Contents

Preface

This is the first of two volumes on the qualitative theory of foliations. It is divided into three parts, the first being a primer on foliated manifolds. The material here could be the basis of a short graduate course or learning seminar, preparing students to begin exploring the literature. The remainder of this volume is a "sampler" of more advanced topics. In Part 2, we focus on foliated manifolds of codimension one. Here, hands–on geometric methods yield a structure theory (the theory of levels) which is reminiscent of the classical Poincaré–Bendixson theory of flows on surfaces. In arbitrary codimension (Part 3) these methods break down and we turn to the techniques of ergodic theory. Again the theory of flows, now in higher–dimensional manifolds, is the model. Specifically, we consider foliations to be generalized dynamical systems and extend to these systems the theory of invariant measures (following J. F. Plante [**107**], D. Sullivan [**127**], *et al.*) and the theory of topological entropy (following E. Ghys, R. Langevin and P. Walczak [**50**]). For the most part, these methods apply equally well to laminations (partial foliations of manifolds) and even to abstract laminations (foliated metric spaces). In Chapter 11, we introduce this abstract setting and most of the subsequent results are proven for foliated metric spaces.

The measures studied in Part 3 are invariant under holonomy. There are interesting conditions guaranteeing the existence of such measures (*cf.* Theorem 12.3.1 and Theorem 13.4.2) and the corresponding ergodic theory has beautiful applications. The foliation cycles of D. Sullivan (Chapter 10) exemplify this. The limitation of this approach is that holonomy–invariant measures do not always exist. On the other hand, the harmonic measures of L. Garnett [**46**] always exist and the ergodic theory for foliations based on them also has profound applications. We plan to treat this in the second volume.

Foliation theory has developed to the point that an encyclopedic treatment is out of the question. Our choice of topics is quite subjective, depending largely on the authors' tastes and expertise. As to topics omitted, we call special attention to the quantitative theory of foliations, culminating in the stunning existence and classification theorems of W. Thurston in the 1970s [**134, 135**]. Likewise, the applications of noncommutative geometry to foliations, pioneered by A. Connes (see [**31**]), and the beautiful study of transverse geometry (see P. Molino [**91**] and P. Tondeur [**137, 138**]) are not treated. Our hope is that the material we do present will whet the reader's appetite for more. In the second volume, in addition to harmonic measures, we will offer introductory treatments of a few other specialized topics, including the exotic characteristic classes of foliations and foliated 3–manifold theory.

Foreword to Part 1

This part is intended to be a primer in foliation theory. A good understanding of its contents should make much of the literature on foliations accessible and will be presupposed in the rest of the book.

We have tried to present a wealth of examples in the conviction that they are essential for understanding abstract theory. Wherever feasible, we have even preceded formal definitions with informal examples, occasionally allowing rigor to be compromised for the sake of clarity. This is especially true in the earlier sections, where foliated manifolds are sometimes constructed by “cut and paste” methods. Careful justifications for these and subsequent constructions are given in Chapter 3.

It will be assumed that the reader has a fairly good background in manifold theory. For example, the material in references [**11**], [**29**], and [**124**] should be adequate for most purposes.

A few remarks on notation are in order. The notation $C^r(M)$ will stand for the space of all real–valued C^r functions on the smooth manifold M, $0 \leq r \leq \infty$ or $r = \omega$ (analytic). The term “smooth”, without further qualification, generally means “smooth of class C^∞”. If $\pi : E \to M$ is a bundle of a given smoothness class C^r, then $\Gamma(E)$ denotes the set of C^r sections of the bundle. The total space of the tangent bundle of M will be denoted by $T(M)$ and the space of smooth vector fields is $\mathfrak{X}(M) = \Gamma(T(M))$. If $\pi : E \to M$ is a vector bundle, E^* denotes the total space of the dual bundle. The notation $\Lambda^k(E)$ denotes the total space of the kth exterior power of $\pi : E \to M$ and $A^k(M) = \Gamma(\Lambda^k(T(M)^*))$ is the space of k–forms on M. The graded exterior algebra of differential forms on M will be denoted by $A^*(M)$.

We will follow the convention that all manifolds are 2nd countable and Hausdorff. Unless otherwise stipulated, all manifolds are smooth, possibly with boundary and corners.

Chapter 1

Foliated Manifolds

1.1. What Is a Foliation?

Roughly speaking, a foliation $\mathcal{F}$ is an equivalence relation on an n–manifold M, the equivalence classes being connected, immersed submanifolds, all of the same dimension k. (By the term "immersed submanifold" we mean the image of a one–to–one immersion $L \hookrightarrow M$.) One further requires that, locally, the decomposition into equivalence classes be modeled on the decomposition of $\mathbb{R}^n$ into the cosets $x + \mathbb{R}^k$ of the standardly imbedded subspace $\mathbb{R}^k$. In particular, this decomposition of $\mathbb{R}^n$ is itself a foliation. The equivalence classes of $\mathcal{F}$ are called the *leaves* of the foliation.

Before making these notions precise, we consider examples.

Example 1.1.1. Let M be an n–manifold, B a manifold of dimension $n-k$, $\partial M = \emptyset = \partial B$, and let

$$\pi : M \to B$$

be a smooth *fiber bundle*. That is, there is a k–manifold F and, for each $x \in B$, an open neighborhood U of x in B and a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\ \varphi\ } & U \times F \\ {\scriptstyle\pi}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ U & \xrightarrow[\text{id}]{} & U \end{array}$$

with φ a diffeomorphism and p the canonical projection onto the first factor. Each subspace $\pi^{-1}(x)$, $x \in B$, is clearly an imbedded k–manifold diffeomorphic to F. The manifold B is the *base* of the fiber bundle and F is the *fiber.* The *total space* of the fiber bundle is M. As x ranges over B, the connected components of the fibers $\pi^{-1}(x)$ range over the leaves of a foliation $\mathcal{F}$. We say that $\mathcal{F}$ has *codimension* $q = n - k = \dim B$ and *dimension* $k = \dim F$. Remark that the leaves are also fibers of a bundle whose base $\widetilde{B}$ is a covering space of B. Indeed, $\widetilde{B} = M/\mathcal{F}$ is the manifold obtained by collapsing each leaf of $\mathcal{F}$ to a point.

Example 1.1.2. A smooth submersion

$$f : M \to B$$

from an n–manifold to a q–manifold (both without boundary) provides a mild generalization of the previous class of examples. The submersion theorem asserts that, for each point $y \in M$, there is a coordinate neighborhood $(U, y^1, \dots, y^n)$ of y in M and a coordinate neighborhood $(V, x^1, \dots, x^q)$ of $f(y)$ in B, relative to which the formula for $f|U$ becomes

$$f(y^1, \dots, y^n) = (y^1, \dots, y^q).$$

It follows that the level sets $f^{-1}(x)$ are properly imbedded submanifolds of M of dimension $k = n - q$ and that, locally, these submanifolds fit together exactly like parallel copies of $\mathbb{R}^k$ in $\mathbb{R}^n$. Thus, the connected components of the nonempty level sets of f are the leaves of a foliation $\mathcal{F}$ of codimension $q = \dim B$. If B is connected and each of these level sets is compact (*e.g.*, if M itself is compact), then

$$f : M \to B$$

is actually a fiber bundle (Exercise 1.1.3).

While a fiber bundle is always a submersion, the converse is false. As an example, consider

$$f : \mathbb{R}^2 \to \mathbb{R},$$
$$f(x, y) = (x^2 - 1)e^y.$$

Here,

$$f_x(x, y) = 2xe^y,$$
$$f_y(x, y) = (x^2 - 1)e^y.$$

The derivative f_x only vanishes along the y–axis, while f_y only vanishes along the lines $x = \pm 1$; so f is a submersion. The reader should check that the level sets of f give the foliation $\mathcal{F}$ in Figure 1.1.1. The vertical lines $x = \pm 1$ are the components of the level set $f = 0$, hence are leaves. Each curve between these lines is a level set $f = c$ with $c < 0$ and is asymptotic to $x = \pm 1$. Each level set $f = c$, $c > 0$, falls into two components, one lying in the region $x > 1$ and asymptotic to $x = 1$, one in the region $x < -1$ and asymptotic to $x = -1$. The leaf space $\mathbb{R}^2/\mathcal{F}$ (again formed by collapsing each leaf to a point and imposing the quotient topology) is locally Euclidean

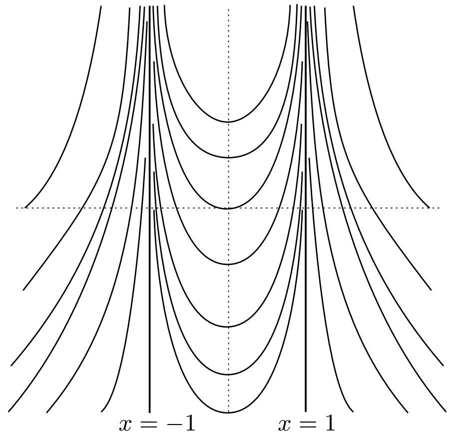

Figure 1.1.1. The foliation by the submersion $(x^2 - 1)e^y$

of dimension 1, but is not Hausdorff (Exercise 1.1.4); so this cannot be the base manifold of a bundle.

We extend the above submersion to a submersion

$$f : \mathbb{R}^{n+1} \to \mathbb{R}$$

by writing $\mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}$ and using cylindrical coordinates (r, z, t). Here, z ranges over the unit sphere $S^{n-1} \subset \mathbb{R}^n$ and the coordinates of any point $v \neq 0$ in $\mathbb{R}^n$ are $(r, z) = (\|v\|, v/\|v\|)$. These coordinates are bad at the origin, where r is not smooth (not even of class C^1) and every pair $(0, z)$ represents the origin. However, r^2 is smooth on all of $\mathbb{R}^n$ and the formula

$$f(r, z, t) = (r^2 - 1)e^t$$

defines the desired submersion. For $n \geq 2$, each level set $f = c$ is connected, hence is a leaf of a foliation $\mathcal{F}$ of $\mathbb{R}^{n+1}$. The level $f = 0$ is exactly the cylinder $S^{n-1} \times \mathbb{R}$, each level $f = c < 0$ is a hypersurface of revolution diffeomorphic to $\mathbb{R}^n$, and each level $f = c > 0$ is a hypersurface of revolution diffeomorphic to $S^{n-1} \times \mathbb{R}$. The foliation of $\mathbb{R}^3$ is depicted in Figure 1.1.2. Since the leaves fall into two distinct diffeomorphism classes, it remains evident that this foliation cannot be a fiber bundle.

Exercise 1.1.3. If $\partial M = \emptyset = \partial B$ and B is connected, prove that a submersion $f : M \to B$ with compact level sets is a fiber bundle.

Exercise 1.1.4. Prove that the leaf space $\mathbb{R}^2/\mathcal{F}$ for the foliation in Figure 1.1.1 is locally Euclidean of dimension 1, but not Hausdorff.

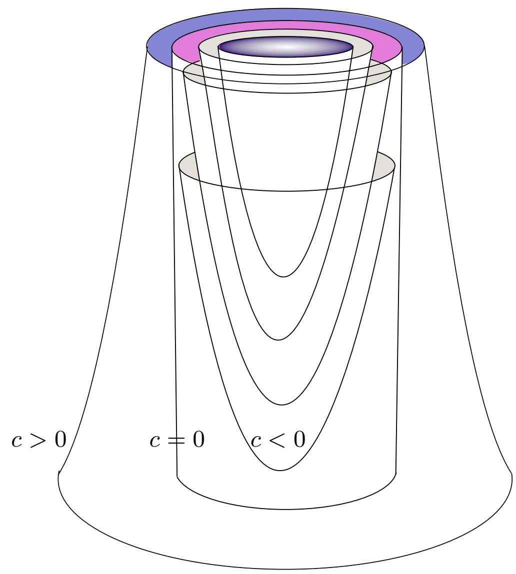

Figure 1.1.2. Foliating $\mathbb{R}^3$ by the level sets $(r^2 - 1)e^t = c$

Example 1.1.5. If $X \in \mathfrak{X}(M)$ is a nonsingular (*i.e.*, nowhere zero) vector field, then the local flow defined by X patches together to define a foliation of dimension 1. Indeed, given an arbitrary point $x \in M$, the fact that X is nonsingular allows us to find a coordinate neighborhood $(U, x^1, \dots, x^n)$ about x such that

$$-\epsilon < x^i < \epsilon, \quad 1 \le i \le n,$$

and

$$\frac{\partial}{\partial x^1} = X|U.$$

Geometrically, the flow lines of $X|U$ are just the level sets

$$x^i = c^i, \quad 2 \le i \le n,$$

where all $|c^i| < \epsilon$. Since we follow the convention that manifolds are 2nd countable, the careful reader may worry that a leaf of this foliation might be the "long line", a notorious example of a connected manifold that is not 2nd countable [**124**, Appendix A]. We will see in the next section that such anomalies are precluded by the 2nd countability of M itself. For the moment, the difficulty can be sidestepped by requiring that X be a complete field (*e.g.*, that M be compact), hence that each leaf be a flowline.

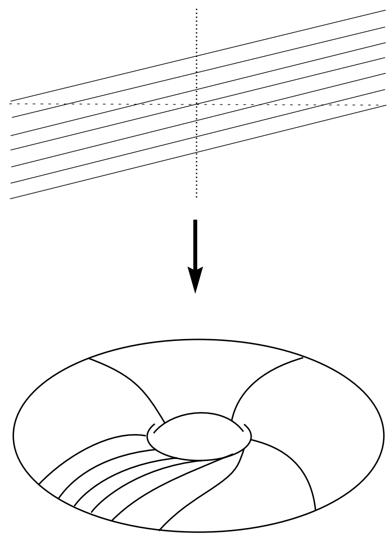

Figure 1.1.3. A linear foliation of T^2

An interesting class of 1–dimensional foliations on the torus T^2 is given by the *linear* foliations. A constant vector field

$$\widetilde{X} \equiv \begin{bmatrix} a \\ b \end{bmatrix}$$

on $\mathbb{R}^2$ is invariant by all translations in $\mathbb{R}^2$, hence passes to a well–defined vector field X on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$. We assume that $a \neq 0$. The foliation $\widetilde{\mathcal{F}}$ on $\mathbb{R}^2$ produced by $\widetilde{X}$ has as leaves the parallel lines of slope b/a. This foliation is also invariant under translations and passes to the foliation $\mathcal{F}$ on T^2 produced by X (Figure 1.1.3). We consider separately the cases that the slope $\rho = b/a$ is rational or irrational.

