Preface

Symbolic dynamics and coding: For many students in mathematics, symbolic dynamics appears first in a general course on dynamical systems, as a symbolic description of Smale's horseshoe, basic sets of Axiom A attractors, or toral automorphisms. For these systems there is a Markov partition and on a symbolic level these are described by a subshift of finite type (SFT). That symbolic dynamics can lead to other subshifts (e.g. Sturmian shifts, β-shifts, kneading theory) often remains beyond view or is only covered in specialized topics courses. Also a standard undergraduate program will not reveal that symbolic dynamics is a flourishing subject by itself, giving very flexible approaches to construct examples with particular topological and ergodic properties. This book is meant to give an overview of many of these aspects of symbolic sequences and coding and allows readers to look beyond their own initial interest. That some topics (in particular ergodic theory and also kneading theory, i.e. symbolic dynamics of unimodal interval maps) are studied at considerably greater depth than others has to do with my own research background and interests.

Because $\{0,1\}^{\mathbb{N}}$ and $\{0,1\}^{\mathbb{Z}}$ are standard representations of the Cantor set, symbolic dynamics is in essence the study of transformations of Cantor sets. When I asked my second thesis supervisor, Jan Aarts, once if there was a general topological classification of such Cantor systems, his answer was "no". If I would have been able to ask this question again, his answer may have been very different. Not only has there been an explosion of new types of subshifts that have been investigated in detail, several unified ways to study them and Cantor systems as a whole have been developed as well. That a complete classification is still lacking seems nowadays less important, or of rather abstract interest, than how modern techniques help us to understand concrete (symbolic) systems.

What you will find in this book:

Chapter 1: Chapter 1 presents the basic notions in symbolic dynamics, including itineraries, word-complexity, and word-frequency (and a first mention of shift-invariant measures), and it presents some of the simplest examples. In Section 1.4 we discuss the Curtis-Hedlund-Lyndon Theorem 1.23 on conjugacy between subshifts given by sliding block codes.

Chapter 2: Chapter 2 gives a brief introduction to topological dynamics. We start with dynamical systems and its basic terminology. Then we present transitive (i.e. dynamically indecomposable) systems followed by minimal systems, which can be characterized by bounded gaps in the sequence of visit times to open sets. We discuss the distinction between expansive and expanding, between expansive and distal (and equicontinuity, which is achieved under the appropriate metric). Mean equicontinuity is a more flexible and important variation of this. In addition to topological entropy and power entropy (which are the logarithmic and polynomial growth rates of the word-complexity), we discuss the fairly new notion of amorphic complexity which is a finer tool than power entropy in distinguishing zero entropy systems, such as constant length substitution shifts and Toeplitz shifts. We discuss several forms of mathematical chaos (in the sense of Devaney, of Li-Yorke, and of Auslander-Yorke for minimal systems) and the Auslander-Yorke dichotomy, with some of its measure-theoretic versions. We present topological mixing in various strengths, and shadowing and specification properties, from Anosov's and Bowen's work to its use in intrinsic ergodicity, i.e. the existence of a unique measure of maximal entropy.

Chapter 3: The main types of subshifts are divided into positive entropy and zero entropy subshifts, although this distinction doesn't apply strictly to every type; for instance zero and positive entropy Toeplitz shifts and $\mathcal{B}$-free shifts might be equally important. Chapter 3 covers the positive entropy subshifts. The main class of these is the subshifts of finite type, followed by sofic shifts. Both can be characterized by transition graphs (vertex-labeled versus edge-labeled) and their topological entropy can be simply computed as the logarithm of the leading (Perron-Frobenius) eigenvalue of their transition matrices, which therefore take specific values only (the logarithms of Perron numbers). More general types of subshifts, in which all values of topological entropy can be achieved, include coded shifts, density shifts, gap-shifts, and spacing shifts, which appear here probably first at textbook level. Directly related to interval maps (β-transformations and unimodal maps) are β-shifts and kneading theory. Our treatment of the latter is broader than in the (by now 40-year-old) monographs of Milnor & Thurston [**420**] or Collet & Eckmann [**164**]. We cover at length Hofbauer's approach of cutting times, which is closely related to the enumeration systems of Section 5.3. Infinitely renormalizable unimodal maps, $*$-products, and (strange) adding machines are deferred to Section 4.7.1 because they fit better in the zero entropy chapter. Finally, we present the square-free subshifts and Dyck shifts, which have more interest in automata theory than dynamical systems.

