following lemma will be useful in the proof of Proposition 2.6.1.

Lemma 2.6.2. Let V, V' be vector spaces and let $\mathsf{V} \xrightarrow{\widetilde{f}_n} \mathsf{V}'$ be a sequence of linear maps converging to $\mathsf{V} \xrightarrow{\widetilde{f}_\infty} \mathsf{V}'$. Let $\widetilde{K} \subset \mathsf{V}$ be a compact subset of $\mathsf{V} \setminus \mathsf{Ker}(\widetilde{f}_\infty)$. Define:

$$\begin{aligned} \mathsf{V} &\xrightarrow{f_i} \mathsf{V}' \\ x &\longmapsto \frac{\widetilde{f}_i(x)}{\|\widetilde{f}_i(x)\|}. \end{aligned}$$

Then f_n converges uniformly to f_∞ on $\widetilde{K}$ as $n \longrightarrow \infty$.

Here uniform convergence is defined by a Euclidean metric on V. Although linear transformations generally do not preserve a Euclidean metric, the uniform structure is preserved, enabling the definition of *uniform convergence.*

Proof. Choose $C > 0$ such that $C \leq \|\widetilde{f}_\infty(x)\|$for $x \in \widetilde{K}$. For $\epsilon > 0$, $\exists N$ such that if $n > N$, then $\forall x \in \widetilde{K}$,

$$\begin{aligned} \|\widetilde{f}_\infty(x) - \widetilde{f}_n(x)\| &< \frac{C\epsilon}{2}, \\ \left|1 - \frac{\|\widetilde{f}_n(x)\|}{\|\widetilde{f}_\infty(x)\|}\right| &< \frac{\epsilon}{2}. \end{aligned} \tag{2.4}$$

Let $x \in \widetilde{K}$. Then

$$\begin{aligned} \Big\| f_n(x) - f_\infty(x) \Big\| &= \left\| \frac{\widetilde{f}_n(x)}{\|\widetilde{f}_n(x)\|} - \frac{\widetilde{f}_\infty(x)}{\|\widetilde{f}_\infty(x)\|} \right\| \\ &= \frac{1}{\|\widetilde{f}_\infty(x)\|} \quad \left\| \frac{\|\widetilde{f}_\infty(x)\|}{\|\widetilde{f}_n(x)\|} \widetilde{f}_n(x) - \widetilde{f}_\infty(x) \right\| \\ &\leq \frac{1}{\|\widetilde{f}_\infty(x)\|} \quad \Bigg(\quad \left\| \frac{\|\widetilde{f}_\infty(x)\|}{\|\widetilde{f}_n(x)\|} \widetilde{f}_n(x) - \widetilde{f}_n(x) \right\| \\ &\qquad\qquad + \quad \|\widetilde{f}_n(x) - \widetilde{f}_\infty(x)\| \quad \Bigg) \\ &= \left|1 - \frac{\|\widetilde{f}_n(x)\|}{\|\widetilde{f}_\infty(x)\|}\right| \quad + \quad \frac{1}{\|\widetilde{f}_\infty(x)\|} \|\widetilde{f}_n(x) - \widetilde{f}_\infty(x)\| \\ &< \quad \frac{\epsilon}{2} + C^{-1}\Big(\frac{C\epsilon}{2}\Big) \ = \ \epsilon \qquad \Big(\text{by } (2.4)\Big), \end{aligned}$$

completing the proof of Lemma 2.6.2. ☐

Proof of Proposition 2.6.1. Suppose g_m is a sequence of locally projective maps defined on a connected domain $\Omega \subset \mathsf{P}$ converging uniformly on all compact subsets of Ω to a map

$$\Omega \xrightarrow{g_\infty} \mathsf{P}'.$$

Lift g_∞ to a linear transformation $\widetilde{g}_\infty$ of norm 1, and lift g_m to linear transformations $\widetilde{g}_m$, also linear transformations of norm 1, converging to $\widetilde{g}_\infty$. Then

$$g_m \longrightarrow g_\infty$$

in $\mathsf{End}(\mathsf{P})$. Conversely if $g_m \longrightarrow g_\infty$ in $\mathsf{End}(\mathsf{P})$ and

