Preface

Since its beginning, the study of calculus has made advances by asking *what if* questions. Leibnitz in 1695 asked about the possible meaning of a half derivative. One hundred and forty years later, the nineteenth century saw the first definitions of general fractional derivatives that supported rigorous analysis. By the middle of the twentieth century, fractional calculus was already an almost fully developed field.

Physics often progresses in a similar way. In the early twentieth century, Einstein presented a mathematical foundation for Brownian motion in terms of a random walk model. This formulation resulted in a law that stated the mean square deviation from the starting position depends linearly on time. This ansatz held sway for over half a century, but soon thereafter examples were turning up of diffusion processes that were poorly described or led to conflicting consequences with this law. Then the question naturally arose of whether Einstein's formulation could be generalised to a fractional power (less than unity) of time, and what this would mean. As the story unfolded, the connection between the physical and mathematical *fractional* questions became apparent. As a consequence, *fractional diffusion* was now supported by an existing mathematical framework, courtesy of the *fractional derivatives*, and *fractional derivatives* now had a physical motivation for further study and transitioned from *pure* mathematics to *applied.*

The *lingua franca* of mathematical physics is2 partial differential equations, and the new modality requires a study of equations containing one or more fractional derivatives. This is one of the primary purposes of this textbook: we ask the usual questions of existence, uniqueness, and regularity of the solutions and try to compare them with the classical counterpart. The short answer is there is much in common, but with some significant and important differences as well as additional technical challenges. There is considerable emphasis in the book towards an analysis of the forwards or direct problem; we seek to determine the solution under the assumption that we know the exact form of the equation including any coefficients and domain boundaries and are given initial or boundary information. However, our main emphasis is on inverse problems. These occur when we have, say, an undetermined coefficient in the equation that we wish to recover together with the solution itself given additional, often boundary, information.

This brings up several questions, but a recurring theme is to answer what differences using the fractional differential equation paradigm bring over the classical one. In some cases these differences can be substantial and can bring in new consequences for the underlying physical model. In others the differences are relatively minor. What we will often find is that the stability of the solutions for integer order and for fractional order derivatives in terms of the given initial/boundary data can differ markedly. Not only does this say something about the effect of the models from a physics standpoint, but it sometimes suggests a way to stabilise a highly ill-conditioned classical inverse problem by approximating it with a "nearby" fractional one. Issues such as these form a major theme and even perhaps **the** primary focus of the second part of the book. We have taken a broad perspective on the types of inverse problems. The reader will see ones where the underlying model is based on elliptic, parabolic, and hyperbolic equations together with what might be viewed as fractional counterparts to these classical equations. This transition to the fractional version can take several forms, each of which can add different types and levels of complexity not only to the forward problem, but also to inverse problems of recovering internal properties and features.

There are many excellent books available, especially at the research level, that emphasise one aspect of fractional calculus or fractional diffusion operators. Our intent was to take a broader scope than the more closely focused research monographs and to select material for a text suitable for a graduate class whose students have a background in analysis and the basic theory of partial differential equations. Inevitably there will be background material required for understanding the more specialised topics in this book, which we provide to make the book self-contained.

We would like to be able to claim that the book can be read completely in sequence. This was impossible as there are the invevitable uncertainties about the reader's background as well as our wish to intermingle the applications and the historical record with a logical mathematical thread. However, we have attempted to minimize the need to "look forward" in the book as much as we thought practical.

The book has two logistical blocks: the study of fractional derivatives and operators together with their motivational and analytical background material in the first block, and wide coverage of inverse problems involving fractional operators in the second block. The two blocks are then quite suitable for a two-term course with a logical break.

For a one-term course there are several options. Those who have a strong background in analysis and are willing to consult the first block as needed can proceed directly to Chapter 8. An alternative is that the first seven chapters together with the first sections of the final chapter could also provide a solid introduction to the components of fractional calculus and operators for differential equations at the expense of omitting the block on inverse problems. However, the authors are both hopeful and optimistic this is not the selection.

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