Chapter 10

Local Structure

In this chapter, we investigate in more detail the local structure of rings—in particular, we study the condition of regularity and the related notion of multiplicity. Regular rings are the algebraic counterpart to smooth algebraic varieties. Smoothness for an algebraic variety is measured by looking at the dimension of its Zariski tangent space—this is at least the dimension of the variety, and equality happens in the smooth case. By Nakayama's lemma, the dimension of the Zariski tangent space at a point $x \in V$ is the same as the minimal number of generators of the maximal ideal of the local ring $R(V)_x$. This notion can readily be generalized to a Noetherian local ring, giving rise to the concept of regular ring.

In the first section, we study the elementary properties of regular rings—we show that from an analytic point of view they are all very similar, and that a regular ring is necessarily integral, which translates to the fact that the union of two algebraic varieties is singular along the intersection. Many more results are known for regular local rings—in particular, they are unique factorization domains—but we are only able to prove the simplest of them, since the most important results require homological techniques and will be proved in the sequel of this book.

Next, we define the notion of multiplicity of a local ring. This is, in some sense, a measure of how much the ring fails to be regular, or how complex a singularity is. In fact, regular rings have multiplicity 1, and the converse is true under some additional hypotheses. The latter implication is not easy, though, and a good part of the chapter builds enough theory to prove at least a special case.

We can also define the multiplicity of a finitely generated module, although the ring case is the most interesting one. The multiplicity of a local ring is related to the degree of the associated graded ring via the *tangent cone construction*.

In the next sections, we develop various results around multiplicity, in particular, the very useful *additivity formula*. We also study the behavior of the multiplicity of rings of the form $A/(a)$, where A is a fixed local ring and $a \in A$ varies. This is a way to express the notion of order of vanishing of the element a, and, in some cases, it can be used to define a valuation on the ring A.

We end the chapter with the famous structure theorem of Cohen, which gives a precise description of complete local Noetherian rings.

10.1. Regular rings

Let A be a local Noetherian ring with maximal ideal $\mathcal{M}$. From the previous chapter we know that A has finite dimension d, and there is an $\mathcal{M}$-primary ideal Q generated by elements $a_1, \ldots, a_d$. In general, though, we cannot just take $Q = \mathcal{M}$.

Definition 10.1.1. Let $A, \mathcal{M}$ be a local Noetherian ring, $a_1, \ldots, a_d$ a system of parameters. We say that $a_1, \ldots, a_d$ is a *regular* system of parameters if $(a_1, \ldots, a_d) = \mathcal{M}$. If $\mathcal{M}$ admits a regular system of parameters, then we say that A is *regular*.

Definition 10.1.2. Let A be a Noetherian ring. We say that A is *regular* if A_P is a regular local ring for all primes $P \subset A$.

Remark 10.1.3. In the definition we require that A is Noetherian, so that by Corollary 9.3.2 we know that A_P has finite dimension for all primes P.

Remark 10.1.4. Let $A, \mathcal{M}$ be a local Noetherian ring. The minimum number of generators of $\mathcal{M}$ is called the *embedding dimension* of A, denoted $\operatorname{embdim} A$. By definition, we have

$$\operatorname{embdim} A \geq \dim A,$$

and A is regular exactly when the above is an equality.

Before giving examples, we can look at the definition from a different angle.

Remark 10.1.5. Let $a_1, \ldots, a_d$ be any system of parameters for $A, \mathcal{M}$. The quotient module $\mathcal{M}/\mathcal{M}^2$ is a vector space over $k := A/\mathcal{M}$. By Nakayama's lemma 1.3.19, $a_1, \ldots, a_d$ is regular if and only if the images $\overline{a_1}, \ldots, \overline{a_d}$ are linearly independent over k.

The above remark makes the geometric meaning of regularity more apparent. As discussed in Section 8.8, in the geometric case this notion corresponds to a nonsingular variety. To be more specific, take a variety V and a point $p \in V$, corresponding to a maximal ideal $\mathcal{M} \subset R(V)$. Assume that V has dimension d in p—then elements $a_1, \dots, a_d \in R(V)_{\mathcal{M}}$ form a regular system of parameters if and only if their linear components are independent over k. This can be guaranteed exactly when the Zariski tangent space T_pV has dimension d. In general, the dimension of T_pV is the embedding dimension of the ring $R(V)_{\mathcal{M}}$.

The name embedding dimension comes from the remark that V cannot be embedded in $\mathbb{A}^n$ for $n < \operatorname{embdim} R(V)_{\mathcal{M}}$, since its tangent space is a subspace of k^n, so the embedding dimension is a lower bound for the dimension of an affine space in which V can be embedded. Notice that in any case this bound is not at all sharp, as the example of a regular variety should immediately show.

Example 10.1.6.

(a) Let A be a local ring of dimension 0. Then A is regular if and only if it is a field.

(b) More generally, let k be a field, $A = k[[x_1, \dots, x_d]]$. Then A is local with maximal ideal $(x_1, \dots, x_d)$. Since $\dim A = d$, A is regular of dimension d. This is our prototypical example: we will prove with Cohen's theorem that the completion of a regular local rings has this form whenever it has the same characteristic as its residue field.

(c) Let $A, \mathcal{M}$ be a regular local ring of dimension d. Then the completion $\widehat{A}$ has dimension d, and generators for $\mathcal{M}$ map to generators of $\widehat{\mathcal{M}}$. This implies that $\widehat{A}$ is regular as well.

(d) Let $A, \mathcal{M}$ be a regular local ring of dimension 1. Then $\mathcal{M}$ has a single generator a. It turns out (we will not prove this here) that a regular local ring is a UFD, and in particular integrally closed. By Proposition 7.3.2, it follows that A is a DVR. You will prove this directly in Exercise 1.

(e) Vice versa, every DVR has dimension 1, and its maximal ideal is principal by Proposition 7.3.1. Hence local regular rings of dimension 1 are the same as discrete valuation rings. This gives a lot of examples of regular rings, such as the p-adic integers $\mathbb{Z}_p$.

(f) For a nongeometric example, take $A = \mathbb{Z}[x]$ and the maximal ideal $\mathcal{M} = (p, x)$ for a prime p. Then $A_{\mathcal{M}}$ has dimension 2, and its maximal ideal is generated by two elements, so $A_{\mathcal{M}}$ is a regular local ring.

Remark 10.1.5 helps understanding when a quotient of a regular ring remains regular.

Proposition 10.1.7. *Let A be a regular local ring, with maximal ideal $\mathcal{M}$. Elements $a_1, \dots, a_i$ are a subset of a regular system of parameters if and only if $A/(a_1, \dots, a_i)$ is regular.*

Proof. Let $I = (a_1, \dots, a_i)$. Notice that $\dim A/I = d - i$ by Proposition 9.4.5. Assume that $a_1, \dots, a_d$ is a regular system of parameters. Then $\overline{a_{i+1}}, \dots, \overline{a_d}$ are a regular system of parameters for the quotient A/I, because their images remain linearly independent over $k = A/\mathcal{M}$. Conversely, any regular system of parameters for A/I lifts to a regular system of parameters for A together with $a_1, \dots, a_i$. □

From an analytic point of view, regular local rings have a very simple structure. This is expected, since they correspond to nonsingular points on a variety. In differential geometry, the inverse function theorem ensures that the nonsingular points of the zero locus of finitely many C^∞ functions admits a local chart, hence the local structure around all nonsingular points looks the same. We have an analogous statement in the algebraic setting.

Proposition 10.1.8. *Let $A, \mathcal{M}$ be a regular local ring of dimension d. Then, as graded rings,*

$$\mathrm{Gr}_{\mathcal{M}}(A) \cong k[x_1, \dots, x_d],$$

where $k = A/\mathcal{M}$.

Proof. Let $a_1, \dots, a_d$ be a regular system of parameters. There is a surjective homomorphism of graded rings $\phi \colon k[x_1, \dots, x_d] \to \mathrm{Gr}_{\mathcal{M}}(A)$, so

$$\mathrm{Gr}_{\mathcal{M}}(A) \cong \frac{k[x_1, \dots, x_d]}{I}$$

for some homogeneous ideal I. Assuming $I \neq 0$, take a homogeneous $f \in I$, say of degree r. Then every multiple of f lies in I, so we can bound the length

$$\ell(\mathrm{Gr}_{\mathcal{M}}(A)_n) \leq \binom{n+d-1}{d-1} - \binom{n+d-r-1}{d-1}$$

for n big enough. But the right-hand side is a polynomial of degree $d-1$, while Theorem 9.6.3 guarantees that $\dim \mathrm{Gr}_{\mathcal{M}}(A) = d$. □

This result has an important corollary.

Theorem 10.1.9. *A regular local ring is an integral domain.*

Proof. Let $A, \mathcal{M}$ be a regular local ring, and take some nonzero $a, b \in A$. By Krull's intersection theorem 7.5.24, $\bigcap_{n \in \mathbb{N}} \mathcal{M}^n = 0$, hence $a \in \mathcal{M}^r \setminus \mathcal{M}^{r+1}$ and $b \in \mathcal{M}^s \setminus \mathcal{M}^{s+1}$ for some $r, s \in \mathbb{N}$. This implies that $\overline{a} \in \mathrm{Gr}_{\mathcal{M}}(A)_r$

and $\overline{b} \in \mathrm{Gr}_{\mathcal{M}}(A)_s$ are not zero. By the above Proposition, $\mathrm{Gr}_{\mathcal{M}}(A)$ is a domain—in particular $\overline{ab} \neq 0$, which implies that $ab \neq 0$. □

In the nonlocal case, we cannot hope to have such a result: a variety made by multiple smooth, not intersecting components has a coordinate ring that is regular but not integral. This is essentially the only thing that can go wrong:

Proposition 10.1.10. *Let A be a regular ring. Then A is a finite direct sum of integral domains.*

Proof. Let $P_1, \dots, P_r$ be the minimal primes of A, so that

$$\mathcal{N}(A) = \sqrt{0} = P_1 \cap \dots \cap P_r.$$

Since A is Noetherian, $\mathcal{N}(A)$ is finitely generated, so $\mathcal{N}(A)^n = 0$ for n big enough.