Each leaf of $\widetilde{\mathcal{F}}$ is of the form

$$\widetilde{L} = \{(x_0 + ta, y_0 + tb)\}_{t\in\mathbb{R}}.$$

When the slope is rational, we can take $a, b \in \mathbb{Z}$. For fixed $t_0 \in \mathbb{R}$, the points of $\widetilde{L}$ corresponding to values of $t \in t_0 + \mathbb{Z}$ all project to the same point of T^2; so the corresponding leaf L of $\mathcal{F}$ is an imbedded circle in T^2. Since L is arbitrary, $\mathcal{F}$ is a foliation of T^2 by circles. It follows rather easily that this foliation is actually a fiber bundle $\pi : T^2 \to S^1$ (Exercise 1.1.6).

When the slope $\rho = b/a$ is irrational, the foliation on T^2 looks radically different. Indeed, each leaf L of $\mathcal{F}$ is a one–to–one immersion of $\mathbb{R}$ and is everywhere dense in T^2 (Kronecker's theorem). To see this, note first that, if a leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$ does not project one–to–one into T^2, there must be a real number $t \neq 0$ such that ta and tb are both integers. But this would imply that $b/a \in \mathbb{Q}$. In order to show that each leaf L of $\mathcal{F}$ is dense in T^2, it is enough to show that, for every $v \in \mathbb{R}^2$, each leaf $\widetilde{L}$ of $\widetilde{\mathcal{F}}$ achieves arbitrarily small positive distances from suitable points of the coset $v + \mathbb{Z}^2$. A suitable translation in $\mathbb{R}^2$ allows us to assume that $v = 0$; so we are reduced to showing that $\widetilde{L}$ passes arbitrarily close to suitable points $(n, m) \in \mathbb{Z}^2$. The line $\widetilde{L}$ has slope–intercept equation

$$y = \rho x + c.$$

So it will suffice to find, for arbitrary $\eta > 0$, integers n and m such that

$$|\rho n + c - m| < \eta.$$

Equivalently, $c \in \mathbb{R}$ being arbitrary, we are reduced to showing that the set $\{\rho n - m\}_{m,n\in\mathbb{Z}}$ is dense in $\mathbb{R}$. This is essentially the criterion of Eudoxus that ρ and 1 be incommensurable (*i.e.,* that ρ be irrational). A more modern proof is suggested in Exercise 1.1.7.

Exercise 1.1.6. Let $\mathcal{F}$ be a linear foliation of T^2, as in Example 1.1.5, with slope b/a rational. Prove that each leaf has a neighborhood U diffeomorphic to $(-\epsilon, \epsilon) \times S^1$ in such a way that, for $-\epsilon < t < \epsilon$, the imbedded circle $\{t\} \times S^1$ is identified with a leaf. Use this to prove that the leaves of the foliation are the fibers of a fiber bundle $\pi : T^2 \to S^1$.

Exercise 1.1.7. If $G \subset \mathbb{R}$ is a nontrivial additive subgroup and if there is a neighborhood $U \subset \mathbb{R}$ of 0 such that $U \cap G = \{0\}$, prove that G is infinite cyclic. Next, show that, if $\rho \in \mathbb{R}$ is irrational, the additive subgroup $G_\rho \subset \mathbb{R}$, generated by $\{1, \rho\}$, is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Use these results to prove that G_ρ is dense in $\mathbb{R}$. Conclude that each leaf of a linear foliation of T^2, with slope b/a irrational, is dense in T^2.

Example 1.1.8. If G is a Lie group and $H \hookrightarrow G$ a connected Lie subgroup, then the collection $\{gH\}_{g\in G}$ of left cosets of H is the set of leaves of a foliation $\mathcal{H}$. If H is a *closed* subgroup, then G/H is a manifold and $\mathcal{H}$ is a foliation with leaves the fibers of a fiber bundle $\pi : G \to G/H$. If H is not closed, the closed subgroup theorem guarantees that $\overline{H}$ will also be a Lie subgroup and, indeed, the closure of each leaf gH is the coset $g\overline{H}$; so the closures of the leaves of $\mathcal{F}$ are the fibers of a bundle $\pi : G \to G/\overline{H}$.

The linear foliations of T^2 are special cases of this. Similarly, the connected Lie subgroups of $T^n = \mathbb{R}^n/\mathbb{Z}^n$ correspond one–to–one to the vector subspaces $V \subset \mathbb{R}^n$. According to the "position" of the k–dimensional subspace $V \subset \mathbb{R}^n$ with respect to $\mathbb{Z}^n$, the leaves in T^n will all be diffeomorphic to $\mathbb{R}^{k-r} \times T^r$, for some integer r, $0 \leq r \leq k$, and the closure of each leaf will be an imbedded torus in T^n of dimension q, $k \leq q \leq n$. The integers

r and q are determined by V and all the possibilities $0 \leq r < k < q \leq n$ can occur. It is clear, however, that $r = k$ if and only if $q = k$, the case in which $V \cap \mathbb{Z}^n$ is a k–dimensional lattice subgroup of V.

Example 1.1.9. Let G be a Lie group, $H \hookrightarrow G$ a connected Lie subgroup, and let $\mathcal{H}$ be the foliation of G by the left cosets of H. Suppose that $\Gamma \subset G$ is a discrete subgroup. Then $M = \Gamma\backslash G$, the space of *right* cosets of Γ, is a manifold of the same dimension as G. If Γ is *cocompact* in G, the manifold M is compact. At any rate, since $\mathcal{H}$ is invariant under left translations and Γ is divided out from the left, the foliation $\mathcal{H}$ passes to a well–defined foliation $\mathcal{H}_\Gamma$ on M. While $\mathcal{H}$ is a fairly tame foliation, $\mathcal{H}_\Gamma$ can be rather wild.

Of some interest is the case in which $G = \mathrm{Sl}(2,\mathbb{R})$, the *special linear group* consisting of all real 2×2 matrices A with $\det(A) = 1$. This is a 3–dimensional Lie group. Let $H \subset \mathrm{Sl}(2,\mathbb{R})$ be the subgroup given by

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right\}_{a>0}.$$

This is a connected, 2–dimensional subgroup. Actually, we are going to work in the group

$$G = \mathrm{PSl}(2,\mathbb{R}) = \mathrm{Sl}(2,\mathbb{R})/\{\pm I\}$$

where H survives intact as a subgroup.

Let $\mathbf{H}$ denote the *hyperbolic plane,* the simply connected Riemannian 2–manifold of constant curvature -1. A standard model of $\mathbf{H}$ is the open upper half plane

$$\mathbf{H} = \{x + iy \in \mathbb{C} \mid y > 0\}.$$

In this model, the metric is

$$\frac{dx^2 + dy^2}{y^2}.$$

This metric is conformal to the Euclidean metric $dx^2 + dy^2$. Thus angles are the same in both geometries but, as one approaches $y = 0$, hyperbolic distances become enormous compared to Euclidean distances. Indeed, a vector tangent to $\mathbf{H}$ at $x + iy$ has hyperbolic length 1 if and only if it has Euclidean length y.

If

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{Sl}(2,\mathbb{R}),$$

one easily checks that the linear fractional transformation

$$z \mapsto \frac{az + b}{cz + d}$$

carries the upper half plane onto itself. In Exercise 1.1.10, you will show that this transformation is an orientation–preserving isometry of the hyperbolic metric and that every orientation–preserving isometry is of this form. Since composition of linear fractional transformations corresponds to multiplication of the corresponding coefficient matrices, we obtain a surjective homomorphism of $\mathrm{Sl}(2,\mathbb{R})$ onto the group $\mathrm{Iso}_+(\mathbf{H})$ of orientation–preserving isometries. The kernel of this projection is clearly the center $\{\pm I\}$ of $\mathrm{Sl}(2,\mathbb{R})$, and so $\mathrm{Iso}_+(\mathbf{H}) = \mathrm{PSl}(2,\mathbb{R})$. In Exercise 1.1.10, you will also show that this isometry group acts (via its differentials) as a smooth transformation group of the manifold

$$T^1(\mathbf{H}) \cong \mathbf{H} \times S^1$$

of unit tangent vectors to $\mathbf{H}$ and that this action is simply transitive. Thus, fixing a basepoint $v_0 \in T^1(\mathbf{H})$, we can identify $T^1(\mathbf{H})$ with

$$\mathrm{PSl}(2,\mathbb{R}) \cdot v_0 \cong \mathrm{PSl}(2,\mathbb{R}).$$

It will be convenient to take the basepoint to be

$$v_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}_i \in T_i^1(\mathbf{H}),$$

the unit vector tangent at $i \in \mathbf{H}$ that points in the positive y direction. We are going to describe H and its left cosets in $\mathrm{PSl}(2,\mathbb{R})$ as sections of the S^1–bundle $T^1(\mathbf{H})$ which can be easily visualized. Indeed, a matrix

$$A = \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \in H$$

carries $i \mapsto a^2 i + ba$ and one readily checks that the correspondence

$$A \mapsto a^2 i + ba$$

is a diffeomorphism between H and $\mathbf{H}$. The differential of this transformation sends

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix}_i \mapsto \begin{bmatrix} 0 \\ a^2 \end{bmatrix}_{a^2 i + ab},$$

imbedding H as the smooth section of $T^1(\mathbf{H})$ which, to each point $z \in \mathbf{H}$, assigns the unit tangent vector at z that points in the positive y direction. This is the unit velocity field for the family of parallel geodesics in $\mathbf{H}$ given by the vertical lines. Under our identification of $T^1(\mathbf{H})$ with $\mathrm{PSl}(2,\mathbb{R})$, the subgroup H "is" this section.

In order to see the cosets gH as sections of $T^1(\mathbf{H})$, we need to recall that the (oriented but otherwise unparametrized) geodesics in hyperbolic geometry are just the intersections of the open upper half plane with oriented circles that meet S_∞, the "circle at infinity", orthogonally. The circle S_∞ is the real axis, compactified by the point ∞. A geodesic, "extended to infinity", meets this circle orthogonally in exactly two points, its "source" and its "target". The geodesics with target $\infty \in S_\infty$ are just the vertical straight line segments, oriented in the positive y direction. This will be called

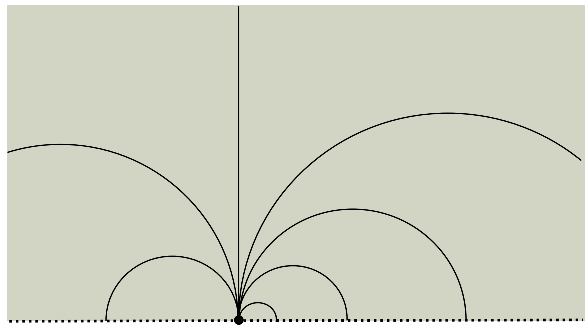

Figure 1.1.4. The pencil $\mathcal{P}_x$

the "geodesic pencil through ∞" and denoted by $\mathcal{P}_\infty$. In Figure 1.1.4, we draw the pencil $\mathcal{P}_x$ of geodesics with finite target $x \in \mathbb{R} \subset S_\infty$. The linear fractional transformation $z \mapsto g(z)$, defined by $g \in \mathrm{PSl}(2, \mathbb{R})$, extends by the same formula to the circle at infinity and, being an isometry of $\mathbf{H}$, it sends $\mathcal{P}_x$ to $\mathcal{P}_{g(x)}$. The subgroup H is exactly the stabilizer of $\infty \in S_\infty$, hence is exactly the subgroup that preserves $\mathcal{P}_\infty$. It follows that the elements of the coset gH are exactly the elements of $\mathrm{Iso}_+(\mathbf{H})$ that send $\mathcal{P}_\infty$ to $\mathcal{P}_x$, where $x = g(\infty)$. Viewing H as the section of $T^1(\mathbf{H})$ consisting of the unit velocity vectors along the pencil $\mathcal{P}_\infty$, we see that gH is identified as the section consisting of the unit velocity vectors along the pencil $\mathcal{P}_x$ (Figure 1.1.5). Every $x \in S_\infty$ occurs, and so we have completely described the leaves of $\mathcal{H}$ in $\mathrm{PSl}(2, \mathbb{R})$.

This foliation is carried to a foliation $\mathcal{H}_\Gamma$ of a compact manifold

$$M = \Gamma \backslash \mathrm{PSl}(2, \mathbb{R})$$

as follows. The compact, orientable surface Σ_g of genus $g > 1$ has a Riemannian metric of constant curvature -1, hence has universal cover $\mathbf{H}$, the group Γ of covering transformations being a group of orientation–preserving isometries of $\mathbf{H}$. This is the subgroup $\Gamma \subset \mathrm{PSl}(2, \mathbb{R})$ and M is the total space of the unit tangent bundle $T^1(\Sigma_g)$, a circle bundle over the surface. The foliation $\mathcal{H}$ is invariant under left translations by Γ, hence descends to a foliation $\mathcal{H}_\Gamma$, the leaves of which are locally sections of $T^1(\Sigma_g)$. That is, this foliation is transverse to the circle fibers (*cf.* Definition 1.1.11). It can be shown that each leaf of $\mathcal{H}_\Gamma$ is dense in M, that exactly a countable infinity of them are diffeomorphic to the cylinder and that all the other leaves are diffeomorphic to the plane.

