Preface

I have learned the basics of commutative algebra from the famous book by Atiyah and MacDonald [**AM69**]. The present text was born from the intention to expand the material therein, and give an alternative organization, although at this point it has grown into its own thing.

In writing this book, I have tried to follow a few principles. First, I have tried not to skim on the basics too rapidly. While it is true that some topics—such as Euclidean domains or unique factorization—are often met in a first algebra course, I feel that they belong in a basic text on commutative rings. Second, and more important, I have tried to present the connections with the most important applications of commutative algebra, such as number theory, algebraic geometry, and computational algebra. This approach makes for a less terse style, but I hope that this is repaid by a wider perspective. Finally, while the exercises present many auxiliary topics, the core of the book should make sense on its own and not depend on them, either logically or because important themes are only left to the exercises.

Commutative algebra is at the crossroad between many fertile areas of mathematics, and I hope that this book conveys the various points of view appropriately. In particular, results in commutative algebra are often clarified by their geometric interpretation. In the present book I have not relied on previous knowledge on the topic, but I have opted instead for a chapter that translates the algebra in geometric terms. Number theory is the other root of commutative algebra, and a constant source of inspiration. Many important topics, such as completion, can be better appreciated by first learning a special case of arithmetic relevance—in the case of completion, this is the construction of the field of p-adic numbers.

Computational algebra has clearly seen an explosion since the advent of computers. Chapter 4 is dedicated to it, but other computational topics are scattered in the text, such as the algorithm to compute Smith's normal form in Chapter 1 or the LLL algorithm in the exercises for Chapter 6. Many other topics could be mentioned, but I could only make some small connections with model theory and invariant theory in the exercises. For the former, see [**Sch99**], while a good introduction to invariant theory is [**Dol03**].

Unfortunately, many topics are not covered here, in particular those that depend on homological algebra techniques. Among them, we can mention flatness, spectral sequences, the study of the Koszul complex and regular sequences, Cohen–Macaulay rings, Gorenstein rings, and duality theory. These are introduced in a subsequent volume, called (without much fantasy) "Homological methods in commutative algebra" [**Fer**].

I should remark that this is an introductory text, so I didn't even try to cover the material of more advanced books, such as Mastumura's books [**Mat70**] and [**Mat86**], or Eisenbud [**Eis95**]. To give an idea, the latter was created as a reference for the famous algebraic geometry book [**Har77**], and ended up being far thicker than it. I advise the reader to consult [**Eis95**] for further reading, and remark that it is actually quite enjoyable despite its appearance; my scope here is far more limited than Eisenbud's.

Here is a brief description of the contents of the book; see the introduction to the various chapters for more details on the topics covered therein.

In the first chapter, we introduce the basics notions of commutative algebra, like rings, ideals, and modules. Moreover, we treat a few basic constructions that we will use throughout the book: quotients, localization, and tensor products. A section about Euclidean rings is also present, to show an example of rings that are especially well-behaved. These topics will probably be familiar from a first algebra course, and the knowledgeable reader can just have a quick glance, as we do nothing fancy in this chapter. Finally, we discuss the language of graded rings and modules.

The core of the book starts with Chapter 2, where we introduce some particular finiteness conditions on our rings and modules. We start by considering rings whose ideals are generated by a single element, and then we generalize to the case of *Noetherian* rings, where ideals are generated by finitely many elements. An equivalent condition is that ascending chains of ideals eventually stabilize. This condition is strong enough to produce a lot of result, but no so stringent, so that most of the rings that we will encounter will satisfy this hypothesis. More generally, we will define Noetherian modules by asking that ascending chains of submodules stabilize, and their symmetric counterpart, *Artinian* modules, where ask the same thing for descending chains. We investigate various operations that preserve these properties, proving in particular the famous Hilbert's basis theorem, that guarantees that if A is a Noetherian ring, so is the polynomial ring $A[x]$. Finally, for modules that are both Noetherian and Artinian, we introduce the important numerical invariant of length.

Chapter 3 is about factorization. We start from the simple case of rings that admit a unique factorization for elements. This is a rather special class of rings, so after that we present the theory of primary decomposition of Lasker, which is a generalization of unique factorization that works over arbitrary Noetherian rings and modules. We then specialize this case to study the important case of Dedekind domains, which have a theory of prime factorization for ideals.

