Preface to the Second Edition

Smooth ergodic theory lies at the core of the modern theory of dynamics and our book is the only currently available source for graduate students to learn the theory, including basic ideas, methods, and examples. Moreover, smooth ergodic theory has numerous and a still-growing number of applications in various areas of mathematics and beyond. As a result there is a growing number of students interested in some aspects of the theory who either major in areas of mathematics outside of dynamics (probability, statistics, number theory, etc.) or in areas outside mathematics (physics, biology, chemistry, engineering, and economy).

Since the publication of our book in 2013, we received many comments and suggestions from colleagues who taught courses in dynamics using the text as well as questions and comments from students who studied the text, which we took into consideration while working on the second edition. We restructured the book and improved its exposition by revising the proofs of some theorems by adding more details and informal discussions. We also included numerous comments on the notions we introduce, added more exercises, and corrected some typos and minor mistakes.

The second edition contains complete proofs of all major results. Moreover, in an attempt to expand its scope and bring the reader closer to the modern research in dynamics, throughout the book we added many remarks which include comments on the notions and results being presented as well as a brief discussion of some recent advances in the field and statements (usually without proofs) of some new results. Here is a brief account of the changes.

In Chapter 1 we included a new Section 1.3 on uniformly hyperbolic sets which are invariant but not compact or compact but not invariant. The goal is to introduce the reader to the special way of setting up the hyperbolicity conditions, which, in a more general form, are used to define nonuniform hyperbolicity. Further, we added the two new Sections 1.4 and 1.8 on the Smale–Williams solenoid and its slow-down version to discuss some features of hyperbolicity for dissipative dynamical systems which we consider later in Section 11.1. We also wrote a proof of Proposition 1.34, which to the best of our knowledge cannot be found in the literature.

In Chapter 2 we added the two new Sections 2.5.2 and 2.5.3 in which we complete our study of the Lyapunov–Perron regularity for sequences of matrices.

In Chapter 5, Section 5.1.1, we added the important Corollary 5.2 and in Section 5.1.2 (see Remark 5.5) we discuss the crucial role the "deterioration conditions" play in nonuniform hyperbolicity and also introduce the important notion of ε-tempered functions. In Section 5.3 we presented a new Theorem 5.10, which establishes some basic properties of regular sets, and added Section 5.3.3 where the Lyapunov inner product is introduced and its properties are described in Theorem 5.15. We also added the important Remark 5.16 where we compare the weak and strong Lyapunov inner products and charts. Section 5.4 on partial hyperbolicity was extended and improved.

In Chapter 7, Section 7.1.3, we simplified the notations and clarified the exposition. The abstract Stable Manifold Theorem in Section 7.2 is now presented in a greater generality of nonlinear operators acting on Banach spaces. Throughout Section 7.3 we added more details and discussions about local stable and unstable manifolds and we substantially improved and expanded Section 7.3.3 on the graph transform property. We also added a new Section 7.3.5 on stable and unstable manifolds for nonuniformly partially hyperbolic sets.

In Chapter 8 the reader will find a new Sections 8.1.1 on foliation blocks and in Section 8.1.3 a new Theorem 8.7 as well as new Remarks 8.5, 8.8, 8.10, and 8.11, which provide some important additional information to the main result of the section, Theorem 8.4. In Section 8.2 we added more details to the proof of the absolute continuity property and, in particular, included a proof of Lemma 8.13. In the new Remark 8.19 we briefly discuss a construction of local stable and unstable manifolds in the setting of partially hyperbolic sets.

In Chapter 9, Section 9.1, the proof of the main result, Theorem 9.2, on ergodic decomposition of hyperbolic smooth measures is substantially improved and we added a new Section 9.1.1 where we introduce conditional measures on local manifolds along with a new Remark 9.8 where we compare two descriptions of ergodic components, one which comes from Theorem 9.2 and another one, which is related to ergodic homoclinic classes, is given by Theorem 9.10. We also included a proof of Theorem 9.6. The main addition to Section 9.3 is Section 9.3.4 where we present the Ledrappier–Young entropy formula and outline the main ingredients of its proof.

In Chapter 10, after the proof of Theorem 10.11, the reader finds a brief discussion of obstacles to proving ergodicity of geodesic flows on surfaces on nonpositive curvature and a new Theorem 10.12 which establishes ergodicity under some additional assumption.

Chapter 11 is entirely new. It contains three sections where we discuss various properties of general hyperbolic measures including the local product structure, existence of periodic points, and approximations by horseshoes. We also describe the important Sinai–Ruelle–Bowen measures and present some recent results on their existence. We complete the section by outlining a crucial new result on constructing Markov partitions and symbolic dynamics for nonuniformly hyperbolic diffeomorphisms.

All of this resulted in approximately 50 new pages.

Acknowledgements. It is a pleasure to thank several colleagues who have helped us in various ways: Snir Ben Ovadia for drafting Section 11.3 on shadowing and Markov partitions and for useful discussions about this topic and related topics on weak and strong Lyapunov inner products and charts; Sebastian Burgos for useful comments on the construction of the slow-down of the Smale–Williams solenoid in Section 1.8; Anton Gorodetsky for useful comments on Section 1.3; and Federico Rodriguez Hertz for many useful comments on Section 1.6.2 and, particularly, on the proof of Proposition 1.34 as well as on Remark 9.7.