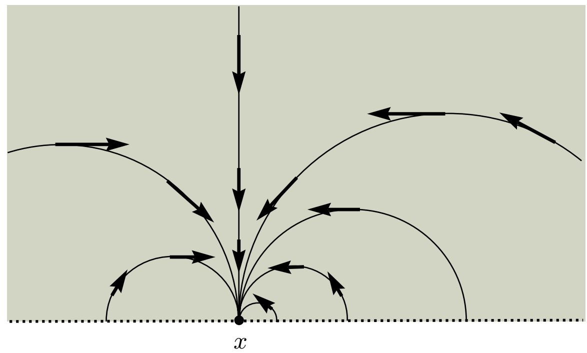

Figure 1.1.5. The section of unit velocity vectors along $\mathcal{P}_x$

Exercise 1.1.10. You are to prove that the group $\mathrm{PSl}(2,\mathbb{R})$, acting as linear fractional transformations of the upper half plane, is exactly the group $\mathrm{Iso}_+(\mathbf{H})$ and that its action on $T^1(\mathbf{H})$, via the differentials of these linear fractional transformations, is simply transitive. Here is a suggested way to proceed:

1. Show that the 2–dimensional subgroup $H \subset \mathrm{PSl}(2,\mathbb{R})$ (as defined in Example 1.1.9) and the 1–dimensional subgroup
$$L = \left\{ \pm \begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} \right\}_{c\in\mathbb{R}}$$
both determine linear fractional transformations of $\mathbf{H}$ that preserve the hyperbolic metric and the orientation of $\mathbf{H}$.
2. Show that L and H generate $\mathrm{PSl}(2,\mathbb{R})$, concluding that this is a subgroup of $\mathrm{Iso}_+(\mathbf{H})$.
3. Show that the action of H on $\mathbf{H}$ is *transitive,* hence that $\mathbf{H}$ is a Riemannian homogeneous space. A standard theorem in Riemannian geometry asserts that such homogeneous spaces are *complete* in their metric. Use these facts to show that an arbitrary isometry $\varphi : \mathbf{H} \to \mathbf{H}$ is uniquely determined by $\varphi(i)$ and φ_{*i}.
4. Show that the subgroup of $\mathrm{PSl}(2,\mathbb{R})$ that fixes i is exactly
$$\mathrm{PSO}(2) = \mathrm{SO}(2)/\{\pm I\}.$$
Use this and the above to prove both that each $\varphi \in \mathrm{Iso}_+(\mathbf{H})$ is an element of $\mathrm{PSl}(2,\mathbb{R})$ and that this group acts simply transitively on $T^1(\mathbf{H})$.

One also allows foliations of manifolds with boundary. If N is a component of ∂M, we demand either that $\mathcal{F}$ be *transverse* to N or *tangent* to N.

Definition 1.1.11. Let $N \subset M$ be a smooth submanifold. We say that $\mathcal{F}$ is transverse to N (and write $\mathcal{F} \pitchfork N$) if, for each leaf L of $\mathcal{F}$ and each point $x \in L \cap N$, $T_x(L)$ and $T_x(N)$ together span $T_x(M)$. At the other extreme, we say that $\mathcal{F}$ is tangent to N if, for each leaf L of $\mathcal{F}$, either $L \cap N = \emptyset$ or $L \subseteq N$.

Remark. We will see that, whenever $\mathcal{F} \pitchfork N$, there is naturally induced on N a foliation of the same codimension as $\mathcal{F}$, each leaf of which is a path component of $N \cap L$, L some leaf of $\mathcal{F}$.

Example 1.1.12. Let

$$D^n = \{v \in \mathbb{R}^n \mid \|v\| \leq 1\},$$

the unit disk in $\mathbb{R}^n$. The manifold $M = D^n \times S^1$ (called a *solid torus*) has boundary $\partial M = S^{n-1} \times S^1$. The foliation $\mathcal{F}_0$ of M by leaves $D^n \times \{y\}$, $y \in S^1$, is transverse to ∂M. In this case, the foliation is modeled locally on the foliation of Euclidean half space

$$\mathbb{H}^{n+1} = \{(x^1, \dots, x^{n+1}) \mid x^1 \leq 0\}$$

by the subspaces parallel to

$$\mathbb{H}^n = \{(x^1, \dots, x^n, 0) \mid x^1 \leq 0\}.$$

This is a fairly trivial example, but it leads to a more interesting foliation, one tangent to the boundary, by a process of "spinning" the leaves asymptotically along ∂M. This is the famous *Reeb foliation* of the solid torus.

The easiest way to visualize the Reeb foliation is to lift it to the solid cylinder $\widetilde{M} = D^n \times \mathbb{R}$ via the universal covering map. The submersion

$$f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R},$$
$$f(r, z, t) = (r^2 - 1)e^t$$

of Example 1.1.2 restricts to a submersion

$$f : D^n \times \mathbb{R} \to \mathbb{H}^1.$$

Here, $0 \leq r \leq 1$, and so $f \leq 0$. The boundary $S^{n-1} \times \mathbb{R}$ is the level set $f = 0$ and each level $f = c < 0$ is a "cup–shaped" manifold diffeomorphic to $\mathbb{R}^n$. The resulting foliation $\widetilde{\mathcal{F}}$ is tangent to $\partial\widetilde{M}$. The cases $n = 1, 2$ are illustrated in Figure 1.1.6.

Translation in the t–coordinate

$$(r, z, t) \mapsto (r, z, t + a)$$

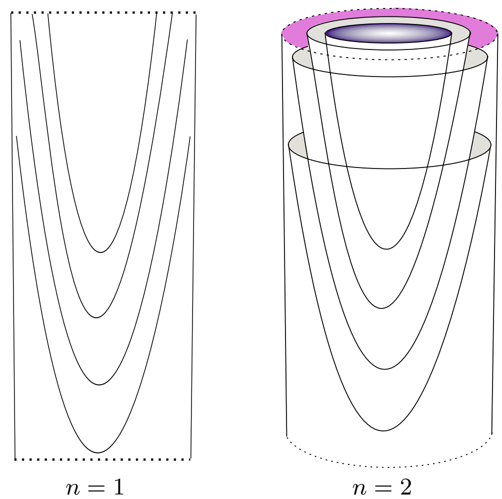

Figure 1.1.6. The foliation $\widetilde{\mathcal{F}}$ of $D^n \times \mathbb{R}$

carries the level set $f = c$ diffeomorphically onto the level set $f = ce^a$. This carries the boundary $f = 0$ diffeomorphically onto itself and, if $a \neq 0$, moves every interior leaf to a different interior leaf. In particular, the $\mathbb{Z}$–action

$$\mathbb{Z} \times (D^n \times \mathbb{R}) \to D^n \times \mathbb{R},$$
$$(k, (r, z, t)) \mapsto (r, z, t + k)$$

preserves $\widetilde{\mathcal{F}}$, inducing a foliation $\mathcal{F}$ on the quotient space

$$D^n \times (\mathbb{R}/\mathbb{Z}) = D^n \times S^1 = M.$$

Each boundary leaf of $\widetilde{\mathcal{F}}$ quotients to a component of ∂M and each interior leaf of $\widetilde{\mathcal{F}}$ is carried diffeomorphically onto an interior leaf of $\mathcal{F}$. The interior leaves still look like cup–shaped Euclidean spaces, but they now wind around inside the solid torus like infinite snakes repeatedly swallowing their own tails (and everyone else's tails as well). These leaves all wind out asymptotically toward the boundary $S^{n-1} \times S^1$. The Reeb foliation of the solid torus is not defined by a submersion, but the foliation of the *interior* $M_0 = \operatorname{int} D^n \times S^1$ is defined by a submersion $g : M_0 \to S^1$ (Exercise 1.1.13). The case $n = 2$ is illustrated in Figure 1.1.7.

Exercise 1.1.13. Let $\mathcal{G}$ be the foliation of $M_0 = \operatorname{int} D^n \times S^1$ obtained by restricting the Reeb foliation (Example 1.1.12) to M_0. Prove that $\mathcal{G}$ is defined by a submersion $g : M_0 \to S^1$. On the other hand, show that, if a submersion f of a manifold to a 1–manifold has a compact level set $f^{-1}(a)$, then every level set that comes sufficiently close to $f^{-1}(a)$ is also compact. Conclude that the Reeb foliation itself cannot be defined by a submersion.

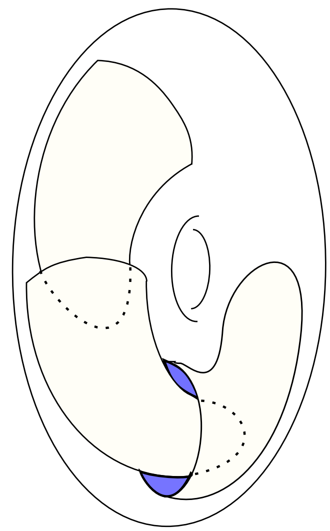

Figure 1.1.7. The Reeb foliation of $D^2 \times S^1$

The submersion $f : D^n \times \mathbb{R} \to \mathbb{H}^1$ admits a deformation f_s, $0 \le s \le 1$, through submersions

$$f_s : D^n \times \mathbb{R} \to \mathbb{H}^1,$$
$$f_s(r, z, t) = f(sr, z, t).$$

Let $\widetilde{\mathcal{F}}_s$ denote the foliation of $D^n \times \mathbb{R}$ defined by f_s. We view this as a deformation of $\widetilde{\mathcal{F}}_0$ to $\widetilde{\mathcal{F}}_1 = \widetilde{\mathcal{F}}$. This deformation projects in a well–defined way to a deformation $\mathcal{F}_s$ of $\mathcal{F}_0$ to $\mathcal{F}_1 = \mathcal{F}$. It is clear that $\mathcal{F}_0$ is the foliation of the solid torus M by the disks $D^n \times \{x\}$, the foliation transverse to ∂M with which we began this example. The intermediate foliations $\mathcal{F}_s$, $0 < s < 1$, are diffeomorphic to $\mathcal{F}_0$, but "look" more and more like $\mathcal{F}_1$ as $s \to 1$. This deformation is one interpretation of "spinning" the leaves of $\mathcal{F}_0$ along ∂M.

Example 1.1.14. Let $\Sigma = \Sigma_2$ denote the 2–holed torus, $C_1, C_2 \subset \Sigma$ the two imbedded circles depicted in Figure 1.1.8. Let $\mathcal{F}$ be the product foliation of the 3–manifold $M = \Sigma \times S^1$ with leaves $\Sigma \times \{y\}$, $y \in S^1$. Note that $N_i = C_i \times S^1$ is an imbedded torus and that $\mathcal{F} \pitchfork N_i$, $i = 1, 2$. Let $\mathrm{Diff}_+(S^1)$ denote the group of orientation–preserving diffeomorphisms of S^1 and choose $f_1, f_2 \in \mathrm{Diff}_+(S^1)$. Cut M apart along N_1 and N_2, letting N_i^+ and N_i^- denote the resulting copies of N_i, $i = 1, 2$. At this point we have a manifold $M' = \Sigma' \times S^1$ with four boundary components $\{N_i^{\pm}\}_{i=1,2}$. The foliation $\mathcal{F}$ has passed to a foliation $\mathcal{F}' \pitchfork \partial M'$, each leaf of which is of the form $\Sigma' \times \{y\}$, $y \in S^1$ (Figure 1.1.9).

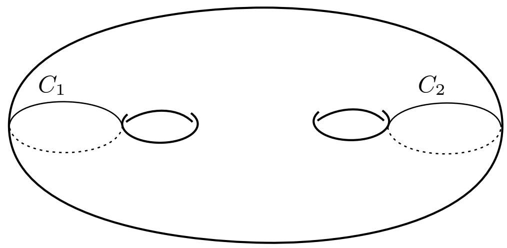

Figure 1.1.8. The 2–holed torus Σ

This leaf meets $\partial M'$ in four circles $C_i^{\pm} \times \{y\} \subset N_i^{\pm}$. If $z \in C_i$, we denote the corresponding points in $C_i^{\pm}$ by $z^{\pm}$ and "reglue" N_i^- to N_i^+ by the identification

$$(z^-, y) \equiv (z^+, f_i(y)), \quad i = 1, 2.$$

Since f_1 and f_2 are orientation–preserving diffeomorphisms of S^1, they are isotopic to the identity and the manifold obtained by this regluing operation is homeomorphic to M (Exercise 1.1.15). The leaves of $\mathcal{F}'$, however, reassemble to produce a new foliation $\mathcal{F}(f_1, f_2)$ of M. If a leaf L of $\mathcal{F}(f_1, f_2)$ contains a piece $\Sigma' \times \{y_0\}$, then

$$L = \bigcup_{g \in G} \Sigma' \times \{g(y_0)\},$$

where $G \subset \operatorname{Diff}_+(S^1)$ is the subgroup generated by $\{f_1, f_2\}$. These copies of Σ' are attached to one another by identifications

$$(z^-, g(y_0)) \equiv (z^+, f_1(g(y_0))), \qquad \forall\, z \in C_1,$$
$$(z^-, g(y_0)) \equiv (z^+, f_2(g(y_0))), \qquad \forall\, z \in C_2,$$

where g ranges over G. The leaf is completely determined by the G–orbit of $y_0 \in S^1$ and can be simple or immensely complicated. For instance, a leaf will be compact precisely if the corresponding G–orbit is finite. As an extreme example, if G is trivial ($f_1 = f_2 = \mathrm{id}_{S^1}$), then $\mathcal{F}(f_1, f_2) = \mathcal{F}$. If an orbit is dense in S^1, the corresponding leaf is dense in M. As an example, if f_1 and f_2 are rotations through rationally independent multiples of 2π, every leaf will be dense. In other examples, which we will see later, some leaf L has closure $\overline{L}$ that meets each factor $\{w\} \times S^1$ in a Cantor set. Similar constructions can be made on $\Sigma \times I$, where I is a compact, nondegenerate interval. Here we take $f_1, f_2 \in \operatorname{Diff}_+(I)$ and, since ∂I is fixed pointwise by all orientation–preserving diffeomorphisms, we get a foliation having the

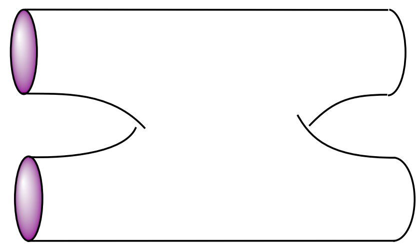

Figure 1.1.9. The typical leaf $\Sigma' \times \{y\}$ of $\mathcal{F}'$

two components of ∂M as leaves. Note that, when we form M' in this case, we get a foliated manifold with corners. In either case, this construction is called the *suspension* of a pair of diffeomorphisms and is a fertile source of interesting examples of codimension–one foliations.

Exercise 1.1.15. Let $f \in \text{Diff}_+(S^1)$ and show that there is a smooth homotopy f_t, such that $f_1 = f$, $f_0 = \text{id}$, and $f_t \in \text{Diff}_+(S^1)$, $0 \le t \le 1$. (We say that f is *isotopic* to the identity.) Use this to justify the assertion in Example 1.1.14 that the regluing of M' described there gives a manifold homeomorphic to M. (In fact, the reglued manifold is *diffeomorphic* to M when its differentiable structure is appropriately defined.)