Chapter 4: Zero entropy subshifts are covered in Chapter 4 starting with linear recurrence, which is rather a property than a class of subshifts. From linear recurrence various properties can be derived, including unique ergodicity and linear complexity, and the absence of arbitrarily high powers of subwords. Substitution shifts are a main class of zero entropy subshift that have been studied in regard to their intriguing ergodic and spectral properties, as demonstrated in Chapter 6. They are central in the theory of tiling spaces, (Rauzy) fractals, and applications such as paper-folding sequences. Their generalization to S-adic shift is powerful enough to describe most minimal subshifts. The second main class of zero entropy subshifts is the Sturmian shifts, which are almost one-to-one extensions of circle rotations (via Denjoy's construction; see Section 4.3.1), but otherwise appearing in many problems in combinatorics. Sturmian sequences have the threefold characterization as symbolic dynamics of irrational circle rotations, sequences of minimal non-trivial word-complexity $p(n) = n+1$, and minimal 1-balanced sequences. We use them to introduce Rauzy graphs (see Section 4.3.4) and also give a detailed account of their representation as S-adic shifts (generalized to Arnoux-Rauzy shifts of torus rotations). Circle rotations generalize naturally to interval exchange transformations (IETs), and the corresponding subshifts have word-complexity $p(n) = (d-1)n+1$, where d is the number of intervals of the IET. We discuss Rauzy induction which leads to their representation as S-adic shifts, and this will later (Section 6.3.5) be used to give a solution of the Keane conjecture on the typical unique ergodicity of IETs, bypassing the use of Teichmüller flows in the original proofs of Masur [**410**] and Veech [**543**]. Sections 4.5 and 4.6 combine adding machines (odometers), which are not actual subshifts as they are not expansive, with Toeplitz shifts and $\mathcal{B}$-free subshifts, because the latter two are, under mild conditions, one-to-one extensions of odometers. The final Section 4.7 comes largely from some of my own research papers (with or without coauthors) in topological dynamics. It shows how some minimal subshifts (known and new) arise naturally in the setting of unimodal interval maps, but contrary to the section on kneading theory, they fit in the zero entropy chapter.

Chapter 5: Chapter 5 discusses minimal Cantor systems that are not necessarily subshift and presents three methods of describing them, namely cutting and stacking, enumeration systems, and Bratteli-Vershik systems. In Section 5.1 we show a basic construction of Kakutani-Rokhlin partitions, probably first described in general form by Herman, Putnam & Skau [**310**], that can be used as a building block to translate (virtually) any minimal system into any of these three descriptions. The cutting and stacking was used by Kakutani, Chacon, and others to produce the earliest examples of uniquely ergodic systems with particular (weak) mixing or spectral properties. Enumeration systems are a generalization of the Ostrowski numeration, see Example 5.22, that is based on the standard continued fraction expansion. Liardet with various coauthors studied these in detail, obtaining among other things novel results on unique ergodicity. Under certain conditions (related to Pisot numbers), the method also gives a way of associating Rauzy fractals to particular minimal Cantor systems; see Section 5.3.1. The discussion of Bratteli-Vershik (BV) systems is the most extensive section in this chapter, partly because we give explicit constructions of various subshifts and also of cutting and stacking and of enumeration systems in terms of BV-systems. In this form, these systems frequently return in Chapter 6 where ergodic and spectral properties of minimal Cantor systems are studied. It is worth mentioning that Gjerde & Johansen's description [**275**] of IETs follows in fact from Rauzy induction described in Section 4.4. The description of Toeplitz shifts in terms of a BV-system is also given by Gjerde & Johansen [**274**], and we largely follow their exposition.

Chapter 6: Chapter 6 treats the properties of invariant measures of subshifts and other (minimal) Cantor systems. After a brief recall of the notions of ergodicity, the Choquet (and Poulsen) simplex of invariant probability measures, and Birkhoff's Ergodic Theorem, we discuss unique ergodicity in Section 6.3. Special topics here are Boshernitzan's results based on low word-complexity, the "classical" proof of unique ergodicity of primitive substitution shifts, unique ergodicity of BV-systems based on contraction in the Hilbert metric, and a proof of unique ergodicity of typical IETs (the Keane conjecture), based on Rauzy induction. In Section 6.4 we discuss measure-theoretic entropy, preparing our discussion of metrically isomorphic systems and Ornstein's Theorem in Section 6.5. We mention the Variational Principle and measures of maximal entropy in Section 6.6 and discuss the notion of entropy denseness. We also present the Shannon-Parry measure (measure of maximal entropy for SFTs).