$$K \subset \mathsf{P} - \mathbb{U}(g_\infty),$$

choose lifts as above and $\widetilde{K} \subset\subset V$ such that $\Pi(\widetilde{K}) = K$. By Lemma 2.6.2,

$$\widetilde{g_m}/\|\widetilde{g_m}\| \rightrightarrows \widetilde{g_\infty}/\|\widetilde{g_\infty}\|$$

on $\widetilde{K}$. Hence $g_m|_K \rightrightarrows g_\infty|_K$, completing the proof of Proposition 2.6.1.

□

2.6.1. Some examples. Let us consider a few examples of this convergence. Consider the case first when $n = 1$. Let $\lambda_m \in \mathbb{R}$ be a sequence converging to $+\infty$ and consider the projective transformations given by the diagonal matrices

$$g_m = \begin{bmatrix} \lambda_m & 0 \\ 0 & (\lambda_m)^{-1} \end{bmatrix}$$

Then $g_m \longrightarrow g_\infty$ where g_∞ is the singular projective transformation corresponding to the matrix

$$g_\infty = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

— this singular projective transformation is undefined at $\mathbb{U}(g_\infty) = \{[0,1]\}$; every point other than $[0,1]$ is sent to $[1,0]$. It is easy to see that a singular projective transformation of P^1 is determined by the ordered pair of points $\big(\mathbb{U}(f), \mathcal{R}(f)\big)$. Note that in the next example, the two points $\mathbb{U}(\phi_\infty), \mathcal{R}(\phi_\infty)$ coincide.

Exercise 2.6.3. Consider the sequence of projective transformations of P^1

$$\phi_n(x) := \frac{x}{1 - nx}, \text{ as } n \longrightarrow +\infty$$

- Show that the pointwise limit equals the constant function 0:

$$\lim_{n\to\infty} \phi_n(x) = 0, \qquad \forall x \in \mathsf{P}^1.$$

- Show that ϕ_n does *not* converge uniformly on any subset $S \subset \mathsf{P}^1$ which contains an infinite subsequence $s_j > 0$ with $s_j \searrow 0$ as $j \longrightarrow \infty$.

- Show that ϕ_n *does* converge to the singular projective transformation ϕ_∞ defined on the complement of $\mathbb{U}(\phi_\infty) = \{0\}$ having constant value 0.
- Use this idea, together with the techniques involved in the proof of Lemma 2.6.2, to prove a statement converse to Proposition 2.6.1:

  If $g_n \in \mathsf{Aut}(\mathsf{P})$ is a sequence of projective transformations converging to a singular projective transformation $g_\infty \in \mathsf{End}(\mathsf{P})$, then, for any open subset $S \subset \mathsf{P}$ which meets $\mathbb{U}(g_\infty)$, the restrictions $g_n|_S$ do not converge uniformly.

2.6.2. Relation to limit sets. A discrete subgroup $\Gamma < \mathsf{PGL}(2, \mathbb{C})$ acts properly on H^3. If $x \in \mathsf{H}^3$, then its orbit $\Gamma \cdot x$ is discrete in H^3. However the action of Γ extends to the compactification $\mathsf{H}^3 \cup \mathbb{C}\mathsf{P}^1$, where $\mathbb{C}\mathsf{P}^1 = \partial \mathsf{H}^3$. Since $\mathsf{H}^3 \cup \mathbb{C}\mathsf{P}^1$ is compact, $\Gamma \cdot x$ has accumulation points in $\mathsf{H}^3 \cup \mathbb{C}\mathsf{P}^1$. Denote by $\Lambda(\Gamma, x)$ the set of accumulation points.

Exercise 2.6.4. Show that if $x, y \in \mathsf{H}^3$, then $\Lambda(\Gamma, x) = \Lambda(\Gamma, y)$.

This set is called the *limit set* of Γ and denoted $\Lambda(\Gamma)$. If its cardinality exceeds two, then it is the unique Γ-invariant closed subset of $\mathbb{C}\mathsf{P}^1$.