Let $\mathcal{M}$ be a maximal ideal of A. Then $A_{\mathcal{M}}$ is a regular local ring, in particular it is a domain. The only minimal prime of $A_{\mathcal{M}}$ is 0, hence $\mathcal{M}$ contains exactly one of the P_i by Corollary 1.6.9. It follows that the primes $P_1, \dots, P_r$ are all coprime. By the Chinese remainder theorem,

$$A \cong \frac{A}{P_1^n} \oplus \dots \oplus \frac{A}{P_r^n}.$$

Let $A_i := A/P_i^n$, and take a divisor of zero $a \in A_i$. By construction, a is nilpotent in A_i. For every maximal ideal $\mathcal{M} \subset A$, the image of a inside $A_{\mathcal{M}}$ is 0, since $A_{\mathcal{M}}$ is a domain. But then $\mathrm{Ann}(a)$ is not contained in any maximal ideal, so $a = 0$. It follows that each A_i is an integral domain, and A is a finite sum of integral domains. □

We have another way to express the fact that the analytic structure of a regular ring is especially simple.

Lemma 10.1.11. *Let A be a local Noetherian ring of dimension d, with a system of parameters $a_1, \dots, a_d$, and let $Q = (a_1, \dots, a_d)$. Let $f \in A[x_1, \dots, x_d]$ be a homogeneous polynomial of degree s, and assume that $f(a_1, \dots, a_d) \in Q^{s+1}$. Then $f \in \mathcal{M}[x_1, \dots, x_d]$.*

Proof. Evaluation at $a_1, \dots, a_d$ gives a surjective homomorphism

$$\tfrac{A}{Q}[x_1, \dots, x_d] \xrightarrow{ev_a} G_Q(A) = \bigoplus \tfrac{Q^n}{Q^{n+1}}.$$

By hypothesis, $ev_a(\overline{f}) = 0$, where $\overline{f}$ is the class of f modulo Q. If $f \notin \mathcal{M}[x_1, \dots, x_d]$, $\overline{f}$ has an invertible coefficient, hence $\overline{f}$ is not a zero divisor.

By Proposition 9.5.2,

$$\dim G_Q(A) \leq \dim \frac{A/Q[\underline{x}]}{(\overline{f})} = d - 1,$$

which is a contradiction since $\dim G_Q(A) = d$. □

The following theorem expresses the fact that a system of local coordinates for a regular ring is given by analytically independent parameters. It is just a special case of the above lemma.

Theorem 10.1.12. *Let A be a regular local ring of dimension d, with a* regular *system of parameters $a_1, \dots, a_d$. Then the a_i are* analytically independent*, that is, if $f \in A[x_1, \dots, x_d]$ is a homogeneous polynomial such that $f(a_1, \dots, a_d) = 0$, then $f \in \mathcal{M}[x_1, \dots, x_d]$.*

When the ring contains a field, this result has a simpler statement.

Corollary 10.1.13. *Let $A, \mathcal{M}$ be a regular local ring of dimension d, with a* regular *system of parameters $a_1, \dots, a_d$. Assume that there is a field $k \subset A$ that maps isomorphically onto $A/\mathcal{M}$. Then the a_i are algebraically independent over k.*

Proof. Take a polynomial $f \in k[x_1, \dots, x_d]$ such that $f(a_1, \dots, a_d) = 0$. Let s be the minimal degree of a monomial of f, so that $f = f_s + g$, where each monomial of g has degree at least $s + 1$. Then $f_s(a_1, \dots, a_d) \in \mathcal{M}^{s+1}$, and Lemma 10.1.11 guarantees that $f_s \in \mathcal{M}[x_1, \dots, x_d]$. Since $f_s \in k[x_1, \dots, x_d]$, this means that $f_s = 0$, hence $f = 0$. □

There are a few properties about regular rings that are too important to omit. However, we cannot prove them with the techniques we have at hand. The proof of these theorems marks the beginning of the usage of homological methods in commutative algebra. We will expand on this circle of ideas in the following volume [**Fer**].

The most important result is that Theorem 10.1.9 can be strengthened considerably:

Theorem 10.1.14 (Auslander–Buchsbaum)**.** *A regular local ring is a unique factorization domain.*

Proof. See [**Eis95**, Theorem 19.19] or [**Fer**, Theorem 8.4.1]. □

Remark 10.1.15. In particular, this means that regular local rings are integrally closed. A singular point of variety having an integrally closed local ring is called *normal*—this is considered a mild form of singularity.

Example 10.1.16. The planar singularity $y^2 = x^2 + x^3$ (a node) is not normal. In fact, this is a rephrasing of the content of Exercise 5 in Chapter 5.

The other important result is

Theorem 10.1.17. *Let A be regular local ring, $P \subset A$ a prime. Then the local ring A_P is regular as well.*

Proof. See [**Eis95**, Corollary 19.14] or [**Fer**, Corollary 8.2.5]. □

A consequence of this fact is that in order for a ring to be regular it is sufficient that all localizations $A_{\mathcal{M}}$ for a maximal ideal $\mathcal{M}$ are regular local rings (that is, it is redundant to ask this for all prime ideals). Some texts even adopt this as the definition of regular ring.

10.2. Multiplicity and degree

In this section, we study the leading coefficient of the Hilbert polynomial in more detail, both in the local and in the graded case. In the local case, this term is an algebraic expression of the multiplicity of a singularity. In particular, it gives an interesting invariant for local Noetherian rings that are not regular. In the graded case, it computes the degree of a projective variety. We also show how the two notions are related via the *tangent cone* construction.

We start with a semilocal Noetherian ring A of dimension d. Let Q be any ideal of definition for A, so that $\sqrt{Q} = \mathcal{J}(A)$, and fix a finitely generated A-module M. Then we know by Corollary 9.2.6 that for n big enough we can express the length of $M/Q^{n+1}M$ with a polynomial

$$\chi_M^Q(n) = \ell\left(\frac{M}{Q^{n+1}M}\right).$$

We can bound the degree of $\chi_M^Q(n)$ since

$$\dim M = \dim A/\operatorname{Ann}(M) \leq d.$$

Hence, we can write

$$\chi_M^Q(n) = \frac{e}{d!}n^d + a_{d-1}n^{d-1} + \cdots + a_0 \tag{10.2.1}$$

for some $e \in \mathbb{N}$.

Definition 10.2.1. Let A be a semilocal Noetherian ring with $\dim A = d$, M a finitely generated A-module. The natural number e in (10.2.1) is called the *Hilbert–Samuel multiplicity*, or simply *multiplicity*, of M at Q, denoted $e(Q, M)$.

As a notation, we will simply write $e(Q) = e(Q, A)$. When A is local with maximal ideal $\mathcal{M}$, we let $e(A) = e(\mathcal{M}) = e(\mathcal{M}, A)$.

In the geometric case, let V be an affine variety, $\mathcal{M} \subset R(V)$ the maximal ideal of a point $p \in V$. We will call $e(R(V)_{\mathcal{M}})$ the multiplicity of V at the point p.

Remark 10.2.2. Some authors introduce a notation for all coefficients of the Hilbert polynomial, and so denote by $e_0(Q, M)$ what we denote by $e(Q, M)$.

Remark 10.2.3. Let A be a regular local ring. Then Proposition 10.1.8 gives the expression

$$\chi_A^{\mathcal{M}}(n) = \binom{n+d}{d} = \frac{1}{d!} n^d + \cdots ,$$

hence $e(A) = 1$. It follows that the multiplicity is an interesting invariant of the ring only in the singular case.

The converse is not true without additional assumptions.

Example 10.2.4. Consider the ring $A = k[x, y]/(x^2, xy)$ and the maximal ideal $M = (x, y) \subset A$. Let $B = A_M$, which is a local ring with maximal ideal $\mathcal{M} = M \cdot B$.

For $n \geq 2$ we have

$$\ell\left(\frac{A}{\mathcal{M}^n}\right) = \dim_k \frac{A}{\mathcal{M}^n} = n + 1,$$

which shows that $\dim A = 1$ and $e(A) = 1$. On the other hand, A is not regular, because it is not even an integral domain (in fact x is nilpotent in A).

We can also consider the $\mathcal{M}$-primary ideal $Q = (y)$. Then, by the same computation,

$$\ell\left(\frac{A}{Q^n}\right) = n + 1,$$

so $e(Q, A) = 1$ as well.

Remark 10.2.5. In the above example, we may be tempted to describe A as the local ring in 0 of the variety defined by the ideal $I = (x^2, xy)$. This is not correct, since I is not radical, and in fact $V(I)$ is just the line $x = 0$. Talking properly of such singularities requires the language of schemes, which are geometric objects that are not fully described by their set of geometric points. In this language, the zero locus of I would be a line with an embedded double point.

Our counterexample is not integral, but this is not the only restriction. For an example of an integral domain of multiplicity 1 which is not regular, see [**Nag62**, Appendix A1, Example 2]. In general, the appendix of Nagata's book is a fantastic source of counterexamples in commutative algebra.

On the other hand, the condition having multiplicity 1 is not too far from being regular. To state this precisely, we need the following definition.

Definition 10.2.6. Let A be a local Noetherian ring, $\widehat{A}$ its completion with respect to the topology defined by its maximal ideal. We say that A is *unmixed* if for every associated prime P of 0 in $\widehat{A}$ we have

$$\dim \frac{\widehat{A}}{P} = \dim \widehat{A} = \dim A.$$

With this definition, we can state the following *multiplicity* 1 *criterion.*

Theorem 10.2.7 (Nagata)**.** *Let A be a local Noetherian ring. Then A is regular if and only if $e(A) = 1$ and A is unmixed.*

The proof of this result is surprisingly subtle, see [**Nag62**, Theorem 40.6]. Following [**Now97**], we will give the proof in the case where the residue field of A has characteristic 0, as a consequence of Cohen's theorem.

We now pass to the graded case, where we can give similar definitions. Let A be a graded ring of dimension d, and assume that A_0 is Artinian. For a finitely generated graded A-module M we have the bound

$$\dim M = \dim A/\operatorname{Ann}(M) \leq d.$$

Hence, for n big enough we can write

$$\ell(M_n) = \frac{e}{d!} n^d + a_{d-1} n^{d-1} + \cdots + a_0 \qquad (10.2.2)$$

for some $e \in \mathbb{N}$.

Definition 10.2.8. Let A be a graded Noetherian ring of dimension d, M a finitely generated graded A-module. The natural number e in (10.2.2) is called the (Hilbert–Samuel) *multiplicity* of M, denoted $e(M)$.