These examples were somewhat informal. We turn to the precise definitions. In what follows, the symbol $\mathbb{F}^p$ denotes either the full Euclidean space $\mathbb{R}^p$ or Euclidean half space

$$\mathbb{H}^p = \{(x^1, x^2, \dots, x^p) \mid x^1 \le 0\}.$$

Definition 1.1.16. A *rectangular neighborhood* in $\mathbb{F}^n$ is an open subset of the form $B = J_1 \times \cdots \times J_n$, where each J_i is a (possibly unbounded) relatively open interval in the ith coordinate axis. If J_1 is of the form $(a, 0]$, we say that B has boundary

$$\partial B = \{(0, x^2, \dots, x^n) \in B\}.$$

In the following, we will consider coordinate charts that have values in $\mathbb{F}^{n-q} \times \mathbb{F}^q$, allowing the possibility of manifolds with boundary and (convex) corners.

Definition 1.1.17. Let M be an n–manifold. A *foliated chart* on M of codimension q is a pair (U, φ), where $U \subseteq M$ is open and $\varphi : U \to B_\tau \times B_\pitchfork$ is a diffeomorphism, $B_\pitchfork$ being a rectangular neighborhood in $\mathbb{F}^q$ and B_τ

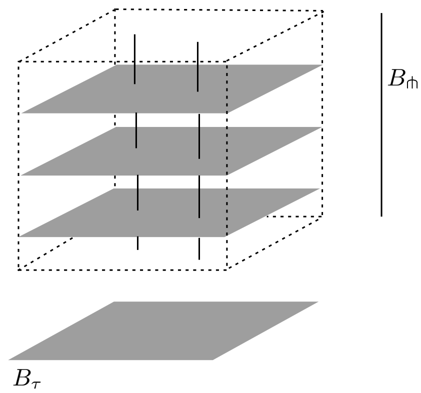

Figure 1.1.10. A 3–dimensional foliated chart

a rectangular neighborhood in $\mathbb{F}^{n-q}$. The set $P_y = \varphi^{-1}(B_\tau \times \{y\})$, where $y \in B_\pitchfork$, is called a *plaque* of this foliated chart. For each $x \in B_\tau$, the set $S_x = \varphi^{-1}(\{x\} \times B_\pitchfork)$ is called a *transversal* of the foliated chart. The set $\partial_\tau U = \varphi^{-1}(B_\tau \times (\partial B_\pitchfork))$ is called the *tangential boundary* of U and $\partial_\pitchfork U = \varphi^{-1}((\partial B_\tau) \times B_\pitchfork)$ is called the *transverse boundary* of U.

In Figure 1.1.10, we show a foliated chart with $n = 3$ and $q = 1$. The plaques are 2–dimensional and the transversals are 1–dimensional.

Remark. A foliated chart will be the basic model for all foliations, the plaques being the leaves. One should read B_τ as "B–tangential" and $B_\pitchfork$ as "B–transverse". One should also pay attention to various possibilities. If both $B_\pitchfork$ and B_τ have empty boundary, the foliated chart models codimension–q foliations of n–manifolds without boundary (Figure 1.1.10). If one, but not both of these rectangular neighborhoods has boundary, the foliated chart models the various possibilities for foliations of n–manifolds with boundary and without corners. Specifically, if $\partial B_\pitchfork \neq \emptyset = \partial B_\tau$, then $\partial U = \partial_\tau U$ is a union of plaques and the foliation by plaques is tangent to the boundary. If $\partial B_\tau \neq \emptyset = \partial B_\pitchfork$, then $\partial U = \partial_\pitchfork U$ is a union of transversals and the foliation is transverse to the boundary. For the case $n = 3$ and $q = 1$, these possibilities are illustrated in Figures 1.1.11 and 1.1.12. Finally, if $\partial B_\pitchfork \neq \emptyset \neq \partial B_\tau$, we obtain a model of a foliated manifold with a corner separating the tangential boundary from the transverse boundary (Figure 1.1.13).

Definition 1.1.18. Let M be an n–manifold, possibly with boundary and corners, and let $\mathcal{F} = \{L_\lambda\}_{\lambda \in \mathfrak{L}}$ be a decomposition of M into connected, topologically immersed submanifolds of dimension $k = n - q$. Suppose that M admits an atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathfrak{A}}$ of foliated charts of codimension q such

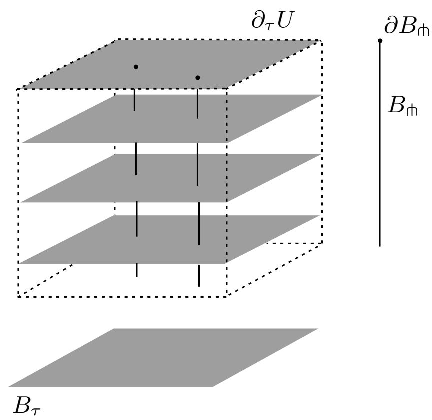

Figure 1.1.11. Foliation tangent to the boundary ($\partial B_\pitchfork \neq \emptyset = \partial B_\tau$)

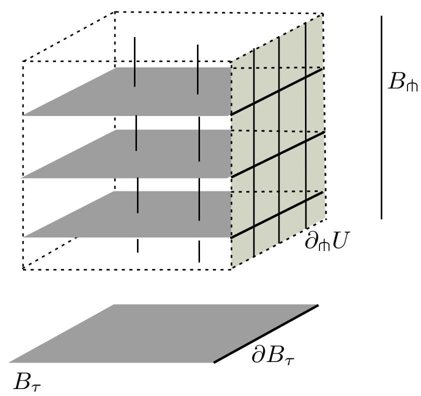

Figure 1.1.12. Foliation transverse to the boundary ($\partial B_\tau \neq \emptyset = \partial B_\pitchfork$)

that, for each $\alpha \in \mathfrak{A}$ and each $\lambda \in \mathfrak{L}$, $L_\lambda \cap U_\alpha$ is a union of plaques. Then $\mathcal{F}$ is said to be a foliation of M of codimension q (and dimension k) and $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathfrak{A}}$ is called a foliated atlas associated to $\mathcal{F}$. Each L_λ is called a leaf of the foliation and the pair $(M, \mathcal{F})$ is called a foliated manifold. If the foliated atlas is of class C^r $(0 \leq r \leq \infty$ or $r = \omega)$, then the foliation $\mathcal{F}$ and the foliated manifold $(M, \mathcal{F})$ are said to be of class C^r.

Remarks. (1) Implicit in this definition is the fact that plaques in distinct foliated charts intersect (if at all) in open subsets of one another. There is no such compatibility requirement for the transversals in distinct foliated charts.

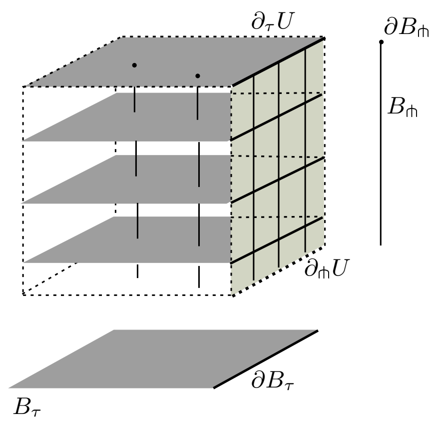

Figure 1.1.13. Foliation with corner ($\partial B_\tau \neq \emptyset \neq \partial B_\pitchfork$)

(2) By a *decomposition* of a set X into a family of subsets $\{X_\lambda\}_{\lambda \in \mathfrak{L}}$, we mean that X is the disjoint union of these subsets. Equivalently, a decomposition is an equivalence relation on X with equivalence classes X_λ. In the case of a foliation $\mathcal{F}$, we can describe this equivalence relation as follows. Let $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathfrak{A}}$ be an associated foliated atlas. If $x, y \in M$, we write $x \sim y$ if and only if there is a finite sequence $\{P_{\alpha_i} \subset U_{\alpha_i}\}_{i=0}^p$ of plaques such that $x \in P_{\alpha_0}$, $y \in P_{\alpha_p}$, and $P_{\alpha_{i-1}} \cap P_{\alpha_i} \neq \emptyset$, $1 \leq i \leq p$. It is trivial that this is an equivalence relation on M. The fact that the equivalence classes are exactly the leaves of $\mathcal{F}$ is left to the reader. In particular, the equivalence relation depends only on $\mathcal{F}$, not on the choice of the associated foliated atlas. Finally, if $\mathcal{F}$ is of class C^r, each leaf is a C^r–immersed submanifold (Exercise 1.1.19).

Exercise 1.1.19. Let L be a leaf in a foliated manifold $(M, \mathcal{F})$ of class C^r. Show that L, in its manifold topology, has a natural C^r structure relative to which the inclusion map $i : L \hookrightarrow M$ is a C^r immersion.

Definition 1.1.20. Let $(M, \mathcal{F})$ be a foliated manifold and let $\{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathfrak{A}}$ be an associated foliated atlas. Then the *tangential boundary* of M is

$$\partial_\tau M = \bigcup_{\alpha \in \mathfrak{A}} \partial_\tau U_\alpha$$

and the *transverse boundary* is

$$\partial_\pitchfork M = \bigcup_{\alpha \in \mathfrak{A}} \partial_\pitchfork U_\alpha.$$

Evidently, $\partial_\tau M$ is a union of leaves of $\mathcal{F}$ and $\mathcal{F} \pitchfork \partial_\pitchfork M$. If $\partial_\tau M \neq \emptyset$ and $\partial_\pitchfork M \neq \emptyset$, then the corners of M are the components (if any) of $\partial_\tau M \cap \partial_\pitchfork M$.

While foliated manifolds with corners are not the primary focus of this book, they occur as intermediate steps in certain constructions of foliated manifolds without corners.

1.2. Foliated Atlases

An equivalent way to define a foliation is to give only the foliated atlas, without reference to a decomposition of M into leaves. Since we have only defined the term "foliated atlas" with reference to such a decomposition, we will need a general definition of this term. In the following, r is allowed to take all values $0 \leq r \leq \infty$ or $r = \omega$.

Definition 1.2.1. A foliated atlas of codimension q and class C^r on the n–manifold M is a C^r–atlas $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathfrak{A}}$ of foliated charts of codimension q which are *coherently foliated* in the sense that, whenever P and Q are plaques in distinct charts of $\mathcal{U}$, then $P \cap Q$ is open both in P and Q.

Example 1.2.2. Let $f : \mathbb{R}^{n-1} \to \mathbb{R}$ be smooth and, for each $c \in \mathbb{R}$, set $f_c = f + c$. Let

$$L_c = \{(x^1, \dots, x^{n-1}, f_c(x^1, \dots, x^{n-1}))\}_{(x^1,\dots,x^{n-1}) \in \mathbb{R}^{n-1}},$$

the graph of f_c in $\mathbb{R}^n$. To simplify notation, set $x = (x^1, \dots, x^{n-1})$ and $y = x^n$, the last coordinate in $\mathbb{R}^n$. Then,

$$\varphi : \mathbb{R}^n \to \mathbb{R}^n,$$
$$\varphi(x, y) = (x, y - f(x))$$

is a diffeomorphism with inverse

$$\varphi^{-1}(x, y) = (x, y + f(x)).$$

Since

$$\varphi(L_c) = \{(x, c)\}_{x \in \mathbb{R}^{n-1}}, \quad \forall\, c \in \mathbb{R},$$

the single coordinate chart $(\mathbb{R}^n, \varphi)$ is a foliated atlas having the graphs L_c as plaques. Since there is only one foliated chart, the condition that any two of the charts are coherently foliated is satisfied vacuously. In this example, the codimension is 1.

Example 1.2.3. For a more interesting case, we return to Example 1.1.5. For simplicity, assume that $\partial M = \emptyset$ and let $X \in \mathfrak{X}(M)$ be a nonsingular vector field. This field is not necessarily complete, hence may not generate a global flow. For each $z \in M$, we can find a coordinate chart $(U, x^1, y^1, \dots, y^{n-1})$ about z in which $X|U = \partial/\partial x^1$. We can choose U so that this $\mathbb{R}^n$–chart is rectangular, obtaining a foliated chart of codimension $n - 1$, where $x = x^1$ ranges over B_τ and $y = (y^1, \dots, y^{n-1})$ ranges over $B_\pitchfork$. The 1–dimensional plaques, defined by $y \equiv$ const., are local flow lines of X. We cover M with such charts, obtaining an atlas $\mathcal{U}$. By the uniqueness of solutions of ordinary differential equations, it follows that the intersection of any two local flow lines is an open subset of each, and so $\mathcal{U}$ is a foliated atlas. Intuitively, it should follow that the full system of plaques links together into a family of connected, immersed submanifolds of dimension 1 which will be the leaves of a foliation. This is not immediately obvious, the main problem being to show that these "leaves" are 2nd countable. In Example 1.1.5, we bypassed the technical problems by assuming that the vector field was complete. The discussion in this section will make clear how the 2nd countability of leaves is forced by the 2nd countability of M.

Example 1.2.4. View $\mathbb{R}^2$ as an additive group, fix $\rho \in \mathbb{R}$, and consider the subgroup $\Lambda_\rho \subset \mathbb{R}^2$ generated by $(0,1)$ and $(1,-\rho)$. These vectors are linearly independent. So $\Lambda_\rho \cong \mathbb{Z}^2$ is a full lattice subgroup and $M = \mathbb{R}^2/\Lambda_\rho$ is diffeomorphic to the torus T^2. The corresponding projection $\mathbb{R}^2 \to M$ is a covering map and, if $B = J_1 \times J_2 \subset \mathbb{R}^2$ is a small enough open rectangular neighborhood, it projects diffeomorphically onto an open set $U \subset M$. The inverse $\varphi : U \to B$ of this diffeomorphism defines a coordinate chart (U, φ) which we view as a foliated chart (U, x, y) of codimension 1. Cover M by an atlas $\{U_\alpha, x_\alpha, y_\alpha\}_{\alpha\in\mathfrak{A}}$ consisting entirely of charts constructed this way. In Exercise 1.2.5, you are asked to verify directly that this is a foliated atlas of class C^ω and is associated to a foliation $\mathcal{F}_\rho$ of M. In Exercise 1.2.6, you will show that there is an analytic diffeomorphism $\theta : M \to T^2$ carrying $\mathcal{F}_\rho$ onto the linear foliation of slope ρ (*cf.* Example 1.1.5).