In Section 6.7 we recall proofs that many minimal subshifts cannot be strongly mixing (i.e. the correlation coefficients cannot tend to zero). This applies to substitution shifts, or more generally linearly recurrent subshifts, and also to cutting and stacking systems with a bounded number of layers of spacer. In contrast, for staircase systems (i.e. cutting and stacking with an unbounded number of slices in between the stacks without bound on the number of layers of spacer) strong mixing can occur. This conjecture by Smorodinsky was proved by Adams, and we present the result in detail. In Section 6.7.3 we discuss the notion of weak mixing, which in spectral terms means that constant functions are the only eigenfunction of the Koopman operator. The search for (continuous or measurable) eigenfunctions for various subshifts, specifically substitution shifts, goes back to Dekking, Keane, Queffélec, and Host, and coauthors. We follow the more recent approach of Durand, Maass, and coauthors and give detailed proofs of how the condition $\|\alpha h_v(n)\| := d(\alpha h_v(n), \mathbb{Z}) \to 0$ as $n \to \infty$ (where $h_v(n)$ are the height vectors of the Bratteli diagram) relates to $e^{2\pi i\alpha}$ being a continuous or measurable eigenvalue of the Koopman operator.

In Section 6.8 we study spectral properties of Cantor systems, that is, the properties that can be expressed in terms of the spectrum of the Koopman operator $U_T : L^2(\mu) \to L^2(\mu)$, $U_T f \mapsto f \circ T$. This includes the spectral measures (Fourier transforms) of observables $f \in L^2(\mu)$ that are important for the spectral decomposition of the Hilbert space $L^2(\mu)$ and the Koopman operator itself. We discuss the spectral measure and spectral type of U_T itself, in particular conditions under which Cantor systems have pure point spectrum or mixed spectrum.

Chapter 7: Chapter 7 discusses ways in which symbol sequences play a role in computer science, information theory, and data transmission. Automata represent a theoretical model for computers and a concrete way of generating sequences, which are hence called automatic sequences. Fixed points of substitutions are automatic, by Cobham's Little Theorem. We discuss automata and automatic sequences in Section 7.1. In Section 7.2 we present the Chomsky hierarchy of formal languages, consisting of four basic levels of complexity of their underlying grammar (production rules) and of the type of automaton that can detect or produce them. The lowest level, regular languages, can be produced by finite automaton and also correspond to sofic subshifts discussed in Section 3.2. Pumping lemmas are a major tool for distinguishing between these levels, and they show that most other types of subshifts discussed in earlier chapters are in the context-sensitive category or beyond.

Chapter 8: Here we give background material to which earlier chapters frequently refer, but some of its sections comprise short topics of independent interest as well. Section 8.1 on Pisot numbers, apart from giving definitions and some historic background, addresses the question of for which $\alpha \in \mathbb{R}$, $\|\alpha h_v(n)\| := d(\alpha G_n, \mathbb{Z}) \to 0$ as $n \to \infty$, where G_n is a sequence of integers satisfying a linear recursion formula (such as the height vectors of the Bratteli diagram do). In Section 8.2 we discuss the standard continued fraction algorithm, as well as Farey arithmetic, and associated ways (Kepler, Calkin-Wilf) of denumbering all the rationals.

Section 8.3 covers uniformly distributed sequences, Weyl's criterion, and Van der Corput's Difference Theorem. Section 8.4 on Diophantine approximation, in addition to definitions and some historic background, discusses Dirichlet's Theorem, the Lagrange spectrum, and Roth's Theorem

Section 8.5 covers the (Banach) density and logarithmic density (important in $\mathcal{B}$-free subshifts) of sequences.

In Section 8.6 we present the Perron-Frobenius Theorem 8.58, on the leading eigenvalue and eigenspace of non-negative matrices, also in connection with the Hilbert metric on projective space. In Section 8.7 we treat countable Markov graphs and matrices and present the Vere-Jones classification into transient, positive recurrent, and null recurrent systems, including Gurevich entropy and refined classifications by Salama and Ruette. We also give some results on the (non-)existence of measures of maximal entropy for countable state Markov chains. Section 8.7.3 presents the rome technique from the paper [**88**] by Block et al., which facilitates entropy computations for countable Markov graphs.

The scope and aim of the book: Although the text grew from two courses I gave on symbolic dynamics at the University of Vienna, there is no attempt to shape it as a textbook for a course on symbolic dynamics. The material is too diverse, hardly balanced in depth and detail, and I have made no attempt to indicate which sections together would constitute such a course. My experience is that hardly any book, however well-conceived and written, completely fits the purpose and taste of the lecturers teaching the actual class. Instead, I hope this book can serve a purpose for topics courses or reading courses or as a reference book for anyone wishing to acquaint him/herself to a particular topic.

Also the exercises are not devised as a testing tool of the students' understanding of the material. Symbolic dynamics is at the intersection of dynamical systems, topological dynamics, combinatorics, and of course coding theory, and there are a lot of trivia to share. Some of these trivia are disguised as exercises. Most of the exercises have solutions in the back of the book. I have given, probably multiple times, proofs of simple results that are used as exercises in comparable books, but I wanted to avoid the annoying situation that you cannot refer to a well-known result because it is only presented as an exercise without solution.