In particular, we are interested in the multiplicity of a graded ring as a module over itself. In this case, we will also call $e(A)$ the *degree* of A, denoted $\deg A$.

Remark 10.2.9. The notions of multiplicity and degree are strictly related. In fact, let $A, \mathcal{M}$ be a local Noetherian ring of dimension d. Then by construction the Hilbert polynomials of $A, \mathcal{M}$ and $\operatorname{Gr}_{\mathcal{M}}(A)$ are the same, since

$$\chi_A^{\mathcal{M}}(n) = \sum_{i=0}^{n} \ell\left(\frac{\mathcal{M}^i}{\mathcal{M}^{i+1}}\right) = \chi_{\operatorname{Gr}_{\mathcal{M}}(A)}(n).$$

In particular

$$e(A) = \deg \mathrm{Gr}_{\mathcal{M}}(A).$$

To better understand this relation, we consider the geometric case. Let $V \subset \mathbb{A}^n(k)$ be an affine variety, p a point of V defined by the maximal ideal $M \subset R(V)$. To this we associate the local ring $A = R(V)_M$, with maximal ideal $\mathcal{M} = M \cdot A$. If we translate the variety so that $p = 0$, each polynomial $f \in I(V)$ has zero constant term, hence we can write

$$f = f_d + f_{d+1} + \cdots ,$$

where f_k is homogeneous of degree k and $d > 0$. In particular, we can consider the homogeneous component of lowest degree, f_d. For the purpose of this section, we denote $H(f) := f_d$.

Definition 10.2.10. Let $V \subset \mathbb{A}^n(k)$ be an affine variety with $0 \in V$. Let $H(I(V))$ be the homogeneous ideal generated by all polynomials $H(f)$ for $f \in I(V)$. The affine variety C_0V defined by $H(I(V))$ is called the *tangent cone* of V in 0. Since $H(I(V))$ is homogeneous, it also defines a projective variety in $\mathbb{P}^{n-1}(k)$ called the *projective tangent cone* of V in 0, and denoted $\mathbb{P}C_0V$.

Remark 10.2.11. Assume that $I(V) = (f_1, \ldots, f_r)$. Then

$$H(I(V)) = (H(f_1), \ldots, H(f_r)).$$

This is especially useful when V is regular in 0. In this case, we can choose a regular system of parameters $f_1, \ldots, f_r$. Then, each f_i has a nonzero linear term, and the tangent cone is exactly the Zariski tangent space to V in 0. However, when V is singular, the tangent cone contains strictly more information.

By construction, the ring associated to the projective tangent cone $\mathbb{P}C_0V$ is exactly the associated graded ring to the local ring $A, \mathcal{M}$. In particular, we can translate Remark 10.2.9 as follows:

Proposition 10.2.12. *Let $V \subset \mathbb{A}^n(k)$ be an affine variety with $0 \in V$. Then the multiplicity of V in 0 is equal to the degree of the projective variety $\mathbb{P}C_0V$.*

Notice that we have defined the notion of tangent cone in the point 0, but using a suitable translation this concept readily generalizes to other points in V.

Example 10.2.13. Let C be the node defined by the equation $y^2 = x^2 + x^3$. The tangent cone is defined by the equation $y^2 = x^2$, so it is the union of two lines. This expresses the fact that—while C itself is irreducible—locally there are two different branches.

The notion of multiplicity has a much simpler interpretation for singularities of a hypersurface, as the degree of vanishing of a singular polynomial. The following Proposition makes this precise.

Proposition 10.2.14. *Let k be a field and $V \subset \mathbb{A}^n(k)$ an affine hypersurface, given by the equation $f = 0$. Assume that $0 \in V$, which means that $f(0) = 0$, and write f as a sum of homogeneous components*

$$f = f_d + f_{d+1} + \cdots ,$$

where $\deg f_k = k$ *and* $f_d \neq 0$. *Let A be the local ring of V in 0. Then $e(A) = d$.*

Proof. Let $\mathcal{M} = I(0) \subset k[x_1, \ldots, x_n]$ be the ideal of the point 0, and let $I = (f)$, $\mathcal{M}' = \mathcal{M}/I$, so that $A = R(V)_{\mathcal{M}'}$. To compute the multiplicity, we have to evaluate the length of $A/\mathcal{M}'^k$.

Let $B_k := k[x_1, \ldots, x_n]/(f, \mathcal{M}^k)$. First, we claim that

$$A/\mathcal{M}'^k \cong B_k.$$

In fact, since quotients and localizations commute, $A/\mathcal{M}'^k$ is a localization of B_k at the multiplicative set S constisting of images of polynomials with nonzero constant term. But such polynomials are already invertible in B_k. In fact, take a polynomial $g \in k[x_1, \ldots, x_n]$ with $g(0) \neq 0$. Multiplication by $\overline{g}$ is an injective map $B_k \to B_k$, and since B_k is a finite-dimensional k-vector space, it must be surjective as well. This means that $\overline{g}$ has an inverse in B_k, proving the claim.

To compute $\dim B_k$, for $k > d$, we look at the map

$$\mu_f \colon \frac{k[x_1, \ldots, x_n]}{\mathcal{M}^{k-d}} \to \frac{k[x_1, \ldots, x_n]}{\mathcal{M}^k}$$

given by multiplication by f. Since $f_d \neq 0$, μ_f is injective, giving an exact sequence

$$0 \longrightarrow \tfrac{k[x_1,\ldots,x_n]}{\mathcal{M}^{k-d}} \longrightarrow \tfrac{k[x_1,\ldots,x_n]}{\mathcal{M}^k} \longrightarrow B_k \longrightarrow 0 \ .$$

This allows us to compute

$$\begin{aligned} \dim \frac{A}{\mathcal{M}'^k} &= \dim B_k = \binom{n+k}{n} - \binom{n+k-d}{n} \\ &= \frac{d}{(n-1)!} k^{n-1} + \text{lower order terms}, \end{aligned} \tag{10.2.3}$$

which implies that $e(A) = d$. □

With exactly the same proof we get an analogous result for the graded case:

Proposition 10.2.15. *Let k be a field and $V \subset \mathbb{P}^n(k)$ a projective hypersurface, given by the equation $f = 0$. Let A be the ring associated to V. Then* $\deg A = \deg f$.

In fact, the notions of multiplicity and degree were originally understood in simple cases such as this, and the definition with the Hilbert polynomial was introduced later [**Sam49**].

Remark 10.2.16. The concepts of multiplicity and degree are both fundamental in intersection theory. To understand what this is about, we recall some notions on the topology of manifolds.

Let M be a compact, oriented smooth manifold of real dimension d. The cup product gives a ring structure on the graded sum of the cohomology groups $H^*(M, A)$, for all rings A. To a compact oriented submanifold $S \subset M$ one can associate a fundamental class $c(S) \in H^*(M, \mathbb{Z})$ using Poincaré duality. If $S, T \subset M$ are two such submanifolds, one can then compute the product $c(S) \cdot c(T) \in H^*(M, \mathbb{Z})$.

Let $s = \dim S$, $t = \dim T$. Assuming $s + t = d$, and that S and T are transverse, the product

$$c(S) \cdot c(T) \in H^d(M, \mathbb{Z}) \cong \mathbb{Z}$$

computes the number of points of intersection between S and T, counted with sign. For a point $p \in S \cap T$, the sign is positive when the natural isomorphism

$$T_pS \oplus T_pT \cong T_pM$$

preserves the orientation on the tangent space, negative otherwise. In the case where S and T are not transverse, one can deform S in its tubular neighborhood to a submanifold S' having $c(S') = c(S)$, in such a way that S' and T *are* transverse, and then the product $c(S') \cdot c(T)$ has this geometric interpretation as the number of signed intersections.

One would like to be able to obtain a similar theory for projective algebraic varieties, but there are some subtleties. First, for fields other than $\mathbb{R}$ or $\mathbb{C}$, there is not an obvious replacement for singular cohomology. In any case, one would like to be able to compute products even inside singular varieties, where something like Poincaré duality cannot be expected to hold. Third, since subvarieties are defined algebraically, there is no obvious way to deform them to obtain tranversality, as one can do in the differentiable case.

It turns out that one can develop intersection theory in this algebraic setting, but some care has to be taken. In particular, the notion of multiplicity is fundamental in computing products in cases where the intersections cannot be made transverse. The degree of a projective variety $V \subset \mathbb{P}^k$, instead, measures the number of intersections (counted with multiplicities) with a generic linear space of dimension $k - \dim V$. For much more about intersection theory, the standard reference is [**Ful84**].

10.3. Formulas for multiplicity

In this section, we are going to investigate some properties of multiplicity. In particular we will derive some convenient formulas to compute the multiplicity in a local ring in terms of multiplicity in smaller rings such as quotients and localizations. We start by mentioning some elementary facts, which are immediate.

Proposition 10.3.1. *Let A be a semilocal Noetherian ring of dimension d, Q and Q' two ideals of definition for A and M a finitely generated A-module.*

(i) $e(Q, M) = \lim_{n\to\infty} \frac{d!}{n^d} \ell\left(\frac{M}{Q^n M}\right)$;

(ii) $e(Q^r, M) = e(Q, M) \cdot r^d$

(iii) *if* $Q \subset Q'$, $e(Q', M) \leq e(Q, M)$.

Also, we can rephrase Proposition 9.2.11 as follows:

Proposition 10.3.2. *Let A be a semilocal Noetherian ring, $Q \subset A$ an ideal of definition. If we have an exact sequence of finitely generated A-modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0\ ,$$

then $e(Q, M) = e(Q, M') + e(Q, M'')$.

Although we have defined the multiplicity for an A-module M, in many cases of interest only the multiplicity of A carries some new information. In particular, we can reduce the computation of the multiplicity of an A-module to that of a ring in many cases. The following result, known as the *additivity formula*, achieves this, and at the same time reduces the computation of multiplicity to the case of integral domains. Some authors call this the associativity formula (see for instance [**Eis95**, Ex. 12.11]), but following [**Lec57**] we reserve this name for Theorem 10.3.8.

Proposition 10.3.3 (Additivity formula)**.** *Let A be a local Noetherian ring, $Q \subset A$ an ideal of definition, M a finitely generated A-module. Let $d = \dim A$ and let $P_1, \dots, P_r$ be the minimal primes of A for which $\dim A/P_i = d$. Then*

$$e(Q, M) = \sum_{i=1}^{r} e(\overline{Q_i}, A/P_i)\ell(M_{P_i}),$$

where $\overline{Q_i} := \frac{Q+P_i}{P_i}$.