Exercise 1.2.5. Consider the atlas $\{U_\alpha, \varphi_\alpha\}_{\alpha\in\mathfrak{A}}$ on $M = \mathbb{R}^2/\Lambda_\rho$, constructed as in Example 1.2.4 relative to a choice of $\rho \in \mathbb{R}$. Show that this is a foliated atlas of codimension 1 and of class C^ω. Verify directly (without appeal to the results of this section) that this foliated atlas is associated to a foliation $\mathcal{F}_\rho$ of M by 1–dimensional leaves.

Exercise 1.2.6. If $M = \mathbb{R}^2/\Lambda_\rho$ is constructed as in Example 1.2.4, find an analytic diffeomorphism $\theta : M \to T^2$ that carries $\mathcal{F}_\rho$ to the linear foliation of T^2 of slope ρ as constructed in Example 1.1.5.

Exercise 1.2.7. We say that a nonsingular (*i.e.,* nowhere vanishing) 1–form $\omega \in A^1(M)$ *defines* a foliation $\mathcal{F}$ of codimension 1 if it vanishes exactly on vectors tangent to the leaves of $\mathcal{F}$. By the previous exercise, view the foliation $\mathcal{F}_\rho$ as a foliation of T^2 and show that it is defined by a *closed* nonsingular 1–form ω_ρ on T^2. The *periods* of this closed form are the numbers that can be obtained by line integrals $\int_\sigma \omega_\rho$ over piecewise smooth closed loops σ on T^2. Show that the set of periods is the additive subgroup $G_\rho \subset \mathbb{R}$ generated by 1 and ρ.

Remarks. Let $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}_{\alpha\in\mathfrak{A}}$ be a C^r foliated atlas of codimension q on the n–manifold M. For $w \in U_\alpha \cap U_\beta$, write

$$\varphi_\alpha(w) = (x_\alpha(w), y_\alpha(w)) \in B^\alpha_\tau \times B^\alpha_\pitchfork,$$
$$\varphi_\beta(w) = (x_\beta(w), y_\beta(w)) \in B^\beta_\tau \times B^\beta_\pitchfork.$$

We often write $(U_\alpha, x_\alpha, y_\alpha)$ for $(U_\alpha, \varphi_\alpha)$, with

$$x_\alpha = (x^1_\alpha, \dots, x^{n-q}_\alpha),$$
$$y_\alpha = (y^1_\alpha, \dots, y^q_\alpha).$$

On $\varphi_\beta(U_\alpha \cap U_\beta)$, consider the change of coordinates formula

$$g_{\alpha\beta}(x_\beta, y_\beta) = \varphi_\alpha \circ \varphi_\beta^{-1}(x_\beta, y_\beta) = (x_\alpha(x_\beta, y_\beta), y_\alpha(x_\beta, y_\beta)).$$

The condition that $(U_\alpha, x_\alpha, y_\alpha)$ and $(U_\beta, x_\beta, y_\beta)$ be coherently foliated means that, if $P \subset U_\alpha$ is a plaque, the connected components of $P \cap U_\beta$ lie in (possibly distinct) plaques of U_β. Equivalently, since the plaques of U_α and U_β are level sets of the transverse coordinates y_α and y_β, respectively, each point $z \in U_\alpha \cap U_\beta$ has a neighborhood in which the formula

$$y_\alpha = y_\alpha(x_\beta, y_\beta) = y_\alpha(y_\beta)$$

is independent of x_β. This is a useful way to reformulate the notion of "coherently foliated" charts.

1.2.A. Coherence and regular foliated atlases. As already mentioned, one wants to link overlapping plaques of a foliated atlas to form the leaves of a foliation. For this and many other purposes, the general definition of "foliated atlas" is a bit clumsy. One problem is that a plaque of $(U_\alpha, \varphi_\alpha)$ can meet multiple plaques of (U_β, φ_β) (Figure 1.2.1). It can even happen that a plaque of one chart meets infinitely many plaques of another chart. We are going to show that no generality is lost in assuming the situation to be much more regular.

Definition 1.2.8. Two foliated atlases $\mathcal{U}$ and $\mathcal{V}$ on M of the same codimension and smoothness class C^r are *coherent* ($\mathcal{U} \approx \mathcal{V}$) if $\mathcal{U} \cup \mathcal{V}$ is a foliated C^r–atlas.

Lemma 1.2.9. *Coherence of foliated atlases is an equivalence relation.*

Proof. Reflexivity and symmetry are immediate. Let $\mathcal{U} \approx \mathcal{V}$ and $\mathcal{V} \approx \mathcal{W}$. Let $(U_\alpha, x_\alpha, y_\alpha) \in \mathcal{U}$ and $(W_\lambda, x_\lambda, y_\lambda) \in \mathcal{W}$ and suppose that there is a point $w \in U_\alpha \cap W_\lambda$. Choose $(V_\delta, x_\delta, y_\delta) \in \mathcal{V}$ such that $w \in V_\delta$. By the above remarks, there is a neighborhood N of w in $U_\alpha \cap V_\delta \cap W_\lambda$ such that

$$y_\delta = y_\delta(y_\lambda) \quad \text{on } \varphi_\lambda(N),$$
$$y_\alpha = y_\alpha(y_\delta) \quad \text{on } \varphi_\delta(N),$$

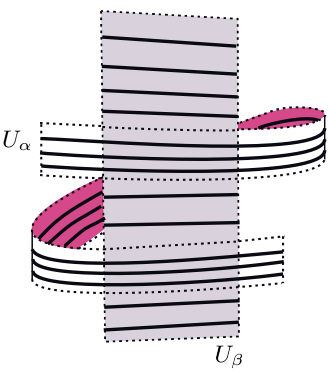

Figure 1.2.1. Plaques of U_α each meet two plaques of U_β.

and hence

$$y_\alpha = y_\alpha(y_\delta(y_\lambda)) \quad \text{on } \varphi_\lambda(N).$$

Since $w \in U_\alpha \cap W_\lambda$ is arbitrary, we conclude that $y_\alpha(x_\lambda, y_\lambda)$ is locally independent of x_λ. We have proven that $\mathcal{U} \approx \mathcal{W}$, hence that coherence is transitive. □

Lemma 1.2.10. *Let $\mathcal{U}$ and $\mathcal{V}$ be foliated atlases on M and suppose that $\mathcal{U}$ is associated to a foliation $\mathcal{F}$. Then $\mathcal{U}$ and $\mathcal{V}$ are coherent if and only if $\mathcal{V}$ is also associated to $\mathcal{F}$.*

Proof. If $\mathcal{V}$ is also associated to $\mathcal{F}$, every leaf L is a union of $\mathcal{V}$–plaques and of $\mathcal{U}$–plaques. These plaques are open subsets in the manifold topology of L, hence intersect in open subsets of each other. Since plaques are connected, a $\mathcal{U}$–plaque cannot intersect a $\mathcal{V}$–plaque unless they lie in a common leaf; so the foliated atlases are coherent.

Conversely, if we only know that $\mathcal{U}$ is associated to $\mathcal{F}$ and that $\mathcal{V} \approx \mathcal{U}$, let Q be a $\mathcal{V}$–plaque. If L is a leaf of $\mathcal{F}$ and $w \in L \cap Q$, let $P \subset L$ be a $\mathcal{U}$–plaque with $w \in P$. Then $P \cap Q$ is an open neighborhood of w in Q and $P \cap Q \subset L \cap Q$. Since $w \in L \cap Q$ is arbitrary, it follows that $L \cap Q$ is open in Q. Since L is an arbitrary leaf, it follows that Q decomposes into disjoint open subsets, each of which is the intersection of Q with some leaf of $\mathcal{F}$. Since Q is connected, $L \cap Q = Q$. Finally, Q is an arbitrary $\mathcal{V}$–plaque, and so $\mathcal{V}$ is associated to $\mathcal{F}$. □

Remark. Let (U, φ) and (W, ψ) be foliated charts such that $\overline{U} \subset W$ and $\varphi = \psi | U$. Then, if $\varphi(U) = B_\tau \times B_\pitchfork$, we see that $\psi|\overline{U}$, written $\overline{\varphi}$, carries $\overline{U}$

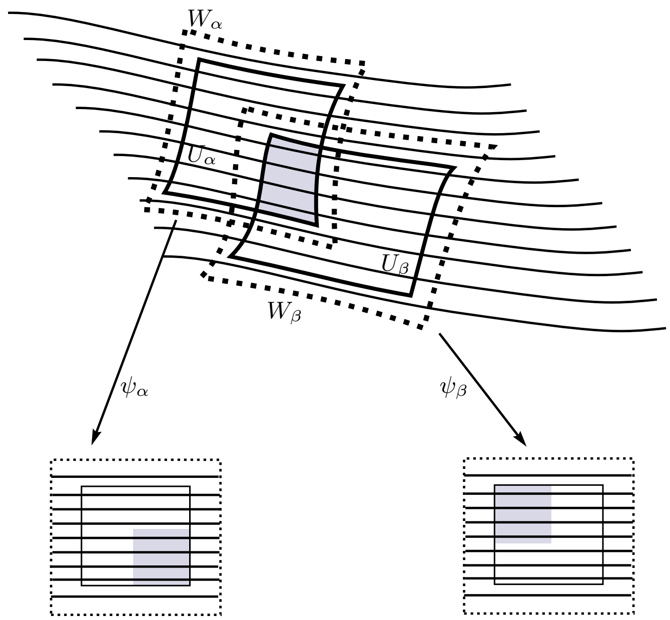

Figure 1.2.2. Sample charts in a regular foliated atlas

diffeomorphically onto $\overline{B}_\tau \times \overline{B}_\pitchfork$. Thus, we can speak of the (closed) plaques and transversals of $\overline{U}$.

Definition 1.2.11. A foliated atlas $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}_{\alpha\in\mathfrak{A}}$ of class C^r is said to be *regular* if

1. for each $\alpha \in \mathfrak{A}$, $\overline{U}_\alpha$ is a compact subset of a foliated chart (W_α, ψ_α) and $\varphi_\alpha = \psi_\alpha|U_\alpha$;
2. the cover $\{U_\alpha\}_{\alpha\in\mathfrak{A}}$ is locally finite;
3. if $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β) are elements of $\mathcal{U}$, then the interior of each closed plaque $P \subset \overline{U}_\alpha$ meets at most one plaque in $\overline{U}_\beta$.

Remarks. Properties (1) and (3) in this definition are illustrated in Figure 1.2.2. By (1), the coordinates x_α and y_α extend to coordinates $\overline{x}_\alpha$ and $\overline{y}_\alpha$ on $\overline{U}_\alpha$ and we write $\overline{\varphi}_\alpha = (\overline{x}_\alpha, \overline{y}_\alpha)$. Property (3) is equivalent to requiring that, if $U_\alpha \cap U_\beta \neq \emptyset$, the transverse coordinate changes

$$\overline{y}_\alpha = \overline{y}_\alpha(\overline{x}_\beta, \overline{y}_\beta)$$

be independent of $\overline{x}_\beta$. That is,

$$\overline{g}_{\alpha\beta} = \overline{\varphi}_\alpha \circ \overline{\varphi}_\beta^{-1} : \overline{\varphi}_\beta(\overline{U}_\alpha \cap \overline{U}_\beta) \to \overline{\varphi}_\alpha(\overline{U}_\alpha \cap \overline{U}_\beta)$$

has the formula

$$\overline{g}_{\alpha\beta}(\overline{x}_\beta, \overline{y}_\beta) = (\overline{x}_\alpha(\overline{x}_\beta, \overline{y}_\beta), \overline{y}_\alpha(\overline{y}_\beta)).$$

A fortiori, similar assertions hold without the overlines. Note that the transverse coordinate map y_α can be viewed as a submersion

$$y_\alpha : U_\alpha \to \mathbb{F}^q = \mathbb{R}^q \text{ or } \mathbb{H}^q$$

and that the formulas $y_\alpha = y_\alpha(y_\beta)$ can be viewed as diffeomorphisms

$$\gamma_{\alpha\beta} : y_\beta(U_\alpha \cap U_\beta) \to y_\alpha(U_\alpha \cap U_\beta).$$

These satisfy the *cocycle conditions.* That is, on $y_\delta(U_\alpha \cap U_\beta \cap U_\delta)$,

(1.2.1) $$\gamma_{\alpha\delta} = \gamma_{\alpha\beta} \circ \gamma_{\beta\delta}$$

and, in particular,

(1.2.2) $$\gamma_{\alpha\alpha} = \mathrm{id}_{y_\alpha(U_\alpha)},$$

(1.2.3) $$\gamma_{\alpha\beta} = \gamma_{\beta\alpha}^{-1}.$$

Definition 1.2.12. The set $\gamma = \{\gamma_{\alpha\beta}\}_{\alpha,\beta\in\mathfrak{A}}$ is the *holonomy cocycle* of the regular foliated atlas.

Remark. The holonomy cocycle plays a central role in foliation theory. Note that, by property (1) in the definition of regular foliated atlas, the set $y_\beta(U_\alpha \cap U_\beta)$ has compact closure in $\mathbb{F}^q$ and, by property (3), that $\gamma_{\alpha\beta}$ has a well–defined, smooth (of class C^r) extension $\overline{\gamma}_{\alpha\beta}$ to this compact set. In particular, the component functions of $\gamma_{\alpha\beta}$ and their derivatives of all orders $\leq r$ are bounded functions.

Exercise 1.2.13. Choosing the atlas in Exercise 1.2.5 with a little care, show that it can be assumed to be regular. Show that the associated holonomy cocycle consists of transformations of the form

$$\gamma_{\alpha\beta}(y) = y + c_{\alpha\beta}$$

and that this regular atlas can be chosen so that the numbers $c_{\alpha\beta}$ generate G_ρ.

Definition 1.2.14. Let $(M, \mathcal{F})$ be smooth of class C^1 and let $\{\gamma_{\alpha\beta}\}_{\alpha\beta\in\mathfrak{A}}$ be the associated holonomy cocycle of a regular foliated atlas for $\mathcal{F}$. We say that $\mathcal{F}$ is *transversely orientable* if $\det J\gamma_{\alpha\beta} > 0$, $\forall\, \alpha, \beta \in \mathfrak{A}$, where $J\gamma$ denotes the Jacobian matrix of the C^1 diffeomorphism γ.