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Preface to the First Edition

This book is a revised and considerably expanded version of our book *Lyapunov Exponents and Smooth Ergodic Theory* [**11**]. When the latter was published, it became the only source of a systematic introduction to the core of smooth ergodic theory. It included the general theory of Lyapunov exponents and its applications to the stability theory of differential equations, nonuniform hyperbolicity theory, stable manifold theory (with emphasis on absolute continuity of invariant foliations), and the ergodic theory of dynamical systems with nonzero Lyapunov exponents, including geodesic flows. In the absence of other textbooks on the subject it was also used as a source or as supportive material for special topics courses on nonuniform hyperbolicity.

In 2007 we published the book *Nonuniform Hyperbolicity: Dynamics of Systems with Nonzero Lyapunov Exponents* [**13**], which contained an up-to-date exposition of smooth ergodic theory and was meant as a primary reference source in the field. However, despite an impressive amount of literature in the field, there has been until now no textbook containing a comprehensive introduction to the theory.

The present book is intended to cover this gap. It is aimed at graduate students specializing in dynamical systems and ergodic theory as well as anyone who wishes to acquire a working knowledge of smooth ergodic theory and to learn how to use its tools. While maintaining the essentials of most of the material in [**11**], we made the book more student-oriented by carefully selecting the topics, reorganizing the material, and substantially expanding the proofs of the core results. We also included a detailed description of essentially all known examples of conservative systems with nonzero Lyapunov exponents and throughout the book we added many exercises.

The book consists of two parts. While the first part introduces the reader to the basics of smooth ergodic theory, the second part discusses more advanced topics. This gives the reader a broader view of the theory and may help stimulate further study. This also provides nonexperts with a broader perspective of the field.

We emphasize that the new book is self-contained. Namely, we only assume that the reader has a basic knowledge of real analysis, measure theory, differential equations, and topology and we provide the reader with necessary background definitions and state related results.

On the other hand, in view of the considerable size of the theory we were forced to make a selection of the material. As a result, some interesting topics are barely mentioned or not covered at all. We recommend the books [**13**, **24**] and the surveys [**12**, **77**] for a description of many other developments and some recent work. In particular, we do not consider random dynamical systems (see the books [**7**, **68**, **75**] and the survey [**69**]), dynamical systems with singularities, including "chaotic" billiards (see the book [**67**]), the theory of nonuniformly expanding maps (see the survey [**76**]), and one-dimensional "chaotic" maps (such as the logistic family; see [**58**]).

Smooth ergodic theory studies the ergodic properties of smooth dynamical systems on Riemannian manifolds with respect to "natural" invariant measures. Among these measures most important are smooth measures, i.e., measures that are equivalent to the Riemannian volume. There are various classes of smooth dynamical systems whose study requires different techniques. In this book we concentrate on systems whose trajectories are hyperbolic in some sense. Roughly speaking, this means that the behavior of trajectories near a given orbit resembles the behavior of trajectories near a saddle point. In particular, to every hyperbolic trajectory one can associate two complementary subspaces such that the system acts as a contraction along one of them (called the stable subspace) and as an expansion along the other (called the unstable subspace).

A hyperbolic trajectory is unstable—almost every nearby trajectory moves away from it with time. If the set of hyperbolic trajectories is sufficiently large (for example, has positive or full measure), this instability forces trajectories to become separated. On the other hand, compactness of the phase space forces them back together; the consequent unending dispersal and return of nearby trajectories is one of the hallmarks of chaos.

Indeed, hyperbolic theory provides a mathematical foundation for the paradigm that is widely known as "deterministic chaos"—the appearance of irregular chaotic motions in purely deterministic dynamical systems. This paradigm asserts that conclusions about global properties of a nonlinear dynamical system with sufficiently strong hyperbolic behavior can be deduced from studying the linearized systems along its trajectories.

The study of hyperbolic phenomena originated in the seminal work of Artin, Morse, Hedlund, and Hopf on the instability and ergodic properties of geodesic flows on compact surfaces (see the survey [**53**] for a detailed description of results obtained at that time and for references). Later, hyperbolic behavior was observed in other situations (for example, Smale horseshoes and hyperbolic toral automorphisms).

The systematic study of hyperbolic dynamical systems was initiated by Smale (who mainly considered the problem of structural stability of hyperbolic systems; see [**107**]) and by Anosov and Sinai (who were mainly concerned with ergodic properties of hyperbolic systems with respect to smooth invariant measures; see [**5**, **6**]). The hyperbolicity conditions describe the action of the linearized system along the stable and unstable subspaces and impose quite strong requirements on the system. The dynamical systems that satisfy these hyperbolicity conditions uniformly over all orbits are called Anosov systems.

In this book we consider the weakest (hence, most general) form of hyperbolicity, known as nonuniform hyperbolicity. It was introduced and studied by Pesin in a series of papers [**87**–**91**]. The nonuniform hyperbolicity theory (which is sometimes referred to as Pesin theory) is closely related to the theory of Lyapunov exponents. The latter originated in works of Lyapunov [**78**] and Perron [**86**] and was developed further in [**35**]. We provide an extended excursion into the theory of Lyapunov exponents and, in particular, introduce and study the crucial concept of Lyapunov–Perron regularity. The theory of Lyapunov exponents enables one to obtain many subtle results on the stability of differential equations.

Using the language of Lyapunov exponents, one can view nonuniformly hyperbolic dynamical systems as those systems where the set of points for which *all* Lyapunov exponents are nonzero is "large"—for example, has full measure with respect to an invariant Borel measure. In this case the Multiplicative Ergodic Theorem of Oseledets [**85**] implies that almost every point is Lyapunov–Perron regular. The powerful theory of Lyapunov exponents then yields a profound description of the local stability of trajectories, which, in turn, serves as grounds for studying the ergodic properties of these systems.

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