In Chapter 4, we focus on the case of polynomial rings, where some explicit computational techniques are available. In particular, we introduce the resultant of two polynomials. Using it, we tackle the problem of elimination, which is about solving polynomial systems in an inductive way. We also use resultants to introduce discriminants and show their basic properties. In the second part of the chapter, we switch to a different approach and introduce Gröbner bases, which are certain special sets of generators of an ideal in a polynomial ring. Again, after having proved their basic properties, and found algorithms to compute them, we use them to study the problem of elimination from a different angle.

Chapter 5 introduces the concept of integral elements. These are the analogue of algebraic elements from field theory in the setting of rings. In fact, much of the theory just mimics what one does for fields. The parallel notion of an algebraic extension of fields is an integral extension of rings. For such an extension $A \subset B$, we present the Cohen–Seidenberg theory that relates prime ideals in A and B in a precise way. We also use these results to give another, more traditional, characterization of Dedekind rings.

Dedekind rings are studied in much more detail in Chapter 6. In particular, we study the properties of factorization of ideals and what happens in integral extensions of Dedekind rings. An important special case here is the class of number rings: these are obtained by taking elements integral over $\mathbb{Z}$ in a finite extension of $\mathbb{Q}$, say of degree n. Such rings have a natural embedding in $\mathbb{R}^n$ that represents them as lattices, and their ideal are sublattices of finite index. One can then use geometric techniques to derive bounds on the index of ideals, and to find elements of small norm in a given ideal. Using these bounds, we prove two important finiteness results in number theory: the fact that the class group of ideals of an algebraic number field is finite, and that the group of units of its ring of integers is finitely generated.

Chapter 7 introduces topological methods. We start by defining absolute values over fields, which are similar to norms in functional analysis. By mimicking the construction of the Cauchy completion of the reals, one can define the completion of a field endowed with an absolute value. In particular, using a suitable absolute value on $\mathbb{Q}$, this construction produces the field of p-adic numbers. A class of absolute values, called nonarchimedean, can be obtained from the more general notion of a valuation, which we study next. Following this, we study the problem of completion from a more algebraic point of view. To do this, we introduce the machinery of direct and inverse limits. The algebraic point of view allows us to define completion of a ring with respect to a topology (called I-adic) determined by an ideal I, a notion that one can also extend to modules. A crucial tool is the Artin–Rees lemma, which relates the I-adic topology of a module and that of its submodules. In the last section, we give yet another generalization of the construction of p-adic numbers, this time using the notion of Witt vectors.

In Chapter 8, while proving few new results, we give the basic definitions about algebraic geometry: affine and projective varieties, Zariski topology, morphisms of varieties and so on. We then show how most of the material covered so far can be used to quickly obtain information about these new objects. We have decided to put this chapter almost at the end of the book, so that most of the text can be read independently of it. Still, one can start reading this chapter even at the beginning, and follow the geometric dictionary while learning new algebraic concepts. The main result from commutative algebra that we introduce in this chapter is Hilbert's Nullstellensatz, which gives a deep link between the points of an algebraic variety and the maximal ideals of its coordinate ring.

Our treatment of algebraic geometry is as elementary as possible; in particular we do not even mention the machinery of schemes. Even in this more limited setting, the interplay between algebra and geometry is veruy fruitful. One can apply results from commutative algebra to some simple rings (usually finitely generated reduced k-algebras) to obtain results which are geometric in nature; conversely the geometry can suggest that a result may be true for some special rings, and often the algebraic result can be proved for a much more general class of rings.

The next chapter is about dimension theory. The concept of dimension is introduced in Chapter 8 from a geometric point of view, but Chapter 9 gives a wide algebraic generalization. We show that a Noetherian local ring has a definite notion of dimension, the main result being that all reasonable definitions suggested by the geometric intuition lead to the same concept. In most of the chapter we work with local or graded rings, and the parallelism here is very strict. The main technical tool to develop the theory is the Hilbert polynomial, which estimates the order of growth of the size of homogeneous components in a graded module, so we begin the chapter by proving its existence and studying its properties.

In Chapter 10 we define regular local rings, which correspond to nonsingular points on a variety. We also study the nonregular case by introducing the concept of multiplicity of a local ring, which is a simple measure of singularity. As it turns out, this is related to the concept of degree of a graded ring. In the geometric case, where a graded ring corresponds to a projective variety, the degree expresses the number of points of intersection with a general linear space of complementary dimension. This chapter develops the theory of multiplicity, and in doing so describes in some detail the structure of local rings, culminating in the celebrated theorem of Cohen, that gives a precise description for complete local Noetherian rings.