Exercise 1.2.15. Let $(M, \mathcal{F})$ be C^1, let $T(\mathcal{F}) \subset T(M)$ be the subbundle tangent to the leaves, and define the *normal bundle* of $\mathcal{F}$ to be the quotient bundle $Q = T(M)/T(\mathcal{F})$ over M. Prove that $\mathcal{F}$ is transversely orientable if and only if Q is an orientable bundle. In case $\mathcal{F}$ is smooth of codimension one, show that this is equivalent to the existence of a nonsingular 1–form that defines $\mathcal{F}$ (*cf.* Exercise 1.2.7). In particular, transverse orientability is independent of the choice of regular foliated atlas for $\mathcal{F}$.

Lemma 1.2.16. *Every regular foliated atlas of codimension q is associated to a unique foliation $\mathcal{F}$ of codimension q.*

Proof. Let $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}_{\alpha \in \mathfrak{A}}$ be a regular foliated atlas of codimension q. Define an equivalence relation on M by setting $x \sim y$ if and only if either there is a $\mathcal{U}$–plaque P_0 such that $x, y \in P_0$ or there is a sequence $\{P_0, P_1, \dots, P_p\}$ of $\mathcal{U}$–plaques such that

$$\begin{aligned} &x \in P_0, \\ &y \in P_p, \\ &P_i \cap P_{i-1} \neq \emptyset, \qquad 1 \le i \le p. \end{aligned}$$

This will be called a *plaque chain of length p* connecting x and y. In the case that $x, y \in P_0$, we say that $\{P_0\}$ is a plaque chain of length 0 connecting x and y. The fact that $\sim$ is an equivalence relation is clear. It is also clear that each equivalence class L is a union of plaques. Since $\mathcal{U}$–plaques can only overlap in open subsets of each other, L is locally a topologically immersed submanifold of dimension $n - q$. The open subsets of the plaques $P \subset L$ form the base of a locally Euclidean topology on L of dimension $n - q$ and L is clearly connected in this topology. It is also trivial to check that L is Hausdorff. The main problem is to show that L is 2nd countable. Since each plaque is 2nd countable, the same will hold for L if we show that the set of $\mathcal{U}$–plaques in L is at most countably infinite. Fix one such plaque P_0. By the definition of a regular, foliated atlas, P_0 meets only finitely many other plaques. That is, there are only finitely many plaque chains $\{P_0, P_1\}$ of length 1. By induction on the length p of plaque chains that begin at P_0, we prove similarly that there are only finitely many of length $\le p$. Since every $\mathcal{U}$–plaque in L is, by the definition of $\sim$, reached by a finite plaque chain starting at P_0, the assertion follows. □

Lemma 1.2.17. *Every foliated atlas has a coherent refinement that is regular.*

Proof. Fix a metric on M and a foliated atlas $\mathcal{W}$. Consider first the case that M is compact. Passing to a subcover, if necessary, we can assume that $\mathcal{W} = \{W_j, \psi_j\}_{j=1}^{\ell}$ is finite. Let $\epsilon > 0$ be a Lebesgue number for $\mathcal{W}$. That is, any subset $X \subseteq M$ of diameter $< \epsilon$ lies entirely in some W_j. For each $x \in M$, choose j such that $x \in W_j$ and choose a foliated chart (U_x, φ_x) such that

$$\begin{gathered} x \in U_x \subseteq \overline{U}_x \subset W_j, \\ \varphi_x = \psi_j | U_x, \\ \operatorname{diam}(U_x) < \epsilon/2. \end{gathered}$$

Suppose that $U_x \subset W_k$, $k \neq j$, and write $\psi_k = (x_k, y_k)$ as usual, where

$$y_k : W_k \to \mathbb{F}^q$$

is the transverse coordinate map. This is a submersion having the plaques in W_k as level sets. Thus, y_k restricts to a submersion

$$y_k : U_x \to \mathbb{F}^q.$$

This is locally constant in x_j; so choosing U_x smaller, if necessary, we can assume that $y_k|\overline{U}_x$ has the plaques of $\overline{U}_x$ as its level sets. That is, each plaque of W_k meets (hence contains) at most one (compact) plaque of $\overline{U}_x$. Since $1 \leq k \leq \ell < \infty$, we can choose U_x so that, whenever $U_x \subset W_k$, distinct plaques of $\overline{U}_x$ lie in distinct plaques of W_k. Pass to a finite subatlas $\mathcal{U} = \{U_i, \varphi_i\}_{i=1}^N$ of $\{U_x, \varphi_x\}_{x \in M}$. If $U_i \cap U_j \neq \emptyset$, then $\operatorname{diam}(U_i \cup U_j) < \epsilon$, and so there is an index k such that $\overline{U}_i \cup \overline{U}_j \subseteq W_k$. Distinct plaques of $\overline{U}_i$ (respectively, of $\overline{U}_j$) lie in distinct plaques of W_k. Hence each plaque of $\overline{U}_i$ has interior meeting at most one plaque of $\overline{U}_j$ and *vice versa.* By construction, $\mathcal{U}$ is a coherent refinement of $\mathcal{W}$ and is a regular foliated atlas.

If M is not compact, local compactness and 2nd countability allows us to choose a sequence $\{K_i\}_{i=0}^\infty$ of compact subsets such that

$$K_i \subset \operatorname{int} K_{i+1}, \qquad \forall\, i \geq 0,$$
$$M = \bigcup_{i=0}^\infty K_i.$$

Passing to a subatlas, we assume that

$$\mathcal{W} = \{W_j, \psi_j\}_{j=0}^\infty$$

is countable and we find a strictly increasing sequence $\{n_\ell\}_{\ell=0}^\infty$ of positive integers such that

$$\mathcal{W}_\ell = \{W_j, \psi_j\}_{j=0}^{n_\ell}$$

covers K_ℓ. Let δ_ℓ denote the distance from K_ℓ to $\partial K_{\ell+1}$ and choose $\epsilon_\ell > 0$ so small that

$$\epsilon_\ell < \min\left\{\frac{\delta_\ell}{2}, \epsilon_{\ell-1}\right\}, \qquad \ell \geq 1,$$
$$\epsilon_0 < \frac{\delta_0}{2},$$

and ϵ_ℓ is a Lebesgue number for $\mathcal{W}_\ell$ (as an open cover of K_ℓ) and for $\mathcal{W}_{\ell+1}$ (as an open cover of $K_{\ell+1}$). More precisely, if $X \subset M$ meets K_ℓ (respectively, $K_{\ell+1}$) and $\operatorname{diam} X < \epsilon_\ell$, then X lies in some element of $\mathcal{W}_\ell$ (respectively, $\mathcal{W}_{\ell+1}$). For each $x \in K_\ell \smallsetminus \operatorname{int} K_{\ell-1}$, construct (U_x, φ_x) as for the compact case, requiring that $\overline{U}_x$ be a compact subset of W_j and that $\varphi_x = \psi_j|U_x$, some $j \leq n_\ell$. Also, require that $\operatorname{diam} \overline{U}_x < \epsilon_\ell/2$. As before, pass to a finite subcover $\{U_i, \varphi_i\}_{i=n_{\ell-1}+1}^{n_\ell}$ of $K_\ell \smallsetminus \operatorname{int} K_{\ell-1}$. (Here, we take $n_{-1} = 0$.)

This creates a regular foliated atlas $\mathcal{U} = \{U_i, \varphi_i\}_{i=1}^{\infty}$ that refines $\mathcal{W}$ and is coherent with $\mathcal{W}$. □

The following is now obvious.

Theorem 1.2.18. *The correspondence between foliations on M and their associated foliated atlases induces a one–to–one correspondence between the set of foliations on M and the set of coherence classes of foliated atlases.*

We now have an alternative definition of the term "foliation".

Definition 1.2.19. A foliation $\mathcal{F}$ of codimension q and class C^r on M is a coherence class of foliated atlases of codimension q and class C^r on M.

By Zorn's lemma, it is obvious that every coherence class of foliated atlases contains a unique maximal foliated atlas. Thus,

Definition 1.2.20. A foliation of codimension q and class C^r on M is a maximal foliated C^r–atlas of codimension q on M.

In practice, we generally use a relatively small foliated atlas to represent a foliation. Usually, we also require this atlas to be regular.

Example 1.2.21. A C^r foliation $\mathcal{F}$ induces a C^r foliation $\mathcal{F}|\partial_\pitchfork M$ on $\partial_\pitchfork M$ having the same codimension q as $\mathcal{F}$. The leaves of this foliation are the connected components of the intersections $L \cap \partial_\pitchfork M$, as L ranges over the leaves of $\mathcal{F}$. Indeed, if $\{U_\alpha, x_\alpha, y_\alpha\}_{\alpha \in \mathfrak{A}}$ is a regular foliated atlas for $\mathcal{F}$, let

$$\mathfrak{A}' = \{\alpha \in \mathfrak{A} \mid W_\alpha = U_\alpha \cap \partial_\pitchfork M \neq \emptyset\}.$$

By notational abuse, let

$$\begin{aligned} x_\alpha &= x_\alpha | W_\alpha, \\ y_\alpha &= y_\alpha | W_\alpha, \end{aligned}$$

$\forall\, \alpha \in \mathfrak{A}'$. It is then elementary to check that $\{W_\alpha, x_\alpha, y_\alpha\}_{\alpha \in \mathfrak{A}'}$ is a regular foliated atlas of codimension q and smoothness class C^r. Evidently, the leaves of the associated C^r foliation of $\partial_\pitchfork M$ are as asserted. Similarly, $\mathcal{F}$ induces a C^r–foliation $\mathcal{F}|\partial_\tau M$ on $\partial_\tau M$ of codimension $q - 1$.

In the case of foliations $\mathcal{F}$ of codimension 1, the components of $\partial_\tau M$ are leaves of $\mathcal{F}$ and the corresponding foliation $\mathcal{F}|\partial_\tau M$ is of little interest, being of codimension 0.

1.2.B. Foliations of class C^0. The case $r = 0$ is rather special. Those C^0 foliations that arise in practice are usually "smooth–leaved". More precisely, they are of class $C^{r,0}$, in the following sense.

Definition 1.2.22. A foliation $\mathcal{F}$ is of class $C^{r,k}$, $r > k \geq 0$, if the corresponding coherence class of foliated atlases contains a regular foliated atlas $\{U_\alpha, x_\alpha, y_\alpha\}_{\alpha\in\mathfrak{A}}$ such that the change of coordinate formula

$$g_{\alpha\beta}(x_\beta, y_\beta) = (x_\alpha(x_\beta, y_\beta), y_\alpha(y_\beta))$$

is of class C^k, but x_α is of class C^r in the coordinates x_β and its mixed x_β partials of orders $\leq r$ are C^k in the coordinates (x_β, y_β).

Remark. Definition 1.2.22 suggests the more general concept of a *foliated space* or *abstract lamination* $(M, \mathcal{F})$. One relaxes the condition that the transversals be open, relatively compact subsets of $\mathbb{R}^q$, allowing the transverse coordinates y_α to take their values in some more general topological space Z. The plaques are still open, relatively compact subsets of $\mathbb{R}^p$, the change of transverse coordinate formula $y_\alpha(y_\beta)$ is continuous and $x_\alpha(x_\beta, y_\beta)$ is of class C^r in the coordinates x_β and its mixed x_β partials of orders $\leq r$ are continuous in the coordinates (x_β, y_β). One usually requires M and Z to be locally compact, second countable and metrizable. This may seem like a rather wild generalization, but there are contexts in which it is useful. In this book, foliated spaces will become important in Part 3.

Example 1.2.23. For foliations of class $C^{r,0}$ and, more generally, for foliated spaces $(M, \mathcal{F})$, the holonomy cocycle is C^0, but the individual leaves come with a C^r structure. The tangent spaces to the leaves are well defined and form a continuous vector bundle $T(\mathcal{F})$ on M, called the tangent bundle of $\mathcal{F}$. Indeed, this bundle is determined by the continuous structure cocycle

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{Gl}(p),$$
$$g_{\alpha\beta}(x_\beta, y_\beta) = \left[\frac{\partial x_\alpha^i}{\partial x_\beta^j}(x_\beta, y_\beta)\right].$$

In the case of a $C^{r,0}$–foliated manifold, the atlas in Definition 1.2.22 only establishes M as a C^0 manifold and one does not have a tangent bundle $T(M)$. Even if there is a differentiable structure on M, there seems to be no natural way to identify $T(\mathcal{F})$ as a p–plane subbundle of $T(M)$. The following slightly stronger requirement on $\mathcal{F}$ remedies this.

Definition 1.2.24. Let $\mathcal{F}$ be of class $C^{r,0}$ on a manifold M. If M has a differentiable structure relative to which each leaf is C^1–immersed and the resulting inclusion $T(\mathcal{F}) \hookrightarrow T(M)$ imbeds $T(\mathcal{F})$ as a C^0 p–plane subbundle of $T(M)$, then $(M, \mathcal{F})$ is said to be of class $C^{r,0+}$. A foliation is said to be *integral to a* C^0 *plane field* if and only if it is of class at least $C^{1,0+}$.

Example 1.2.25. Let $(M, \mathcal{F})$ be a foliated n–manifold of codimension 1 and suppose that M is smooth and $\mathcal{F}$ is integral to a C^0 hyperplane field $E \subset T(M)$. It follows that $\mathcal{F}$ is of class $C^{\infty,0+}$. We outline the main ideas here, leaving details until Section 5.1. A Riemannian metric on M defines a C^0 line field $E^\perp$ which, by a small homotopy, becomes a smooth subbundle of $T(M)$ transverse to E. With care, one can guarantee that this line field is tangent to $\partial_\pitchfork M$. By the Frobenius theorem (Theorem 1.3.8), smooth line fields on M are integrable, giving rise to a smooth, 1–dimensional foliation $\mathcal{L}$ of M that is transverse to $\mathcal{F}$ and tangent to $\partial_\pitchfork M$. In this situation, it is possible to choose an atlas $\{U_\alpha, x_\alpha, y_\alpha\}_{\alpha\in\mathfrak{A}}$ that is simultaneously a regular foliated atlas for $\mathcal{F}$ and $\mathcal{L}$. That is, x_α is $\mathbb{R}^{n-1}$–valued, y_α is $\mathbb{R}$–valued, the level sets of x_α are plaques of $\mathcal{L}$ and the level sets of y_α are plaques of $\mathcal{F}$. The change of coordinates on $U_\alpha \cap U_\beta$ takes the form

$$x_\alpha = x_\alpha(x_\beta),$$
$$y_\alpha = y_\alpha(y_\beta),$$

where $x_\alpha(x_\beta)$ is of class C^∞ but $y_\alpha(y_\beta)$ may only be continuous. This proves that the foliated manifold is of class $C^{\infty,0}$. The hypothesis that $\mathcal{F}$ is integral to E means that each leaf of $\mathcal{F}$ is a C^1–immersed submanifold of the smooth manifold M, the vector bundle $E \subset T(M)$ being identified via these immersions with $T(\mathcal{F})$. Furthermore, if the holonomy cocycle $\gamma_{\alpha\beta}(y_\beta) = y_\alpha(y_\beta)$ is of class C^r, $1 \le r \le \infty$, $\forall\, \alpha, \beta \in \mathfrak{A}$, then $(M, \mathcal{F})$ is a C^r–foliated manifold.