Proof. We use an induction over $s(M) := \sum \ell(M_{P_i})$.

When $s(M) = 0$, we must have $M_{P_i} = 0$ for all i, that is, $P_i \notin \operatorname{Supp}(M)$ for all i. But this means that $\operatorname{Ann}(M)$ is not contained in any minimal prime P_i of A such that $\dim A/P_i = d$. It follows that $\dim(M) = \dim(A/\operatorname{Ann}(M)) \leq d - 1$, and in this case $e(Q, M) = 0$.

For the inductive step, choose a minimal prime $P \in \operatorname{Supp}(M)$ that is one of $P_1, \dots, P_r$, say $P = P_1$. By Corollary 3.3.15, the minimal primes of $\operatorname{Supp}(M)$ and $\operatorname{Ass}(M)$ are the same, hence P is associated to M. This means that we can find $N \subset M$ such that $N \cong A/P$. By Proposition 10.3.2, we have

$$e(Q, M) = e(Q, N) + e(Q, M/N). \tag{10.3.1}$$

Remark that $N_P \cong \frac{A_P}{PA_P}$, so $\ell(N_P) = 1$, while $N_{P_i} = 0$ for $i \geq 2$, since these are different minimal primes. This allows us to compute

$$s(M) = \sum_{i=1}^{r} \ell(N_{P_i}) + \sum_{i=1}^{r} \ell((M/N)_{P_i}) = 1 + s(M/N).$$

By inductive hypothesis,

$$e(Q, M/N) = \sum_{i=1}^{r} e(\overline{Q_i}, A/P_i)\ell((M/N)_{P_i}).$$

Moreover, by our choice of N we have $e(Q, N) = e(Q, A/P) = e(\overline{Q_1}, A/P)$. Equation (10.3.1) becomes

$$e(Q, M) = e(\overline{Q_1}, A/P) + \sum_{i=1}^{r} e(\overline{Q_i}, A/P_i)\ell((M/N)_{P_i}).$$

For all $i \geq 2$, we have $(M/N)_{P_i} \cong M_{P_i}$, while for $i = 1$ we have $\ell((M/N)_{P_1}) = \ell(M_{P_1}) - 1$, and the thesis follows. ☐

Corollary 10.3.4. *Under the hypothesis of Proposition* 10.3.3*, assume that A is an integral domain. Then $e(Q, M) = e(Q)\operatorname{rk}(M)$.*

Proof. In this case, the only minimal prime is 0, hence $e(Q, M) = e(Q)\ell(M_0)$ and $\ell(M_0) = \dim_k M \otimes k$, where k is the fraction field of A. ☐

In particular, if A is a regular local ring, the multiplicity just measures the rank of an A-module, and gives no new information.

Remark 10.3.5. Proposition 10.3.3 can also be used taking $M = A$. In this case, it reduces the computation of $e(Q, A)$ to that of $e(Q, A/P)$ for various primes P.

Consider an affine variety V and a point $p \in V$. From a geometric point of view, the additivity formula allows us to compute the multiplicity of V at

p in term of the multiplicity of the components of V of maximal dimension passing through p.

Remark 10.3.6. The previous remark does not add anything in the case where V is irreducible. But even then, the additivity formula can be useful. In fact, let A be a local Noetherian ring with maximal ideal $\mathcal{M}$, and $\widehat{A}$ its completion in the $\mathcal{M}$-adic topology. By Corollary 7.5.18, A and $\widehat{A}$ have the same associated graded ring, hence the same Hilbert polynomial. It follows that $e(A) = e(\widehat{A})$, and it can happen that A is an integral domain while $\widehat{A}$ is not. Geometrically, this can happen if an irreducible variety has two branches at a point that are analytically separable.

Example 10.3.7. Let k be a ring of characteristic different from 2. The curve C defined by $y^2 = x^2 + x^3$ in $\mathbb{A}^2(k)$ is irreducible. But in the ring $k[[x, y]]$ one has the factorization $y^2 = (x\sqrt{1+x})^2$, where

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \cdots,$$

so the completion of the local ring of C at 0 is not an integral domain, and Proposition 10.3.3 applies. Again, we see that while C is irreducible, around 0 we can consider C as composed by two different components.

Related to the additivity formula, there is the following *associativity formula* of Lech from [**Lec57**], generalizing a previous result from [**Che45**].

Theorem 10.3.8 (Lech)**.** *Let $A, \mathcal{M}$ be a local ring, $a_1, \ldots, a_d$ a system of parameters for A and $Q = (a_1, \ldots, a_d)$. Fix a natural number $m \le d$ and let $Q_1 = (a_1, \ldots, a_m)$ and $Q_2 = (a_{m+1}, \ldots, a_d)$. Then*

$$e(Q) = \sum_P e\left(\frac{Q_1 + P}{P}\right) e\left(Q_2 \cdot A_P\right),$$

where the sum ranges over the primes $P \subset A$ that are minimal over Q_2 and that satisfy $\dim A/P + \operatorname{ht} P = \dim A$.

Notice that the elements of the formula are well defined: for such a prime P, $\overline{a_1}, \ldots, \overline{a_m}$ is a system of parameters for A/P, so $(Q_1 + P)/P$ is $\mathcal{M}/P$-primary, and $a_{m+1}/1, \ldots, a_d/1$ is a system of parameters for A_P, so $Q_2 \cdot A_P$ is $P \cdot A_P$-primary.

Let us set up some terminology. Just for the purpose of this proof, we will call a prime $P \subset A$ *balanced* if $\dim A/P + \operatorname{ht} P = \dim A$, and a chain

$$P_d \subsetneq P_1 \subsetneq \cdots \subsetneq P_0$$

compatible with $a_1, \ldots, a_d$ if for each $k = 0, \ldots, d$ we have

$$(a_{k+1}, \ldots, a_d) \subset P_k.$$

This allows us to state some simple lemmas.

Lemma 10.3.9. *Each prime P appearing in a chain compatible with $a_1, \dots, a_d$ is balanced.*

Proof. The chain can be split into a chain for A/P and one for A_P. □

Lemma 10.3.10. *The set of chains compatible with $a_1, \dots, a_d$ is finite.*

Proof. The hypothesis implies that P_k is a minimal prime of the ideal $(a_{k+1}, \dots, a_d)$, and minimal primes of an ideal are finite in a Noetherian ring. □

Lemma 10.3.11. *The set of chains compatible with $a_1, \dots, a_d$ is not empty.*

Proof. There exists a minimal prime of 0 which is balanced, call it P_d. Recursively choose P_k as a minimal prime of $P_{k+1}+(a_{k+1})$ which is balanced. □

Lemma 10.3.12. *Each balanced prime P_k such that $(a_{k+1}, \dots, a_d) \subset P_k$ and $\dim A/P_k = k$ appears in a chain compatible with $a_1, \dots, a_d$.*

Proof. Apply the previous lemma to find a chain for A/P_k and one for A_{P_k}, and join them. □

We are now ready to prove the associativity formula.

Proof of Theorem 10.3.8. Let Σ be the set of chains compatible with $a_1, \dots, a_d$, which is finite and nonempty by the previous lemmas. A repeated application of the additivity formula (Proposition 10.3.3) gives

$$e(Q, A) = \sum_{\underline{P} \in \Sigma} \ell\left((A/P_1)_{P_0}\right) \cdots \ell\left((A/P_d)_{P_{d-1}}\right) \ell(A_{P_d}). \tag{10.3.2}$$

A similar repeated application, just stopped earlier, gives

$$e(Q, A) = \sum_{\underline{P} \in \Sigma} e\left(\frac{Q + P_m}{P_m}\right) \ell\left((A/P_{m+1})_{P_m}\right) \cdots \ell\left((A/P_d)_{P_{d-1}}\right) \ell(A_{P_d}).$$

By applying (10.3.2) to A_{P_m}, we simplify this to

$$e(Q, A) = \sum_{P_m} e\left(\frac{Q + P_m}{P_m}\right) e(Q_{P_m}, A_{P_m}),$$

where the sum is over all balanced primes that contain $(a_{m+1}, \dots, a_d)$ appearing in a member of Σ. By the previous lemmas, these are exactly the balanced primes minimal over $(a_{m+1}, \dots, a_d)$, and we get the conclusion. □

We conclude this section with two classical result by Samuel. The first one ([**Sam53**]) relates length and multiplicity.

Proposition 10.3.13 (Samuel)**.** *Let $A, \mathcal{M}$ be a local Noetherian ring of dimension d, Q an ideal of definition generated by the system of parameters $a_1, \ldots, a_d$. Then*

$$e(Q, A) \leq \ell\left(\frac{A}{Q}\right).$$

Proof. Evaluation at $a_1, \ldots, a_d$ gives a surjective homomorphism of graded rings

$$\frac{A}{Q}[x_1, \ldots, x_d] \to \mathrm{Gr}_Q(A).$$

This gives a corresponding inequality between their Hilbert polynomials, and since both rings have dimension d, an inequality between their first coefficient

$$e\left(\frac{A}{Q}[x_1, \ldots, x_d]\right) \geq e(\mathrm{Gr}_Q(A)) = e(Q, A).$$

To conclude, we just note that the Hilbert polynomial of the first ring is

$$\ell\left(\frac{A}{Q}\right)\binom{n+d}{d}.$$

□

The next theorem gives a way to relate multiplicities in integral extensions. It appears in [**ZS76b**, Theorem 24, Chapter 10], but we give a slightly simplified statement.

Theorem 10.3.14. *Let $A \subset B$ be Noetherian integral domains, and assume that A is local with maximal ideal $\mathcal{M}$. Let $Q \subset A$ be an ideal of definition, and assume that B is integral over A. Take a primary decomposition*

$$Q \cdot B = \bigcap_{i=1}^{r} Q_i,$$

where Q_i is P_i-primary in B. Then the polynomials

$$[\mathcal{F}(B) : \mathcal{F}(A)]\chi_A^Q(n) \tag{10.3.3}$$

and

$$\sum_{i=1}^{r}[B/P_i : A/\mathcal{M}]\chi_{B_{P_i}}^{Q_i}(n) \tag{10.3.4}$$

have the same degree and leading term.