Exercise 2.6.5. If H^3/Γ is compact, then $\Lambda(\Gamma) = \mathbb{C}\mathsf{P}^1$.

Compare Kapovich [**210**], Mumord–Series–Wright [**277**], Marden [**251**], or Thurston [**324**].

Exercise 2.6.6. (1) The projective group $\mathsf{PGL}(2, \mathbb{R}) = \mathsf{Aut}(\mathbb{R}\mathsf{P}^1)$ is an open dense subset of $\mathsf{End}(\mathbb{R}\mathsf{P}^1) \approx \mathbb{R}\mathsf{P}^3$. Its complement naturally identifies with the Cartesian product $\mathbb{R}\mathsf{P}^1 \times \mathbb{R}\mathsf{P}^1$ under the correspondence

$$\begin{aligned} \mathsf{End}(\mathbb{R}\mathsf{P}^1) \setminus \mathsf{Aut}(\mathbb{R}\mathsf{P}^1) &\longleftrightarrow \mathbb{R}\mathsf{P}^1 \times \mathbb{R}\mathsf{P}^1 \\ [f] &\longleftrightarrow \big(\mathbb{U}(f), \mathcal{R}(f)\big). \end{aligned}$$

(2) Prove the analogous statements for $\mathsf{PGL}(2, \mathbb{C})$ and $\mathbb{C}\mathsf{P}^1$, that is, when $\mathbb{R}$ is replaced by $\mathbb{C}$.

(3) Show that if $\Gamma < \mathsf{PGL}(2, \mathbb{C})$ is a nonelementary discrete subgroup with limit set $\Lambda \subset \mathbb{C}\mathsf{P}^1$, then $\overline{\Gamma} \backslash \Gamma$ identifies with $\Lambda \times \Lambda \subset \mathbb{C}\mathsf{P}^1 \times \mathbb{C}\mathsf{P}^1$.

2.6.3. Higher-dimensional projective maps. More interesting phenomena arise when $n = 2$. Let $g_m \in \mathsf{Aut}(\mathsf{P}^2)$ be a sequence of diagonal matrices

$$\begin{bmatrix} \lambda_m & 0 & 0 \\ 0 & \mu_m & 0 \\ 0 & 0 & \nu_m \end{bmatrix}$$

where $0 < \lambda_m < \mu_m < \nu_m$ and $\lambda_m\mu_m\nu_m = 1$. Corresponding to the three eigenvectors (the coordinate axes in $\mathbb{R}^3$) are three fixed points

$$p_1 = [1,0,0], \quad p_2 = [0,1,0], \quad p_3 = [0,0,1].$$

They span three invariant lines

$$l_1 = \overleftrightarrow{p_2\, p_3}, \quad l_2 = \overleftrightarrow{p_3\, p_1}, \quad l_3 = \overleftrightarrow{p_1\, p_2}.$$

Since $0 < \lambda_m < \mu_m < \nu_m$, the collineation has a repelling fixed point at p_1, a saddle point at p_2 and an attracting fixed point at p_3. Points on l_2 near p_1 are repelled from p_1 faster than points on l_3 and points on l_2 near p_3 are attracted to p_3 more strongly than points on l_1.

Suppose that g_m does not converge to a nonsingular matrix; it follows that $\nu_m \longrightarrow +\infty$ and $\lambda_m \longrightarrow 0$ as $m \longrightarrow \infty$ (after extracting a subsequence). Suppose that $\mu_m/\nu_m \longrightarrow \rho$; then g_m converges to the singular projective transformation g_∞ determined by the matrix

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which, if $\rho > 0$, has undefined set $\mathbb{U}(g_\infty) = p_1$ and range l_1; otherwise $\mathbb{U}(g_\infty) = l_2$ and $\mathcal{R}(g_\infty) = p_3$.

2.6.4. Limits of similarity transformations. Convergence to singular projective transformations is perhaps easiest for translations of affine space, or, more generally, Euclidean isometries.