1.2.C. Transverse structures. Let $\gamma = \{\gamma_{\alpha\beta}\}_{\alpha,\beta\in\mathfrak{A}}$ be the holonomy cocycle associated to a regular foliated atlas on M and let $\mathcal{F}$ denote the corresponding foliation. The cocycle consists of C^r diffeomorphisms between open subsets of $\mathbb{R}^q$. Suppose that $\mathcal{S}$ is a geometric structure on an open subset U of $\mathbb{R}^q$ containing the domains and ranges of all $\gamma_{\alpha\beta} \in \gamma$ and let $G \subseteq \operatorname{Diff}^r(U)$ be the group of automorphisms of $\mathcal{S}$.

Definition 1.2.26. If each $\gamma_{\alpha\beta} \in \gamma$ is the restriction to $\operatorname{dom}\gamma_{\alpha\beta}$ of an element of G, then the holonomy cocycle γ is said to be a transverse G–structure for the foliated manifold $(M, \mathcal{F})$.

We have already seen an example of this in Definition 1.2.14. A transversely orientable foliation is one admitting a transverse G–structure with $G = \operatorname{Diff}^r_+(U)$.

Example 1.2.27. Let $\mathcal{S}$ be a Riemannian metric on an open subset $U \subseteq \mathbb{R}^q$, $G = \operatorname{Iso}(U)$ the group of isometries of this metric. A foliated manifold $(M, \mathcal{F})$ admitting a transverse $\operatorname{Iso}(U)$ structure is said to be (transversely) Riemannian. While every q–manifold admits a Riemannian metric, transverse Riemannian structures for foliations are rare, imposing rigid symmetries on the foliated manifold. The theory of Riemannian foliations is highly developed [**91**] and very interesting, but will not be pursued in this book.

Example 1.2.28. Another important example is the class of (transversely) Lie foliations. Let G be a q–dimensional Lie group. Using local coordinate charts on G, one can suppose that the domain $y_\beta(U_\alpha \cap U_\beta)$ of $\gamma_{\alpha\beta}$ is identified with an open subset $V_{\alpha\beta} \subset G$, $\forall\, \alpha, \beta \in \mathfrak{A}$. If this can be done in such a way that each $\gamma_{\alpha\beta}$ is identified with (the restriction to $V_{\alpha\beta}$ of) left translation in G by an element of G, the foliation is (transversely) Lie, modeled on the Lie group G. Again, this places an enormous restriction on the foliated manifold $(M, \mathcal{F})$. Remark that there is a left–invariant Riemannian metric on G and that this induces a transversely Riemannian structure on $(M, \mathcal{F})$.

Example 1.2.29. Similarly, one can define (transversely) homogeneous foliations modeled on a q–dimensional homogeneous space G/K. Here, the domains $V_{\alpha\beta}$ of the cocycle elements $\gamma_{\alpha\beta}$ are identified with open subsets of G/K in such a way that $\gamma_{\alpha\beta}$ becomes (the restriction to $V_{\alpha\beta}$ of) a left translation in G/K by an element of G.

By this point, readers can extend the list of examples for themselves. Every example of a geometric structure on q–manifolds yields an analogous transverse structure for foliated manifolds of codimension q. This important topic is one of many that will not be pursued systematically in this book. The interested reader is referred to [**52**, Chapitre III].

1.3. The Frobenius Theorem

Foliations are sometimes given by infinitesimal data, either by a C^∞ k–plane distribution $E \subseteq T(M)$ or by the annihilator ideal $I^*(E) \subseteq A^*(M)$ of such a distribution. This infinitesimal data amounts to the global analogue of a system of first–order partial differential equations of constant rank. Generally, such a system is *overdetermined* and corresponds to a foliation if and only if it is *completely integrable.* The integrability condition is the classical result known as the Frobenius theorem. Here, we only review this theorem. Proofs will be found, for example, in [**11**, pp. 159–161], [**29**, pp. 118–123 and pp. 178–180], [**15**, Appendix], [**126**, pp. 130–136], and [**147**, pp. 42–45].

Let E be a k–plane distribution on a C^∞ n–manifold M. That is, E is the total space of a C^∞, k–dimensional subbundle of the tangent bundle $\pi : T(M) \to M$. A C^∞ foliation $\mathcal{F}$ of dimension k gives rise to such a distribution, the vector subspace $E_x \subseteq T_x(M)$ being the tangent space to the leaf of $\mathcal{F}$ through x, $\forall\, x \in M$. We have called E the *tangent bundle to the foliation* and have denoted it by $T(\mathcal{F})$ (*cf.* Example 1.2.23). We are interested in the inverse problem: given the k–plane distribution E, can we find a foliation $\mathcal{F}$ with k–dimensional leaves such that $E = T(\mathcal{F})$?

Example 1.3.1. Let $E \subset T(\mathbb{R}^3)$ be a 2–plane distribution spanned by C^∞ vector fields of the form

$$X = \frac{\partial}{\partial x} + g(x,y)\frac{\partial}{\partial z},$$
$$Y = \frac{\partial}{\partial y} + h(x,y)\frac{\partial}{\partial z}.$$

Let $\rho : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection $\rho(x,y,z) = (x,y)$ and assume that $E = T(\mathcal{F})$ for some 2–dimensional foliation $\mathcal{F}$. Since

$$\rho_{*(x,y,z)}(X_{(x,y,z)}) = \left.\frac{\partial}{\partial x}\right|_{(x,y)},$$
$$\rho_{*(x,y,z)}(Y_{(x,y,z)}) = \left.\frac{\partial}{\partial y}\right|_{(x,y)},$$

the inverse function theorem implies that a leaf of $\mathcal{F}$ through any point (x_0,y_0,z_0) has a neighborhood of (x_0,y_0,z_0) that is carried by ρ diffeomorphically onto an open subset of $\mathbb{R}^2$. That is, a leaf of $\mathcal{F}$ is locally a graph $z = f(x,y)$ of a C^∞ function f and

$$X_{(x,y,f(x,y))} = \begin{bmatrix} 1 \\ 0 \\ g(x,y) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x}(x,y) \end{bmatrix},$$
$$Y_{(x,y,f(x,y))} = \begin{bmatrix} 0 \\ 1 \\ h(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y}(x,y) \end{bmatrix}.$$

The locally–defined function $f(x,y)$ solves the (overdetermined) system of partial differential equations

$$\frac{\partial f}{\partial x} = g(x,y),$$
$$\frac{\partial f}{\partial y} = h(x,y).$$

This implies that

$$\frac{\partial g}{\partial y} = \frac{\partial^2 f}{\partial y\,\partial x} = \frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial h}{\partial x}.$$

That is, a necessary condition for the existence of the foliation such that $E = T(\mathcal{F})$ is that

$$(*) \qquad \frac{\partial g}{\partial y} = \frac{\partial h}{\partial x}.$$

Conversely, this condition implies that $\omega = g\,dx + h\,dy$ is closed, hence locally exact. Local solutions of $df = \omega$ provide locally–defined functions f, unique up to an additive constant, whose graphs are local leaves tangent to E. In this way, one easily produces a foliated atlas on $\mathbb{R}^3$, hence a foliation $\mathcal{F}$ such that $E = T(\mathcal{F})$. Condition $(*)$, which we call an *integrability condition*, can also be written in terms of brackets. Indeed,

$$[X, Y] = \left(\frac{\partial h}{\partial x} - \frac{\partial g}{\partial y}\right)\frac{\partial}{\partial z},$$

and so equation $(*)$ becomes

$$[X, Y] = 0$$

(the fields *commute*). Arbitrary fields $Z_1, Z_2 \in \Gamma(E)$ (the space of C^∞ sections $s : \mathbb{R}^3 \to E$) are $C^\infty(\mathbb{R}^3)$–linear combinations of X and Y, and hence commutativity of X and Y implies that $[Z_1, Z_2] \in \Gamma(E)$. Our integrability condition becomes: *the space* $\Gamma(E)$ *of* C^∞ *sections of* E *is a Lie subalgebra of* $\mathfrak{X}(\mathbb{R}^3)$*.* This exemplifies the *vector field* version of the Frobenius theorem.

Remark. In the above example, the words “local” and “locally” can be dropped throughout. One proves this using the simple connectivity of $\mathbb{R}^3$ and $\mathbb{R}^2$.

Throughout this section, all manifolds, distributions, vector fields, forms, *etc.,* will be of class C^∞. Thus, $\mathfrak{X}(M) = \Gamma(T(M))$ is closed under the bracket, and hence is a Lie algebra. Also, the exterior algebra $A^*(M)$ is closed under exterior differentiation, and hence is a differential graded algebra. For simplicity of exposition, we will also assume that $\partial M = \emptyset$, although the reader can easily adapt our treatment to the more general case.

Definition 1.3.2. A k–plane distribution E on the C^∞ manifold M is the total space of a C^∞, k–dimensional subbundle of the tangent bundle $T(M)$.

Definition 1.3.3. A k–plane distribution E on M is *involutive* if the space $\Gamma(E)$ of C^∞ sections of E is a Lie subalgebra of $\mathfrak{X}(M)$.

Definition 1.3.4. Let E be a k–plane distribution on M, $w \in M$. An *integral manifold* to E through w is a smoothly immersed submanifold N of M of dimension k such that $w \in N$ and $E_x = T_x(N)$, $\forall\, x \in N$.

Definition 1.3.5. A k–plane distribution E on M is *completely integrable* if, through each point $w \in M$, there passes an integral manifold to E.

In the following, $\Lambda^p(\Gamma(E))$ denotes the pth exterior power of $\Gamma(E)$ as a $C^\infty(M)$–module.

Definition 1.3.6. Let E be a k–plane distribution on M. The *annihilator* $I^*(E) \subseteq A^*(M)$ is the graded ideal $I^*(E) = \{I^p(E)\}_{p=0}^\infty$ defined by the condition: $\omega \in I^p(E) \Leftrightarrow \omega(v) \equiv 0$, $\forall\, v \in \Lambda^p(\Gamma(E))$.

Remark that $\Lambda^0(\Gamma(E)) = C^\infty(M)$, and hence $I^0(E) = \{0\}$.

Definition 1.3.7. A *differential graded ideal* $I^* \subseteq A^*(M)$ is a graded ideal such that $d(I^*) \subseteq I^*$.

Theorem 1.3.8 (Frobenius Theorem)**.** *For a k–plane distribution E on M, the following are equivalent*:

1. *E is completely integrable;*
2. *$I^*(E)$ is a differential graded ideal;*
3. *$d(I^1(E)) \subseteq I^2(E)$;*
4. *E is involutive;*
5. *there is a C^∞ foliated atlas on M of codimension $q = n - k$, every plaque of which is an integral manifold to E.*

Remarks. The implications (1) ⇒ (2), (2) ⇒ (3), and (5) ⇒ (1) are trivial. The implication (3) ⇒ (4) is an elementary consequence of the well–known formula

$$d\omega(X \wedge Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y]), \tag{1.3.1}$$

where $\omega \in A^1(M)$ and $X, Y \in \mathfrak{X}(M)$. The difficult step is to prove (4) ⇒ (5). As remarked earlier, this is a standard result in manifold theory and will not be proven in this book.

As it stands, the Frobenius theorem is a local result. But Theorem 1.2.18 and property (5) of Theorem 1.3.8 imply the following global consequence.

Corollary 1.3.9. *The k–plane distribution on M is completely integrable if and only if $E = T(\mathcal{F})$ for a C^∞ foliation $\mathcal{F}$ of codimension $q = n - k$.*

The following examples show some of the ways that the Frobenius theorem is used in foliation theory.

Example 1.3.10. We return to Example 1.1.8. If G is a Lie group and $H \hookrightarrow G$ is a connected Lie subgroup, let $\mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{g}$ be the corresponding Lie algebras of left–invariant vector fields on G. Then $\mathfrak{h}$ spans a distribution $E \subset T(G)$ and the fact that $\mathfrak{h}$ is closed under the Lie bracket implies that E is involutive. Hence $E = T(\mathcal{H})$ for a C^∞ foliation $\mathcal{H}$. Evidently, the distribution E is invariant under left translations in the group; so $\mathcal{H}$ is also invariant under left translations. It is standard from elementary Lie theory that the leaf through the identity $e \in G$ is a connected Lie group, which is easily seen to be an open subgroup of H. Since H is connected, it is the only open subgroup of itself; so the leaves of $\mathcal{H}$ are the left cosets of H.

Example 1.3.11. Let $\mathcal{F}$ be a C^∞, transversely–orientable foliation of codimension 1, $E = T(\mathcal{F})$, and let $\omega \in A^1(M)$ be a nonsingular form that defines $\mathcal{F}$ (*cf.,* Exercise 1.2.7). Thus, $\omega \in I^1(E)$ and, by Theorem 1.3.8, $d\omega \in I^2(E)$.

In Exercise 1.3.12, you will show that $I^*(E)$ is the principal ideal generated by ω; so there exists a form $\eta \in A^1(M)$ such that

$$d\omega = \omega \wedge \eta.$$

But

$$0 = d^2\omega = d(\omega \wedge \eta) = \omega \wedge \eta \wedge \eta - \omega \wedge d\eta = -\omega \wedge d\eta.$$

Since ω is nonsingular, it follows that there is a form $\gamma \in A^1(M)$ such that

$$d\eta = \omega \wedge \gamma.$$

Then

$$d(\eta \wedge d\eta) = d\eta \wedge d\eta = \omega \wedge \gamma \wedge \omega \wedge \gamma = 0.$$

The closed 3–form $\eta \wedge d\eta$ on M determines a de Rham cohomology class

$$\mathrm{gv}(\mathcal{F}) = [\eta \wedge d\eta] \in H^3(M),$$

called the *Godbillon–Vey class* of $\mathcal{F}$. Although ω is only determined by $\mathcal{F}$ up to multiplication by nowhere 0 functions $f \in C^\infty(M)$, and η is not uniquely determined by ω, you will show (Exercise 1.3.13) that $\mathrm{gv}(\mathcal{F})$ depends only on $\mathcal{F}$. Much has been learned about this "exotic characteristic class" since its introduction in the early 1970s [**53**], but the experts are still exploring its mysteries.