The statement can be simplified when all localizations B_{P_i} have the same dimension as A. In this case, all summands contribute to the leading term, and we get

Corollary 10.3.15. *In the above theorem, assume that* $\operatorname{ht} P_i = \operatorname{ht} \mathcal{M}$ *for all primes* P_i. *Then*

$$[\mathcal{F}(B) : \mathcal{F}(A)]e(Q, A) = \sum_{i=1}^{r} [B/P_i : A/\mathcal{M}]e(Q_i, B_{P_i}).$$

Remark 10.3.16. If we assume that A is integrally closed, then we can apply the above corollary. In fact, for every prime P_i as in Theorem 10.3.14, we have $Q \subset P_i \cap A$, which implies that $P_i \cap A = \mathcal{M}$. In this case, we can apply the going down Theorem 5.2.13 to conclude that $\operatorname{ht} P_i = \operatorname{ht} \mathcal{M}$.

Remark 10.3.17. If B is local, there is only one prime P_i that appears in the sum, and the corollary applies again.

Proof of Theorem 10.3.14.

For a given M, we denote $\ell_A(M)$ the length of M as an A-module and $\ell_B(M)$ the length of M as a B-module. Take n big enough, and let $d = \chi_{B_{P_i}}^{Q_i}(n)$. By Theorem 2.5.16, there is a chain of B_{P_i}-modules

$$Q_i^n = M_d \subsetneq M_{d-1} \subsetneq \cdots \subsetneq M_1 = P_i \cdot B_{P_i}$$

such that $\ell_{B_{P_i}}(M_k/M_{k+1}) = 1$. This is also a chain of B-modules, and their length as B-modules is the same—and in fact it is the same as $\dim_{B/P_i}(M_k/M_{k+1})$. As $A/\mathcal{M}$-vector spaces, though, each term M_k/M_{k+1} has dimension $[B/P_i : A/\mathcal{M}]$. This implies that

$$[B/P_i : A/\mathcal{M}]\chi_{B_{P_i}}^{Q_i}(n) = [B/P_i : A/\mathcal{M}]d = \ell_A(B/Q_i^n).$$

Moreover, $Q^n \cdot B = \bigcap_{i=1}^r Q_i^n$, and by the Chinese remainder theorem we have

$$\sum_{i=1}^{r} \ell_A(B/Q_i^n) = \ell_A(B/Q^n \cdot B).$$

Putting the two equations together, we recognize that the sum in (10.3.4) is just $\ell_A(B/Q^n \cdot B)$. On the other hand, the term in (10.3.3) amounts to $[\mathcal{F}(B) : \mathcal{F}(A)]\ell_A(A/Q^n)$, so we need to compare these two lengths.

Let $k = [\mathcal{F}(B) : \mathcal{F}(A)]$. If B is a free module over A, its rank must be k, in which case the equality

$$\ell_A(B/Q^n \cdot B) = k \cdot \ell_A(A/Q^n)$$

holds trivially. In general we cannot assume that, but we can find $b_1, \ldots, b_k \in B$ that span the field $\mathcal{F}(B)$ over $\mathcal{F}(A)$, and then one can fit B in

$$C \subset B \subset \frac{1}{d}C,$$

where $C = \langle b_1, \ldots, b_n \rangle_A$ and $d \in A$. We use this to compare $\ell_A(B/Q^n \cdot B)$ with $\ell_A(C/Q^n \cdot C)$.

Namely, there is a surjection

$$\frac{C}{Q^n \cdot C} \to \frac{C + Q^n \cdot B}{Q^n \cdot B},$$

which implies the inequality

$$\begin{aligned} \ell_A\left(\frac{C}{Q^n \cdot C}\right) &\geq \ell_A\left(\frac{C + Q^n \cdot B}{Q^n \cdot B}\right) \geq \ell_A\left(\frac{d \cdot B + Q^n \cdot B}{Q^n \cdot B}\right) \\ &= \ell_A\left(\frac{B}{Q^n \cdot B}\right) - \ell_A\left(\frac{B}{d \cdot B + Q^n \cdot B}\right). \end{aligned}$$

By a symmetric reasoning,

$$\ell_A\left(\frac{C}{Q^n \cdot C}\right) \leq \ell_A\left(\frac{B}{Q^n \cdot B}\right) + \ell_A\left(\frac{C}{d \cdot C + Q^n \cdot C}\right).$$

Hence, to understand the difference between $\ell_A(B/Q^n \cdot B)$ and $\ell_A(C/Q^n \cdot C)$, we look at the terms

$$\ell_A\left(\frac{B}{d \cdot B + Q^n \cdot B}\right)$$

and

$$\ell_A\left(\frac{C}{d \cdot C + Q^n \cdot B}\right).$$

In both cases, these are Hilbert functions for modules over the ring A/dA, which has dimension strictly less than $\dim A$, so $\ell_A(B/Q^n \cdot B)$ and $\ell_A(C/Q^n \cdot C)$ have the same degree and leading coefficient. Since for C we have

$$\ell_A(C/Q^n \cdot C) = k \cdot \ell_A(A/Q^n),$$

the theorem is proved. □

10.4. Multiplicity and valuations

In this section, we investigate the meaning of multiplicity as the order of vanishing of a suitable element. The prototypical result in this spirit is Proposition 10.2.14, which links the order of vanishing of a polynomial f in 0 to the multiplicity of the hypersurface defined by f in 0. We are going to greatly generalize this result, and in doing so, we state some conditions under which the order of vanishing of an element can be interpreted as a valuation.

Let A be a Noetherian ring, I an ideal. Then by Krull intersection theorem 7.5.24 we have

$$\bigcap_{n=0}^{\infty} I^n = 0,$$

hence for each element $a \in A \setminus \{0\}$ we can find a unique n such that $a \in I^n \setminus I^{n+1}$.

Definition 10.4.1. Let A be a Noetherian ring, I an ideal. If $a \in I^n \setminus I^{n+1}$, we denote $v_I(a) = n$. The function $v_I \colon A \setminus \{0\} \to \mathbb{N}$ is called the *order function* of I. When A is local with maximal ideal $\mathcal{M}$, we denote $v_A = v_{\mathcal{M}}$.

Now assume that A is local with maximal ideal $\mathcal{M}$. By construction, we have the inequality

$$v_A(a+b) \geq \min\{v_A(a), v_A(b)\}, \tag{10.4.1}$$

which makes v_A something similar to a discrete valuation, but this is not always the case. For one thing, valuations are defined on a field, and A need not be an integral domain, so it may not have a fraction field. But even if A is a domain, v_A can fail to be multiplicative, hence it is not always a valuation.

Remark 10.4.2. In fact, we always have the inequality

$$v_A(ab) \geq v_A(a) + v_A(b),$$

but it can be strict. The condition that we always have equality is equivalent to saying that the associated graded ring $A' = \mathrm{Gr}_{\mathcal{M}}(A)$ is an integral domain. In fact let $a \in \mathcal{M}^r \setminus \mathcal{M}^{r+1}$ and $b \in \mathcal{M}^s \setminus \mathcal{M}^{s+1}$. Then $\overline{a}$ is a nonzero element of A'_r and $\overline{b}$ is nonzero in A'_s, so their product is nonzero in A'_{r+s} if and only if $v_A(ab) = v_A(a) + v_A(b)$.

In this section, following [**Hor76**], we want to relate the order function of A to the function that measures multiplicity of an element $a \in A$.

Definition 10.4.3. Let $A, \mathcal{M}$ be a Noetherian local ring of dimension d. Given $a \in \mathcal{M} \setminus \{0\}$, we define

$$\mu_A(a) = e\left(\frac{A}{(a)}\right)$$

if $\dim A/(a) = d-1$, and $\mu_A(a) = \infty$ otherwise. We define $\mu(a) = 0$ for $a \in A \setminus \mathcal{M}$ (this is consistent, since in this case $A/(a)$ is trivial). The function

$$\mu_A \colon A \setminus \{0\} \to \mathbb{N} \cup \{\infty\}$$

is called the *multiplicity function* of A.

Remark 10.4.4. By Proposition 9.4.5, $\dim A/(a) \geq d-1$, so the condition $\mu_A(a) = \infty$ means that $\dim A/(a) = d$. This can only happen if a is a divisor of 0 by Proposition 9.5.2. Hence, on an integral domain, the multiplicity function takes values in $\mathbb{N}$.

It is no wonder that the functions v_A and μ_A are related, since both measure some kind of order of vanishing of an element. In some cases, we have already established a relation between the functions v_A and μ_A.

Example 10.4.5.

(a) Let A be a DVR. In this case, v_A is exactly the discrete valuation on A. Moreover, a quotient $A/(a)$ has dimension 0, and if $v_A(a) = n$, then $\ell(A/(a)) = n$ as well, since all ideals are powers of the maximal ideal. Hence $\mu_A = v_A$ in this case.

(b) Let k be a field, let $\mathcal{M} \subset k[x_1, \ldots, x_n]$ be the ideal of 0, and let $A = k[x_1, \ldots, x_n]_{\mathcal{M}}$. Then we can rephrase Proposition 10.2.14 by saying that $\mu_A = v_A$. Moreover, it is immediate that v_A is multiplicative by looking at the monomials of lowest total degre. Hence, we can extend v_A to a valuation on $k(x_1, \ldots, x_n)$.

In both examples, things behave as well as we can hope: the functions v_A and μ_A agree and both are valuations. We will prove in this section that these two phenomena are strictly related. In fact, while v_A satisfies (10.4.1), the multiplicity function μ_A is almost always multiplicative.

Proposition 10.4.6. *Let $A, \mathcal{M}$ be a local Noetherian ring. If $\mu_A(a) = \infty$ or a is not a divisor of zero, then*

$$\mu_A(ab) = \mu_A(a) + \mu_A(b).$$

The hypothesis that a is not a divisor of zero is necessary; see Exercise 8.

Proof. In the case where $\mu_A(a) = \infty$, there is a minimal prime $P \subset A$ such that $a \in P$ and $\dim A/P = \dim A$. Hence, $ab \in P$ as well, and $\mu_A(ab) = \infty$.