Exercise 2.6.7. Suppose $g_m \in \mathsf{Isom}(\mathsf{E}^n)$ is a divergent sequence of Euclidean isometries. Show that $\exists p \in \mathsf{P}^{n-1}_\infty$ and a subsequence g_{m_k}, that $g_{m_k}|_K \rightrightarrows p$ for every compact $K \subset \mathsf{E}^n$.

Indeed the boundary of the translation group V of A^n is the projective space P^{n-1}_∞. More generally the boundary of $\mathsf{Isom}(\mathsf{E}^n)$ identifies with P^{n-1}_∞.

Exercise 2.6.8. Suppose $g_m \in \mathsf{Sim}(\mathsf{E}^n)$ is a divergent sequence of similarities of Euclidean space. Then $\exists$ a subsequence g_{m_k}, and a point

$$p \in \mathsf{E}^n \coprod \mathsf{P}^{n-1}_\infty$$

such that one of three possibilities occurs:

- $p \in \mathsf{P}^{n-1}_\infty$ and $g_{m_k}|_K \rightrightarrows p$, $\forall K \subset\subset \mathsf{E}^n$;
- $p \in \mathsf{P}^{n-1}_\infty$ and $\exists q \in \mathsf{E}^n$ such that

$$g_{m_k}|_K \rightrightarrows p,$$
$$g_{m_k}^{-1}|_K \rightrightarrows q, \qquad \forall K \subset\subset \mathsf{E}^n \setminus \{q\};$$

- $p \in \mathsf{E}^n_\infty$ and
$$g_{m_k}|_K \rightrightarrows p, \ \forall K \subset\subset \mathsf{E}^n.$$

The scale factor homomorphism $\mathsf{Sim}(\mathsf{E}^n) \xrightarrow{\lambda} \mathbb{R}^+$ defined in §1.4.1 of Chapter 1 controls the asymptotics of linear similarities. The two latter cases occur when $\lim_{k\to\infty} \lambda(g_{m_k}) = \infty$ and $\lim_{k\to\infty} \lambda(g_{m_k}) = 0$, respectively.

These results will be used in Fried's classification of closed similarity manifolds in §11.4 .

2.6.5. Normality domains. Convergence to singular projective transformations closely relates to the notion of *normality,* introduced by Kulkarni–Pinkall [**237**], and extending the classical notion of *normal families* in complex analysis. Let G be a group acting on a space X strongly effectively. A point $x \in X$ is a *point of normality* with respect to G if and only if x admits an open neighborhood W such the set of restrictions
$$G|_W := \{g|_W \mid g \in G\}$$
is a precompact subset of $\mathsf{Map}(W, X)$ with respect to the compact-open topology on $\mathsf{Map}(W, X)$. (This means that $G|_W$ is a *normal family* in the sense of Montel.) Denote the set of points of normality by $\mathsf{Nor}(G, X)$. Clearly $\mathsf{Nor}(G, X)$ is a G-invariant open subset of X, called the *normality domain.*

Proposition 2.6.9. Suppose that $\Gamma < \mathsf{Aut}(\mathsf{P})$ is a discrete group of collineations of a projective space P. Let $\overline{\Gamma} \subset \mathsf{End}(\mathsf{P})$ denote its closure in the set of singular projective transformations. Then the normality domain $\mathsf{Nor}(\Gamma, \mathsf{P})$ consists of the complement
$$\mathcal{U}_\Gamma := \mathsf{P} \setminus \bigcup_{\overline{\gamma}\in\overline{\Gamma}} \mathbb{U}(\overline{\gamma})$$
in P of the union $\bigcup_{\overline{\gamma}\in\overline{\Gamma}} \mathbb{U}(\overline{\gamma})$.

Proof. Observe first that $\bigcup_{\overline{\gamma}\in\overline{\Gamma}} \mathbb{U}(\overline{\gamma})$ is compact since each projective subspace $\mathbb{U}(\overline{\gamma})$ is compact and the parameter space $\overline{\Gamma}$ is compact. In particular $\overline{\Gamma}$ is closed so its complement
$$\mathcal{U}_\Gamma := \mathsf{P} \setminus \bigcup_{\overline{\gamma}\in\overline{\Gamma}} \mathbb{U}(\overline{\gamma})$$
is open. We claim that $\mathcal{U}_\Gamma = \mathsf{Nor}(\Gamma, \mathsf{P})$.