The necessary background for the book varies: for most of it a solid knowledge of real analysis and linear algebra and first courses in probability and measure theory (a few times, conditional expectations are used, and martingales in the proof of Theorem 6.118), metric spaces, number theory, topology, and set theory suffice. Chapter 6 is not meant as an introduction to ergodic theory, so a course in ergodic theory and in Hilbert spaces and Fourier analysis for Section 6.8 are probably necessary. Section 8.1 uses some Galois theory.

By adding an extensive index and cross-references within the main text, I tried to enable the reader to study the chapters independently. However, readers without any prior knowledge of symbolic dynamics should not skip (the first halves of) Chapters 1 and 2. To follow Chapter 5 one should have a good understanding of SFTs (Section 3.1), substitution shifts (Section 4.2), and Sturmian shifts (Section 4.3). To follow Chapter 6, an additional understanding of BV-systems (Section 5.4) is required. Chapter 8 can be read largely independently of the rest, except for several examples with direct references to earlier parts in the text.

The book is largely, but not entirely, self-contained. That would go too far, because various topics are covered much better in other textbooks. General books on ergodic theory are [**165**, **277**, **305**, **346**, **408**, **456**, **479**, **509**, **551**]; for proofs of Birkhoff's Ergodic Theorem we suggest [**341**, **346**, **349**, **551**], and for the proof of the Variational Principle we recommend [**551**]. In topological dynamics the book by Auslander is a classic (although difficult to navigate). Other texts are [**12**, **22**, **198**, **199**, **284**, **381**]. In symbolic dynamics there are Kitchens [**364**] and Lind & Marcus [**398**], both specializing in SFTs, and the more general book by Kůrka [**381**], and topic collections by Blanchard et al. [**85**] and Bedford et al. [**57**]. Substitution shifts have the expert monographs Queffélec [**465**], "Phyteas Fogg" [**249**], and other groups of authors [**68**]. I should not fail to mention Viana's monograph [**548**] on interval exchange transformations. Continued fractions and Diophantine approximation are the subject of monographs [**98**, **175**, **360**, **377**]. For general dynamical systems there are [**17**, **113**, **346**, **474**], and [**22**, **87**, **116**, **164**, **414**, **462**] for one-dimensional dynamics in particular.

Various milestones in the theory have been treated extensively by people far more expert and well-placed than I am. For example, this holds for Williams's Theorem on conjugacy in subshifts of finite type; see [**364**, **398**]. This also holds for Ornstein's Theorem [**438**] that entropy is a complete invariant among invertible Bernoulli shifts, which we only briefly introduce in Section 6.5 because the full proof is beyond the scope of this book. We refer to [**208**, **209**, **216**, **352**, **456**] for further developments, new (more conceptual) proofs, and more detailed expositions.

How much detail is given for each topic relied on my own taste and judgment, and if I overstretched the reader and/or my own abilities and understanding, then so be it. My apologies to the true experts. Thus I decided to include Li-Yorke chaos but not distributional chaos, some version of entropy, but no dynamical ζ-functions, no IP-sets, only a few variations of shadowing and specification (see [**461**] for a monograph on shadowing), no higher-dimensional shifts, and no automorphism groups of shift spaces. Although the first use of symbolic dynamics by Hadamard ([**296**] in 1898) was for geodesic flows on modular surfaces, this topic does not appear in this book; see e.g. [**494**]. We cover Kakutani-Rokhlin partitions, cutting and stacking, S-adic transformation, Bratteli-Vershik systems but no graph covers, although they describe the Cantor system in even further generality. See [**273**, **500**–**502**] for their constructions of some of the earliest uses.

Partly, this book is a compendium of bits of knowledge and curiosities that are scattered over the literature if not on Wikipedia pages. The material that I present is not equally up to date. For instance, notions such as $\mathcal{B}$-free shifts (or at least the current state) and amorphic complexity are from the past decade or even less, whereas in the section on SFTs, the material is all from before 1990. Topics that are to my knowledge new to textbook and monograph literature include gap shifts, spacing shifts, power-free shifts, $\mathcal{B}$-free shifts, amorphic complexity, enumeration systems. The breadth of topics allows one to see similarities of methods of proof in different subfields of symbolic dynamics. I hope that my extensive treatment of Bratteli-Vershik systems and unique ergodicity, as well as my treatment of Sturmian shifts and Rauzy fractals (redoing and sometimes reproving results of Arnoux), have some added value. The sections on weak mixing for Bratteli-Vershik systems and on Adams's Theorem of mixing staircase systems, grew out of topics I set for Master's Theses for Silvia Radinger and Kathrin Peticzka, respectively.

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