Thus, we can assume that $\mu_A(a)$ and $\mu_A(b)$ are both finite. Since a is not a divisor of 0, multiplication by a induces an isomorphism

$$\frac{A}{(b)} \cong \frac{(a)}{(ab)}.$$

Using the exact sequence

$$0 \longrightarrow \frac{(a)}{(ab)} \longrightarrow \frac{A}{(ab)} \longrightarrow \frac{A}{(a)} \longrightarrow 0\ ,$$

we deduce the equality

$$\ell\left(\frac{A}{(ab)}\right) = \ell\left(\frac{A}{(a)}\right) + \ell\left(\frac{A}{(b)}\right). \tag{10.4.2}$$

Let $d = \dim A$. We can specialize the additivity formula in Proposition 10.3.3 to get

$$e\left(\frac{A}{(a)}\right) = \sum_P e\left(\frac{A}{P}\right) \ell\left(\frac{A_P}{a \cdot A_P}\right),$$

where the sum ranges over all primes $P \ni a$ such that $\dim A/P = d-1$. We can also include primes P that do not contain a, since in that case $A_P/(a \cdot A_P)$ is 0. A similar formula holds for b and ab.

Using additivity of the lengths (10.4.2) in the rings of the form A_P, we obtain the conclusion. □

Corollary 10.4.7. *Let $A, \mathcal{M}$ be a local Noetherian integral domain. Then the multiplicity function μ_A is multiplicative.*

We now investigate under what conditions v_A is multiplicative as well. A simple case is the following.

Proposition 10.4.8. *Let A be a regular local ring. Then the order function v_A is a discrete valuation on A.*

Remark 10.4.9. Notice that A need not be a discrete valuation ring—in fact this cannot happen unless $\dim A = 1$. What we mean is just that v_A extends to a valuation with values in $\mathbb{Z}$ on the fraction field $\mathcal{F}(A)$. Here we are implicitly using the fact that A is an integral domain (Theorem 10.1.9).

Proof. By Remark 10.4.2, it is enough to prove that $\mathrm{Gr}_{\mathcal{M}}(A)$ is a domain, and Proposition 10.1.8 tells us that $\mathrm{Gr}_{\mathcal{M}}(A)$ is isomorphic to $k[x_1, \dots, x_d]$, where $d = \dim A$ and $k = A/\mathcal{M}$. □

In order to connect the functions v_A and μ_A, we start with a lemma that simplifies the computation of μ_A.

Lemma 10.4.10. *Let A be a semilocal Noetherian ring, Q an ideal of definition, $a \in Q$. Then*

$$\chi_A^Q(n) - \chi_{A/(a)}^{Q/(a)}(n) = \ell\left(\frac{A}{(Q^n : a)}\right).$$

Proof. This is just a computation:

$$\begin{aligned}\chi_A^Q(n) - \chi_{A/(a)}^{Q/(a)}(n) &= \ell\left(\frac{A}{Q^n}\right) - \ell\left(\frac{A}{Q^n + (a)}\right) \\ &= \ell\left(\frac{Q^n + (a)}{Q^n}\right) = \ell\left(\frac{(a)}{Q^n \cap (a)}\right) \\ &= \ell\left(\frac{(a)}{(a) \cdot (Q^n : a)}\right) = \ell\left(\frac{A}{(Q^n : a)}\right).\end{aligned}$$

The last equality uses the fact that, while multiplication by a is not necessarily injective, its kernel is contained in $(Q^n : a)$ anyway. □

Using this lemma, we see that to control the difference between $e(Q, A)$ and $e(Q/(a), A/(a))$ we need to understand the ideal $(Q^n : a)$. Assume that

$a \in Q^s$. Then we have the inclusion $Q^{n-s} \subset (Q^n : a)$. Samuel introduced in [**Sam53**] the following definition to capture the case where we are able to control the behavior of $(Q^n : a)$.

Definition 10.4.11. Let A be a semilocal Noetherian ring, Q an ideal of definition. We say that the element $a \in Q^s$ is *superficial* (of order s) for Q if there exist an integer c such that

$$(Q^n : a) \cap Q^c = Q^{n-s}$$

for all n large enough.

When $A, \mathcal{M}$ is a local ring, we call the element superficial of order s if it is so for $\mathcal{M}$. Notice that this implies that $a \notin \mathcal{M}^{s+1}$, so in fact the order $v_A(a) = s$. This gives a hint that superficial elements can be used to connect v_A to μ_A. We can specialize the previous lemma to the case of superficial elements:

Proposition 10.4.12. *Let A be a semilocal Noetherian ring, Q an ideal of definition, a superficial of order s for Q. Then there exists an integer c such that*

$$\chi_A^Q(n) - \chi_A^Q(n-s) \leq \chi_{A/(a)}^{Q/(a)}(n) \leq \chi_A^Q(n) - \chi_A^Q(n-s) + \chi_A^Q(c)$$

for n large enough.

Proof. By definition of superficial element, we get

$$\begin{aligned}\ell\left(\frac{(Q^n : a)}{Q^{n-s}}\right) &= \ell\left(\frac{(Q^n : a)}{(Q^n : a) \cap Q^c}\right) \\ &= \ell\left(\frac{Q^c + (Q^n : a)}{Q^c}\right) \leq \ell\left(\frac{A}{Q^c}\right),\end{aligned}$$

which we can translate to

$$0 \leq \chi_A^Q(n-s) - \ell\left(\frac{A}{(Q^n : A)}\right) \leq \chi_A^Q(c).$$

We conclude by Lemma 10.4.10. □

By comparing just the first coefficients in the Proposition, we get

Corollary 10.4.13. *Let A be a semilocal Noetherian ring of dimension $d > 1$, $a \in Q^s$ superficial of order s for the ideal of definition Q. Then $e(Q/(a), A/(a)) = e(Q, A) \cdot s$.*

Proof. We need to compute the first coefficient of the polynomial $\chi_A^Q(n) - \chi_A^Q(n-s)$. The coefficients in the same degree cancel each other, so the first nonzero monomial is in degree $d-1$ and is given by

$$\frac{e(Q,A)}{d!}(n^d - (n-s)^d) = \frac{e(Q,A)}{d!} \cdot dsn^{d-1} + \cdots,$$

giving $e(Q/(a), A/(a)) = e(Q, A) \cdot s$. □

Corollary 10.4.14. *Let $A, \mathcal{M}$ be a Noetherian local ring. Then $A[[x]]$ is local as well, and $e(A[[x]]) = e(A)$.*

Proof. The ring $A[[x]]$ is clearly local with maximal ideal $(\mathcal{M}, x)$. Moreover, it is easy to check that x is superficial of order 1. If $\dim A > 0$, then Corollary 10.4.13 applies and gives the conclusion. If $\dim A = 0$, then $e(A) = \ell(A)$ and the Hilbert polynomial of $A[[x]]$ is $\chi_{A[[x]]}(n) = \ell(A) \cdot n$. □

We can finally prove the result that links v_A to μ_A.

Theorem 10.4.15. *Let $A, \mathcal{M}$ be a local Noetherian integral domain. Then v_A is a valuation if and only if $v_A = k \cdot \mu_A$, in which case $k = e(A)$.*

Proof. If v_A is a multiple of μ_A, then it is multiplicative by Corollary 10.4.7, hence it is a valuation.

Conversely, assume that v_A is multiplicative. Then the ring $G_{\mathcal{M}}(A)$ is an integral domain, and this implies that every nonzero element $a \in A$ is superficial (this should be clear, but see Lemma 10.5.2). Assuming $\dim A > 1$, by Corollary 10.4.13, we get

$$\mu_A(a) = e\left(\frac{A}{(a)}\right) = v_A(a) \cdot e(A).$$

For the case where $\dim A = 1$, apply the result to $A[[x]]$. It is a simple check that the order function $v_{A[[x]]}$ is also a valuation, hence $v_{A[[x]]} = k \cdot \mu_{A[[x]]}$. For $a \in A$, we have $v_{A[[x]]}(a) = v_A(a)$, and by Corollary 10.4.14 also $\mu_{A[[x]]}(a) = \mu_A(a)$, so we get the conclusion. □

Putting this together with Proposition 10.4.8, we get

Corollary 10.4.16. *Let A be a regular local integral domain. Then $v_A = \mu_A$, and both functions are valuations.*

In the case where v_A and μ_A disagree, we cannot expect that μ_A is a valuation, but still we have the following result that we quote without proof (see [**Hor76**, Theorem 4]).

Theorem 10.4.17. *Let A be a local Noetherian integral domain. Then there exist r discrete valuations $v_1, \dots, v_r$ on $\mathcal{F}(A)$ and corresponding integers $n_1, \dots, n_r$ such that*

$$\mu_A = n_1 v_1 + \cdots + n_r v_r.$$

10.5. Superficial elements

Let A be a semilocal ring, $Q \subset A$ an ideal of definition. In the previous section, we defined an element $a \in Q^s$ to be *superficial* of order s if it has the property that

$$(Q^n : a) \cap Q^c = Q^{n-s}$$

for a fixed $c \in \mathbb{N}$ and for all n large enough. We used this to prove Corollary 10.4.13 that states that in this case we have $e(Q/(a), A/(a)) = e(Q, A) \cdot s$, provided $\dim A > 1$. This allows us to prove properties of multiplicity by induction on the dimension.

In this section, we investigate this technical tool in more detail, starting from some existence results for superficial elements, and use it to prove some deeper properties of multiplicity. Most of this material is taken from [**ZS76b**]. The main existence result is

Theorem 10.5.1. *Let A be a semilocal Noetherian ring, $Q \subset A$ an ideal of definition. Then there exists a superficial element of order s for some $s \geq 1$.*

In order to prove it, we rephrase the condition of being superficial.

Lemma 10.5.2. *Let A be a semilocal Noetherian ring, $Q \subset A$ an ideal of definition, $A' = G_Q(A)$ the associated graded ring. Given an element $a \in Q^s$, let $\overline{a} \in A'_s$ be its image. Then a is superficial of order s if and only if $\operatorname{Ann}_{A'}(\overline{a}) \subset A'_{<c}$ for some $c \in \mathbb{N}$.*

Proof. This is just a matter of spelling out the definitions.

To say that a is *not* superficial (with constant c) means that we can find $b \in Q^c \setminus Q^{n-s}$ such that $ab \in Q^n$. Rephrasing in terms of the order function v_Q of Q, this means that $c \leq v_Q(b) < n - s$ and $v_Q(ab) \geq n$.

To say that $\operatorname{Ann}_{A'}(a) \not\subset A'_{<c}$ means that we can find b with $v_Q(b) \geq c$ such that $\overline{b} \in A'_{v_Q(b)}$ satisfies $\overline{a}\overline{b} = 0$, which is the same as $ab \in Q^{s+v_Q(b)+1}$.