We first show any point $x \in \mathcal{U}_\Gamma$ is a point of normality. To this end, we show that the set of restrictions $\Gamma|_{\mathcal{U}_\Gamma}$ is precompact in $\mathsf{Map}(\mathcal{U}_\Gamma, \mathsf{P})$. This follows immediately from Proposition 2.6.1 as follows. Consider an infinite sequence $\gamma_n \in \Gamma$. Proposition 2.6.1 ensures a subsequence γ_n and a singular projective transformation $\overline{\gamma}_\infty \in \mathsf{End}(\mathsf{P})$, such that

$$\gamma_n|_K \rightrightarrows \gamma_\infty|_K, \qquad \forall K \subset\subset \mathfrak{U}_\Gamma,$$

as desired. (Since $K \cap \mathbb{U}(\gamma_\infty) = \emptyset$, the restriction $\gamma_\infty|_K$ is defined.)

Conversely suppose that $x \in \mathbb{U}(\overline{\gamma})$ for some $\overline{\gamma} \in \overline{\Gamma}$. Choose a sequence $g_n \in \Gamma$ converging to $\overline{\gamma}$, and a precompact open neighborhood $S \ni x$. By Exercise 2.6.3, the restrictions $g_n|_S$ do not converge uniformly, and the restrictions to the closure $\overline{S} \subset \mathsf{P}$ do not converge uniformly. Thus $x \notin \mathsf{Nor}(\Gamma, \mathsf{P})$ as claimed. $\square$

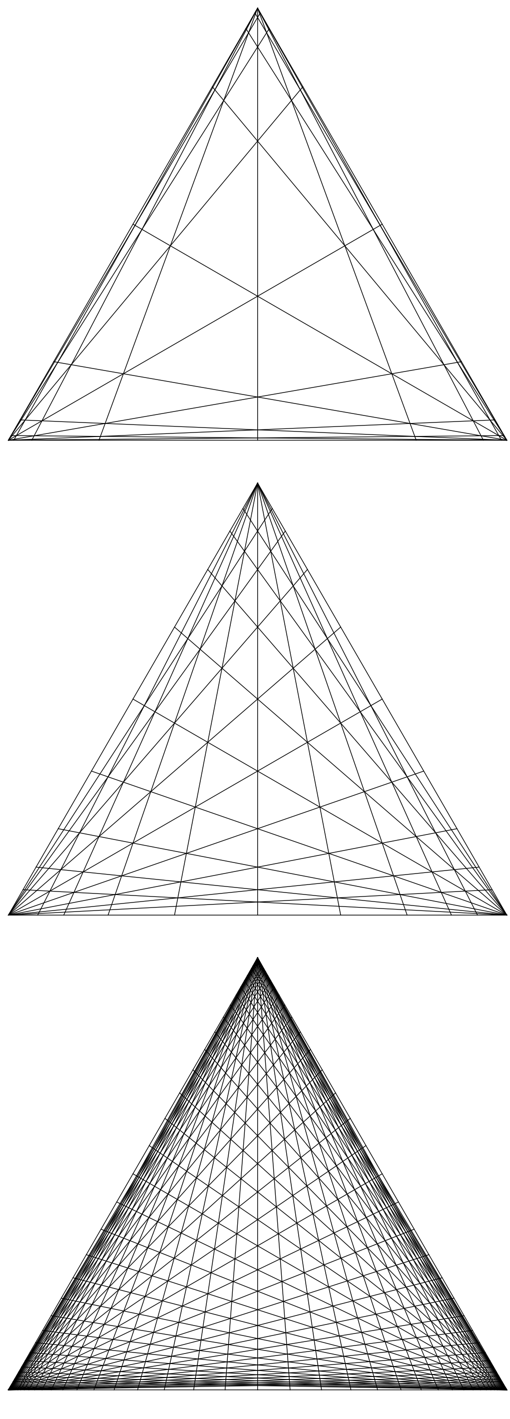

Figure 2.5. Non-Euclidean tessellations by equilateral triangles