The Appendix consists of an exposition of the classical Galois and Kummer theory of field extensions. It aims to give background for the field theoretic results used in the rest of the book (in particular, Galois theory of finite extensions and the notion of separability) but covers more ground than it is strictly needed. In fact, it can be read independently of the rest of the book as a short introduction to field theory. While fields are in many respects simpler than general rings, they also present many new phenomena, and a familiarity with fields is certainly part of the study of commutative algebra in a broad sense.

The prerequisites for reading the book are not many. We assume that the reader is more or less familiar with algebraic objects, and some acquaintance with linear algebra is assumed—for instance, it is useful if the reader has some familiarity with the tensor product construction in the context of vector spaces. Finally, from Chapter 7 we make some use of elementary topology and some notions about metric spaces.

The book is suitable for a semester on algebra at the introductory graduate level. It could also be used to support a shorter course on algebraic number theory, introducing global and local fields, Dedekind rings with their factorization theory and completions.

To help the reader orient themselves, we suggest some possible paths through the book, other than reading it cover to cover.

A standard introduction to commutative algebra, along the lines of [**AM69**], would start from the basics in Chapters 1 to 3, then go through integral extensions in Chapter 5, topological methods in Chapter 7 (going quickly over Sections 7.1 and 7.2), prove the Nullstellensatz in Sections 8.1 and 8.2, then introduce dimension theory in Chapter 9 and at least the basics on regular rings in Section 10.1.

An introduction to algebraic number theory could cover factorization in Chapter 3, the basics of field theory from Appendix A, the first half of Chapter 4 to introduce discriminants, and parts of Chapter 5 to characterize Dedekind rings; then Chapter 6 covers the global theory and Chapter 7 the local theory, especially Sections 7.1, 7.2, and 7.3.

The reader that wants a quick introduction to the methods of computational algebra can just follow Chapter 2, Section 3.1, and Chapter 4, although we advise to complement this with other texts on the matter, such as [**KR00**]. In a similar way, Appendix A could be used, together with other material, in a minicourse on Galois theory.

Finally, a geometrically minded reader could learn the basics in Chapters 1 and 2, then go through Chapter 8 and learn the necessary commutative algebra along the way. Section 8.5 requires learning about completions in Sections 7.4 and 7.5, while the decomposition in irreducible components of Section 8.7 makes use of the primary decomposition of Section 3.2. Finally, Section 8.9 makes use of the theory of Dedekind rings developed in Chapters 5 and 6. The ideas about dimension are then expanded in Chapter 9, while the notion of regularity comes again in Section 10.1. This way of reading the book will require tracking back the prerequisites for some results, but has the advantage of giving geometrical reasons to introduce algebraic constructs.

Examples in the text usually require only trivial verifications. They are part of the core of the text and should not be skipped; some definitions are actually given inside the examples (for instance, the basic operations on ideals). When an example requires more work, it should be considered as an exercise.

On the other hand, exercises vary from simple to hard, and there is (intentionally) no indication to distiguish the level. So try to do as many exercises as you can, and don't feel frustrated if some of them look too hard. Maybe you can come back later, when you are familiar with more techniques.

In general, I have tried to avoid depending on exercises for the main body of the text. The cases where I have done so should be easy verifications. On the other hand, many important and subtle counterexamples are presented as series of exercises.

No contribution in this book is original, except of course the usual amount of errors, that should be attributed only to the author. If you spot some of them, you can send an email to ferrettiandrea@gmail.com.

I hope that you will enjoy reading this book as much as I enjoyed writing it!

If I was able to write this book, it is because Massimo Gobbino and Paolo Tilli, when I was young and did not know better, believed in me and persuaded me to undertake the study of mathematics. This turned out ot be one of the best choices I made, and I have to really thank them for this. Thanks to Roberto Dvornicich, who instilled in me a lasting love for algebra. I take the opportunity to thank the AMS for the editorial support, especially Ina Mette, who believed in the project and followed it with great patience through many years. Most of all, I want to thank my wife Sbambi, who with her love shows me everyday what is really important, and with her patience and understanding has given me the time and peace of mind to do mathematics and finish this book. $\ddot\smile$