The two conditions are now seen to be equivalent. □

Proof of Theorem 10.5.1.

Let $A' = \operatorname{Gr}_Q(A)$ be the associated graded ring. By the lemma, we need to understand divisors of 0 in A', so we investigate the associated primes of 0.

Let A'_+ be the irrelevant ideal, and partition the associated primes $P_1, \dots, P_r$ of 0 in A', so that the last $r - h$ contain A_+ and the first h do not. We claim that we can find a homogeneous element $\overline{a} \in A'_+$ such that $\overline{a} \notin P_i$ for $i = 1, \dots, h$.

To prove this, choose homogeneous elements $b_i \in A'_+ \setminus P_i$ and $c_{ij} \in P_j \setminus P_i$ for $i, j = 1, \dots, h$, and then take

$$a_i = b_i \prod_{j \neq i} c_{ij},$$

so that $a_i \in A'_+$, $a_i \notin P_i$ but at the same time $a_i \in P_j$ for all $j \neq i$. Finally, letting

$$\overline{a} = \sum_{i=1}^{h} a_i^{d_i}$$

for suitable d_i such that $\overline{a}$ is homogeneous, we have the desired element.

Say $\overline{a}$ is homogeneous of order s, and let $a \in Q^s$ be a representative of $\overline{a}$. We want to show that a is superficial of order s. Using Lemma 10.5.2, we look at an element $b \in \mathrm{Ann}(\overline{a})$. We can assume that b is homogeneous.

Consider a primary decomposition of 0

$$0 = \bigcap_{i=1}^{r} Q_i,$$

where Q_i is P_i-primary. For a suitable exponent c we have

$$(A'_+)^c \subset \bigcap_{i=h+1}^{r} Q_i. \tag{10.5.1}$$

By our choice of $\overline{a}$ we have $b \in P_i$ for $i = 1, \ldots, h$. If we assume $b \in A'_{\geq c}$, we find from (10.5.1) that $b \in P_i$ for $i = h+1, \ldots, r$, hence $b = 0$. This proves that $\mathrm{Ann}(\overline{a}) \subset A'_{<c}$, and the theorem. □

Unfortunately, we are not free to choose the order s at will in the above theorem. This is not a limitation of our approach.

Example 10.5.3. Let $k = \mathbb{Z}/2\mathbb{Z}$ and consider the ring

$$A := \frac{k[[x,y]]}{(xy(x+y))}.$$

Then A does not have superficial elements of order 1 for its maximal ideal $\mathcal{M}$. For if $a \in A$ is such an element, then $a \in \mathcal{M} \setminus \mathcal{M}^2$. But there are only 3 such elements: x, y and $x+y$, and each of them is annihilated by a homogeneous element of arbitrarily large order in $\mathrm{Gr}_{\mathcal{M}}(A)$.

We can however guarantee the existence of superficial elements of any order, assuming the residue field is infinite.

Theorem 10.5.4. *Let $A, \mathcal{M}$ a Noetherian local ring of positive dimension, and assume that $k = A/\mathcal{M}$ is infinite. Let Q be a $\mathcal{M}$-primary ideal. Then for any $s \geq 1$ there exists a superficial element of order s for Q. Moreover, we can take such an element outside all minimal primes of* 0 *in A.*

Proof. As in Theorem 10.5.1, we consider the graded ring $A' := G_Q(A)$ and take a primary decomposition of 0 in A':

$$0 = \bigcap_{i=1}^{r} Q_i,$$

where Q_i is P_i-primary, and the indices have been chosen so that $A'_+ \subset P_i$ exactly for $i = h+1, \dots, r$, for some $h \le r$.

Let $D_1, \dots, D_t$ be the minimal primes of 0 in A. Given a nonzero element $a \in A$ with $v_Q(a) = s$, denote $\overline{a} \in A'_s$ its class, and given an ideal $I \subset A$ denote

$$\overline{I} := (\overline{a} \mid a \in I, a \neq 0).$$

We claim that $A'_s \not\subset \overline{D_i}$ for any $i = 1, \dots, t$ and any $s \in \mathbb{N}$. Otherwise, $Q^s \subset Q^{s+1} + D_i$, and by Nakayama's lemma $Q^s \subset D_i$. Since D_i is prime, this implies that $\mathcal{M} \subset D_i$, which can only happen if $\dim A = 0$.

Now choose any $s > 0$. Then A_s is not contained in any of the $\overline{D_i}$, as well as in P_j for $j \le h$. Let $M = A'_s/A'_{s+1}$, which we regard as a module over $A/Q \cong A'/A'_+$. Every ideal J of A' defines a submodule $J \cap A'_s/A'_{s+1}$, and in particular the ideals $\overline{D_i}$ and P_j for $j \le h$ define a collection of submodules N_i, $i = 1, \dots, h+t$. By what we have just said, we have strict inclusions $N_i \subsetneq M$.

If we prove that $M \neq \bigcup_{i=1}^{h+t} N_i$, then we can find $\overline{a} \in A'_s$ such that $a \notin D_i$ for $i = 1, \dots, t$ and $\overline{a} \notin P_j$ for $j = 1, \dots, h$. As in the proof of Theorem 10.5.1, this means that a is superficial of order s, and we are done.

It remains to prove that $M \neq \bigcup N_i$. This is clear when $Q = \mathcal{M}$, for in this case M is a vector space over the infinite field k, hence it cannot be the union of finitely many subspaces. In general, we reduce to this case as follows. Choose g such that $\mathcal{M}^g \subset Q$. Given a submodule $N \subsetneq M$, we have

$$N + \frac{\mathcal{M}}{Q} M \subsetneq M,$$

otherwise, we would have the chain

$$\begin{aligned} M = N + \frac{\mathcal{M}}{Q} M &= N + \frac{\mathcal{M}}{Q}\left(N + \frac{\mathcal{M}}{Q} M\right) = N + \frac{\mathcal{M}^2}{Q} M \\ &= \cdots = N + \frac{\mathcal{M}^g}{Q} M = N. \end{aligned}$$

Therefore, as vector spaces,

$$\frac{N + \mathcal{M}M}{\mathcal{M}M} \subsetneq \frac{M}{\mathcal{M}M}.$$

In particular, $(N_i + \mathcal{M}M)/\mathcal{M}M$ is a strict subspace of $M/\mathcal{M}M$, and the union over all i cannot be the whole space. $\square$

Superficial elements are the basis for many results on multiplicities, including the proof of the multiplicity 1 criterion (Theorem 10.2.7). We will just give an application in this section.

Proposition 10.5.5. *Let $A, \mathcal{M}$ be a Noetherian local ring, $Q \subset A$ an ideal of definition, and assume that $k = A/\mathcal{M}$ is infinite. Then there exists an ideal of definition $Q' \subset Q$ generated by a system of parameters such that $e(Q', A) = e(Q, A)$.*

Proof. Induction on $d := \dim A$. The case $d = 0$ is trivial, since we can take $Q' = 0$. The difficult case is $d = 1$. Choose an element $a \in Q$ which is superficial of order 1 for Q. By definition we have

$$(Q^n : a) \cap Q^c = Q^{n-1} \tag{10.5.2}$$

for a fixed $c \in \mathbb{N}$. Now apply the Artin–Rees lemma to the ideal (a). The Q-adic topology on (a) and the topology induced by the Q-adic topology on A have bounded difference, so

$$Q^n \cap (a) \subset Q^{n-k} \cdot (a) \tag{10.5.3}$$

for some $k \in \mathbb{N}$.

Assume first that a is not a divisor of zero, so that the multiplication map

$$\mu_a \colon A \to a \cdot A$$

is injective. In this case, $\mu_a^{-1}(a \cdot Q^{n-k}) = Q^{n-k}$, while by definition

$$\mu_A^{-1}(Q^n \cap (a)) = (Q^n \cap (a) : a) = (Q^n : a).$$

So, by taking preimages in (10.5.3), we get

$$(Q^n : a) \subset Q^{n-k}.$$

When $n - k \geq c$, we can put this together with (10.5.2) and get

$$(Q^n : a) = Q^{n-1}.$$

Using Lemma 10.4.10 we get

$$\chi_{A/(a)}^{Q/(a)}(n) = \chi_A^Q(n) - \chi_A^Q(n-1) = e(Q, A),$$

since $\dim A = 1$. In particular, $\dim A/(a) = 0$, so the left-hand side is just $\ell(A/(a))$. Since a is not a zero divisor,

$$\ell\left(\frac{(a^n)}{(a^{n+1})}\right) = \ell\left(\frac{A}{(a)}\right) = e(Q, A),$$

which proves that $e((a), A) = e(Q, A)$ and we can take $Q' = (a)$. This ends the proof for the case $d = 1$, assuming a is not a divisor of 0.

For the general case—still with $d = 1$—we first notice that all chains of primes in A have the form $P \subset \mathcal{M}$, where P is a minimal prime of 0. Choose any superficial element $a \in Q$ of order 1 that lies outside the minimal primes, so that $\dim A/(a) = 0$. If a is not a divisor of 0 we are done, otherwise we can apply the above result to $A/\operatorname{Ann}(a)$. To this end, define $I := \operatorname{Ann}(a)$, $\overline{A} := A/I$ and $\overline{Q} := (Q+I)/I$. Then we can compute

$$\begin{aligned}\chi_{\overline{A}}^{\overline{Q}}(n) &= \ell\left(\frac{A}{Q^n + I}\right) \\ &= \ell\left(\frac{A}{Q^n}\right) - \ell\left(\frac{Q^n + I}{Q^n}\right) \\ &= \chi_A^Q(n) - \ell\left(\frac{I}{Q^n \cap I}\right).\end{aligned}$$

Now, I is a finitely generated module over $A/(a)$, which is Artinian, so it has finite length. Moreover, $\bigcap_n Q^n \cap I = 0$ by Krull's intersection theorem, so we have $Q^n \cap I = 0$ for n big enough. In this case, we can simplify the last equation to

$$\chi_{\overline{A}}^{\overline{Q}}(n) = \chi_A^Q(n) - \ell(I).$$

This implies that $\dim \overline{A} = 1$ and that $e(\overline{Q}, \overline{A}) = e(Q, A)$. The same reasoning, applied to (a), show that $e(\overline{(a)}, \overline{A}) = e((a), A)$. Since $\overline{a}$ is not a divisor of 0 in $\overline{A}$, we can apply the first part of the proof to $\overline{A}$ and conclude that $e(\overline{(a)}, \overline{A}) = e(\overline{Q}, \overline{A})$, so finally $e((a), A) = e(Q, A)$ and we can take $Q' = (a)$. This ends the proof for the case $d = 1$.