Exercise 1.3.12. Let $\omega \in A^1(M)$ be a nonsingular form and let E be the $(n-1)$–plane distribution annihilated by ω. Prove that $I^*(E)$ is the principal ideal in $A^*(M)$ generated by ω.

Exercise 1.3.13. Let $\omega \in A^1(M)$ be a nonsingular form defining the foliation $\mathcal{F}$.

1. If
$$d\omega = \omega \wedge \eta = \omega \wedge \widetilde{\eta},$$
prove that $\eta \wedge d\eta$ and $\widetilde{\eta} \wedge d\widetilde{\eta}$ are cohomologous 3–forms.
2. If $f \in C^\infty(M)$ is nowhere zero and
$$\widetilde{\omega} = f\omega,$$
$$d\omega = \omega \wedge \eta,$$
prove that
$$d\widetilde{\omega} = \widetilde{\omega} \wedge \widetilde{\eta},$$
where η and $\widetilde{\eta}$ differ by an exact form.
3. Using these computations, prove that $\mathrm{gv}(\mathcal{F})$ depends only on $\mathcal{F}$, not on the allowable choices of η and ω.

Example 1.3.14. Topologists were only convinced that $\mathrm{gv}(\mathcal{F})$ was an interesting characteristic class when foliations were found for which $\mathrm{gv}(\mathcal{F}) \neq 0$. The first example was due to R. Roussarie [**53**]. Later, in [**132**], W. Thurston showed that $\mathrm{gv}(\mathcal{F}) \in H^3(S^3) = \mathbb{R}$ takes all real values as $\mathcal{F}$ varies over all foliations of S^3 of codimension 1. We sketch the Roussarie example.

Let $M = \Gamma\backslash G$, where $G = \mathrm{PSl}(2, \mathbb{R})$ and Γ is a discrete, cocompact subgroup. As in Example 1.1.9, The foliation $\mathcal{H}$ of G by the left cosets of

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & a^{-1} \end{bmatrix} \right\}_{a>0}$$

passes to a foliation $\mathcal{H}_\Gamma$ of M.

Let $\mathfrak{sl}(2, \mathbb{R})$ denote the Lie algebra of $\mathrm{Sl}(2, \mathbb{R})$, hence also of G. This is identified with the vector space of 2×2 real matrices of trace 0, the bracket operation being given by the commutator product $[A, B] = AB - BA$. The Lie subalgebra $\mathfrak{h} \subset \mathfrak{sl}(2, \mathbb{R})$ of H has basis

$$X = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$Y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and this is completed to a basis of $\mathfrak{sl}(2, \mathbb{R})$ by adjoining

$$Z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The Lie brackets of these fields are

$$[X, Y] = 2Y,$$
$$[X, Z] = -2Z,$$
$$[Y, Z] = X.$$

Let X^*, Y^*, Z^* denote the dual basis. Since X, Y, Z can be viewed as left–invariant vector fields on G, the duals can be viewed as left–invariant 1–forms on G. Since X and Y span $\mathfrak{h}$, the form $\omega = Z^*$ defines the foliation of G by the left cosets of H. By Equation (1.3.1),

$$d\omega(X \wedge Y) = 0,$$
$$d\omega(X \wedge Z) = 2,$$
$$d\omega(Y \wedge Z) = 0.$$

It follows that

$$d\omega = 2X^* \wedge Z^* = \omega \wedge \eta,$$

where

$$\eta = -2X^*.$$

Similar computations to the above give

$$d\eta = 2Y^* \wedge Z^*,$$

and hence

$$\eta \wedge d\eta = -4X^* \wedge Y^* \wedge Z^*,$$

a nowhere vanishing 3–form on G. Since ω and η are invariant under left translations, they descend to well–defined forms on $M = \Gamma\backslash G$ which (by abuse) we continue to call ω and η. The form $\omega \in A^1(M)$ defines $\mathcal{H}_\Gamma$, $d\omega = \omega \wedge \eta$ and $\eta \wedge d\eta$ is a nowhere vanishing 3–form on M. But M is a compact, orientable 3–manifold without boundary. So

$$\int_M \eta \wedge d\eta \neq 0,$$

implying that

$$\mathrm{gv}(\mathcal{H}_\Gamma) = [\eta \wedge d\eta] \neq 0.$$

Exercise 1.3.15. If $\Gamma \subset \mathrm{PSl}(2,\mathbb{R})$ is a discrete subgroup such that $\mathbf{H}/\Gamma$ is a compact, orientable surface of genus $g \geq 2$, show that

$$\int_M \mathrm{gv}(\mathcal{H}_\Gamma) = \pm 4\pi(1-g).$$

(Hint. Show that $\pm 4X^* \wedge Y^* \wedge Z^*$ is the Riemannian volume form for a left–invariant metric hyperbolic along the leaves of $\mathcal{H}$.)

Exercise 1.3.16. You are to generalize the construction of $\mathrm{gv}(\mathcal{F})$ to foliations of higher codimension, proceeding as follows.

1. Let $\mathcal{F}$ be a smooth, transversely–oriented (Definition 1.2.14) foliation of codimension q tangent to the $(n-q)$–plane field E. Prove that there is a nonsinglar q–form ω on M such that, for each $x \in M$,

   $$\omega_x(v_1 \wedge \cdots \wedge v_q) = 0 \Leftrightarrow \text{at least one } v_i \in E_x.$$

   Clearly $\omega \in I^q(E)$ and we say that ω defines $\mathcal{F}$.
2. Prove that $d\omega = \omega \wedge \eta$, for some $\eta \in A^1(M)$.
3. Prove that $d\eta \in I^2(E)$ and that $\eta \wedge (d\eta)^q$ is a closed $(2q+1)$–form.
4. Show that $\mathrm{gv}(\mathcal{F}) = [\eta \wedge (d\eta)^q] \in H^{2q+1}(M;\mathbb{R})$ depends only on the foliation $\mathcal{F}$, not on the allowable choices of ω and η.

The class $\mathrm{gv}(\mathcal{F})$ is called the generalized Godbillon–Vey class of $\mathcal{F}$.

Example 1.3.17. Let $\omega \in A^1(M)$ be a closed, nonsingular 1–form on a compact, connected n–manifold M. Since the form is nonsingular, there is an $(n-1)$–plane distribution E such that ω generates the principal ideal $I^*(E)$ (Exercise 1.3.12). Since ω is closed,

$$d(f\omega) = df \wedge \omega;$$

so $d(I^1(E)) \subseteq I^2(E)$ and E is integrable. Let $\mathcal{F}$ be the corresponding foliation. Since ω is nonsingular, there is a vector field $X \in \mathfrak{X}(M)$ such that $\omega(X) > 0$ everywhere. We normalize X so as to assume $\omega(X) \equiv 1$ and let Φ_t denote the flow of X. This flow is transverse to $\mathcal{F}$. If $Y \in \Gamma(E)$, Equation (1.3.1) gives

$$0 = d\omega(X \wedge Y) = X \underbrace{\omega(Y)}_{\equiv 0} - Y \underbrace{\omega(X)}_{\equiv 1} - \omega([X,Y]) = -\omega([X,Y]),$$

implying that $[X, \Gamma(E)] \subseteq \Gamma(E)$. If L is a leaf of $\mathcal{F}$, the fact that L is $(n-1)$–dimensional and the flow is transverse to $\mathcal{F}$ implies that

$$V_L = \bigcup_{t\in\mathbb{R}} \Phi_t(L)$$

is an open subset of M. By Proposition 1.3.21 below, every leaf in V_L is diffeomorphic to L. We have proven that the diffeomorphism classes of leaves form open (necessarily disjoint) subsets of M. So the connectivity of M implies that all of the leaves of $\mathcal{F}$ are diffeomorphic. Finally, since $d\omega = 0$, we can take $\eta = 0$ in the definition of $\mathrm{gv}(\mathcal{F})$ and conclude that $\mathrm{gv}(\mathcal{F}) = 0$. By reasoning similar to that of Exercise 1.2.7, the foliations of T^n by cosets of connected, $(n-1)$–dimensional Lie subgroups are particular cases of this example.

Generally, one cannot expect a compact manifold to admit a closed, nowhere vanishing 1–form ω. For example, the presence of such a form implies that $H^1(M) \neq 0$. Indeed, every exact 1–form df must vanish at a maximum or minimum of $f \in C^\infty(M)$; so ω cannot be exact. In fact, a theorem of D. Tischler [**136**], which we will prove later (Theorem 9.4.2), asserts that M admits such a form ω if and only if M is the total space of a fiber bundle $\pi : M \to S^1$.

Exercise 1.3.18. Let ω, $\mathcal{F}$ and Φ_t be as in Example 1.3.17 and let $P(\omega) \subset \mathbb{R}$ be the group of periods of ω (*cf.* Exercise 1.2.7). Let L be an arbitrary leaf of $\mathcal{F}$ and show that $P(\omega)$ is exactly the set of real numbers t such that $\Phi_t(L) = L$. Proceed as follows.

1. Let L and L' be leaves of $\mathcal{F}$. Show that the set $P(L) \subset \mathbb{R}$ of numbers t such that $\Phi_t(L) = L$ is an additive subgroup of $\mathbb{R}$ and prove that $P(L) = P(L')$. Call this group P.
2. Show that an arbitrary piecewise C^∞ loop σ in M is homotopic to a piecewise C^∞ loop, each C^∞ segment of which lies either in a flow line of Φ_t or in a leaf of $\mathcal{F}$.
3. If $a \in P$, construct a piecewise C^∞ loop s such that $\int_s \omega = a$.

4. If s is a piecewise C^∞ loop and
$$\int_s \omega = a,$$
use step 2 to prove that $\Phi_a(L) = L$, for some leaf L of $\mathcal{F}$.

Exercise 1.3.19. Let M be a compact, connected n–manifold, $\mathcal{F}$ a foliation of codimension 1. Prove that $\mathcal{F}$ is defined by a closed, nonsingular 1–form ω if and only if there is a regular foliated atlas associated to $\mathcal{F}$ such that the associated holonomy cocycle $\gamma = \{\gamma_{\alpha\beta}\}_{\alpha,\beta\in\mathfrak{A}}$ is of the form
$$\gamma_{\alpha\beta}(y) = y + c_{\alpha\beta},$$
for suitable constants $c_{\alpha\beta} \in \mathbb{R}$. Show that this foliated atlas can be chosen so that the numbers $\{c_{\alpha\beta}\}_{\alpha,\beta\in\mathfrak{A}}$ generate the group $P(\omega)$.

Exercise 1.3.20. Let M be a compact connected n–manifold. A codimension 1 foliation $\mathcal{F}$ of M is said to be transversely affine if there is a regular foliated atlas associated to $\mathcal{F}$ whose holonomy cocycle $\gamma = \{\gamma_{\alpha\beta}\}$ is of the form
$$\gamma_{\alpha\beta}(y) = a_{\alpha\beta}y + b_{\alpha\beta},$$
where $a_{\alpha\beta} > 0$ and $b_{\alpha\beta}$ are constants.

1. If $\mathcal{F}$ is transversely affine, show that there is a 1–form ω defining $\mathcal{F}$, together with a closed 1–form η, such that $d\omega = \omega \wedge \eta$. Therefore, $\mathrm{gv}(\mathcal{F}) = 0$.
2. If M admits a transversely–affine foliation, then $H^1(M;\mathbb{R}) \neq 0$.

Remark. The converse of the first part of the above exercise is also true (for a proof of both implications, see [**52**, III-3.11]). These are exactly the foliations having a transverse G–structure (Subsection 1.2.C), where $G = \mathrm{Aff}^+(\mathbb{R})$ is the group of orientation–preserving affine transformations
$$t \mapsto at + b, \quad a > 0,$$
on the real line.

Here is the proposition that was used in Example 1.3.17.

Proposition 1.3.21. *Let Φ_t be a flow on the n–manifold M with infinitesimal generator $X \in \mathfrak{X}(M)$. Let $\mathcal{F}$ be a foliation (of codimension $q = n - k$) and let $E = T(\mathcal{F})$. Then, Φ_t carries leaves of $\mathcal{F}$ diffeomorphically onto leaves of $\mathcal{F}$ if and only if*
$$[X, \Gamma(E)] \subseteq \Gamma(E).$$

Proof. If Φ_t carries leaves to leaves, it follows that E is also invariant under the flow. Thus, if $Y \in \Gamma(E)$,
$$[X, Y] = \lim_{t\to 0} \frac{\Phi_{-t*}(Y) - Y}{t} \in \Gamma(E).$$

Conversely, if $[X, \Gamma(E)] \subseteq \Gamma(E)$ and if (U, x, y) is a foliated chart for $\mathcal{F}$, we will show that Φ_t carries the plaques $y =$ const. to other such plaques for all small values of t. The fact that Φ_t carries leaves to leaves for *all* values of t is an easy consequence. Write

$$X|U = \sum_{i=1}^{k} f_i \frac{\partial}{\partial x^i} + \sum_{j=1}^{q} g_j \frac{\partial}{\partial y^j}.$$

By hypothesis,

$$\left[\frac{\partial}{\partial x^i}, X\right] \in \Gamma(E|U), \quad 1 \leq i \leq k;$$

so $g_j(x, y) = g_j(y)$ is independent of x, $1 \leq j \leq q$. In U, the flow

$$\Phi_t(a, b) = (x(a, b, t), y(a, b, t))$$

satisfies the system

$$\begin{aligned} \frac{dx^i}{dt} &= f_i(x, y), \qquad 1 \leq i \leq k, \\ \frac{dy^j}{dt} &= g_j(y), \qquad 1 \leq j \leq q. \end{aligned}$$

Consequently, given the initial condition $(a, b) \in U$, the last q coordinates $y(a, b, t)$ of the flow line $\Phi_t(a, b)$ depend only on b and t. That is, the plaque $y = y(b, 0) = b$ is carried to the plaque $y = y(b, t)$. □