For $d > 1$, we perform an inductive step. Using Theorem 10.5.4, choose an element $a \in Q$ which is superficial of order 1 for Q. Let $\overline{A} = A/(a)$ and $\overline{Q} = Q/(a)$; then $e(\overline{Q}, \overline{A}) = e(Q, A)$ and by induction we find a system of parameters $\overline{a_2}, \dots, \overline{a_d}$ for $\overline{A}$ such that

$$e((\overline{a_2}, \dots, \overline{a_d}), \overline{A}) = e(Q, A).$$

Then $\{a, a_2, \dots, a_d\} \subset Q$ is the desired system of parameters for A. In fact, let $Q' = (a, a_2, \dots, a_d)$. Then using Lemma 10.4.10 we deduce that

$$e(Q', A) \leq e(Q'/(a), \overline{A}) = e(\overline{Q}, \overline{A}) = e(Q, A),$$

and vice versa $e(Q', A) \geq e(Q, A)$ in any case, so we must have equality. □

10.6. Cohen's structure theorem

In this section, we discuss a famous result of Cohen ([**Coh46**]) that gives a precise description of complete local rings. In particular, we will be able to exactly describe the structure of a complete, *regular* local ring. Under an additional hypothesis, which we will describe below, such a ring is isomorphic to a ring of power series over its residue field. Geometrically, this is essentially an algebraic version of the inverse function theorem.

In differential geometry, the inverse function theorem describes the local structure of the zero locus V of k smooth functions $f_1, \ldots, f_k$ defined on an open set in $\mathbb{R}^n$. Namely, if p is a point in V, and the differentials of the functions $\{f_i\}$ are independent in p, then V admits a local chart $\phi\colon \mathbb{R}^{n-k} \to V$ around p, giving to V the structure of an embedded differentiable manifold.

In the algebraic setting, one does not have the luxury of a local chart. Still, let $V \subset \mathbb{A}^n(k)$ be an affine variety of dimension d, p a point of V, and $A, \mathcal{M}$ the local ring of V in p. If V is regular in p, we will be able to say that *the completion* $\widehat{A}$ of A with respect to the $\mathcal{M}$-adic topology is isomorphic to $k[[x_1, \ldots, x_d]]$. In a sense, from an analytic point of view V has something that resembles a local chart, even though the Zariski topology is too coarse to express this.

Our treatment follows in part [**Katb**] and [**Mur**], but the reader can also consult tag 0323 in [**dJea20**]. The equicharacteristic case is Theorem 7.8 in [**Eis95**].

In order to state our results we need

Definition 10.6.1. Let $A, \mathcal{M}$ be a local ring with residue field $k = A/\mathcal{M}$. We say that the field $K \subset A$ is a *coefficient field* if the induced map $K \to A/\mathcal{M} = k$ is surjective (hence an isomorphism).

This condition cannot always be ensured. For a simple example, A may be of characteristic 0, while k may have characteristic $p > 0$ (as it happens with the ring $\mathbb{Z}_p$). In this case, a coefficient field cannot exist.

In fact, a few cases can arise. If $\operatorname{char}(k) = 0$, we must have $\operatorname{char}(A) = 0$ as well. If instead $\operatorname{char}(k) = p > 0$, then either $\operatorname{char}(A) = 0$ or $\operatorname{char}(A) = p^n$ for some n. In fact, A is local and for every prime $q \neq p$ we have $q \notin \mathcal{M}$, so q is invertible in A.

Definition 10.6.2. Let $A, \mathcal{M}$ be a local ring with residue field $k = A/\mathcal{M}$. We say that A is *equicharacteristic* if $\operatorname{char}(A) = \operatorname{char}(k)$.

By the above discussion, if A is not equicharacteristic then $\operatorname{char}(k) = p > 0$ and either $\operatorname{char}(A) = 0$ or $\operatorname{char}(A) = p^n$ for some $n > 1$. If A admits a coefficient field, then A is equicharacteristic. To handle the other cases, we need a more subtle definition.

Definition 10.6.3. Let $A, \mathcal{M}$ be a local ring with residue field $k = A/\mathcal{M}$. We say that the ring $C \subset A$ is a *coefficient ring* if

(1) C is a complete, Hausdorff local ring
(2) the ideal $C \cap \mathcal{M}$ is generated by $p = \operatorname{char}(k)$
(3) the induced map $C \to A/\mathcal{M} = k$ is surjective.

Remark 10.6.4. From 3), it follows that the maximal ideal of C is $C \cap \mathcal{M} = p \cdot C$. Moreover, since C is Hausdorff in the topology defined by its maximal ideal, $\bigcap p^k \cdot C = 0$. If $I \subset C$ is any ideal, we can find a biggest k such that $I \subset p^k \cdot C$. In particular there exists $x \in I \setminus (p^{k+1} \cdot C)$, and we can write $x = p^k \cdot y$ for some $y \in C$. But $y \notin p \cdot C$, hence y is invertible in C and $I = p^k \cdot C$. It follows that C is a principal ideal domain, and all ideals of C are powers of its maximal ideal.

If moreover, $p \notin (C \cap \mathcal{M})^2$, it follows by Nakayama's lemma that all ideals $p^k \cdot C$ are distinct. In this case, one can define a valuation v on $C \setminus \{0\}$ by declaring that $v(x) = n$ if $x \in p^n \cdot C$ but $x \notin p^{n+1} \cdot C$ and extend it to the fraction field of C. In this case, C is a discrete valuation ring.

The following proposition allows us to get some structure from the mere existence of a coefficient field or ring.

Proposition 10.6.5. *Let $A, \mathcal{M}$ be a Noetherian complete local ring, $C \subset A$ a coefficient field or ring. Then there is a surjective map*

$$\phi\colon C[[x_1, \dots, x_t]] \to A.$$

More precisely if $\mathcal{M} = (a_1, \dots, a_t)$, the map defined by $\phi(x_i) = a_i$ is surjective, hence one can take $t = \operatorname{embdim} A$.

Proof. For $n \geq 0$, take an element $a \in \mathcal{M}^n$, and write

$$a = \sum b_i m_i,$$

where $b_i \in A$ and m_i is a monomial of degree n in $a_1, \dots, a_t$. By definition of a coefficient field or ring, we can write $b_i = c_i + d_i$, where $c_i \in C$ and $d_i \in \mathcal{M}$. Hence we obtain a decomposition $a = c + d$, where $d \in \mathcal{M}^{n+1}$ and c is a form of degree n in $a_1, \dots, a_t$ with coefficients in C.

Now given any $a \in A$ we apply this construction repeatedly, starting from

$$a = c_1 + d_1,$$

with c_1 a linear term in $a_1, \dots, a_t$ with coefficients in C and $d_1 \in \mathcal{M}$. We repeat this writing

$$d_1 = c_2 + d_2,$$

and so on. After n steps, we have

$$a = (c_1 + c_2 + \cdot + c_n) + d_n,$$

where the term in parenthesis is a polynomial in $a_1, \dots, a_t$ with coefficients in C. Taking the limit (which exists and is unique since A is complete and Noetherian) we get

$$a = \sum_{i=1}^{\infty} c_i,$$

which is a power series in $a_1, \dots, a_t$ with coefficients in C.

We can define ϕ by sending x_i to a_i—then ϕ is well defined since A is complete, and is surjective by the above argument. □

In view of this result, it makes sense to ask when a complete local ring admits a coefficient field or a coefficient ring. This beautiful result of Cohen gives a pretty complete answer.

Theorem 10.6.6 (Cohen)**.** *Let A be a complete Noetherian local ring.*

(i) *If A is equicharacteristic, then A admits a coefficient field.*

(ii) *If A is not equicharacteristic, then A admits a coefficient ring that is the image of a DVR.*

Before going into the proof of the theorem, we will state and prove some important consequences.

Corollary 10.6.7. *Let $A, \mathcal{M}$ be a complete regular local ring of dimension d, with residue field k.*

(i) *If A is equicharacteristic, then*

$$A \cong k[[x_1, \ldots, x_d]].$$

(ii) *If A is not equicharacteristic,* $\operatorname{char}(k) = p > 0$ *and $p \notin \mathcal{M}^2$, then there exists complete DVR C such that*

$$A \cong C[[x_1, \ldots, x_{d-1}]].$$

Proof. Let us first consider the equicharacteristic case. By Cohen's theorem, there is a coefficient field $k \subset A$. If $a_1, \ldots, a_d$ is a regular system of parameters, using Proposition 10.6.5, we find a surjection

$$\phi\colon k[[x_1, \ldots, x_d]] \to A$$

such that $\phi(x_i) = a_i$. It remains to prove that ϕ is injective. There are two ways to see this. For one thing, Proposition 10.1.8 implies that ϕ induces an isomorphism between the associated graded rings, and then one can use Remark 10.4.2. Alternatively, one sees that

$$\dim k[[x_1, \ldots, x_d]] = \dim A = d.$$

If $I := \ker \phi$ was not trivial, the dimension of A would be less than d since $k[[x_1, \ldots, x_d]]$ is integral, a contradiction.

The same proof works in the mixed characteristic case with slight modifications. In this case, we can take a regular system of parameters of the form $p, a_1, \ldots, a_{d-1}$. By Cohen's theorem, we find a coefficient ring C, and using Proposition 10.6.5 a surjection

$$\phi\colon C[[x_1, \ldots, x_{d-1}]] \to A$$

such that $\phi(x_i) = a_i$. By Remark 10.6.4, C is in fact a DVR, so the dimension on the two sides are equal, and ϕ is injective as in the previous case. □

Corollary 10.6.8. *Let A be a complete Noetherian local ring. Then A is the image of a complete* regular *local ring.*

Proof. This follows immediately by Proposition 10.6.5 and Theorem 10.6.6, since the rings $k[[x_1, \ldots, x_d]]$, where k is a field, and $C[[x_1, \ldots, x_{d-1}]]$, where C is a DVR, are complete and regular (see Exercise 10). □

Corollary 10.6.9. *Let $A, \mathcal{M}$ be a complete, equicharacteristic local ring of dimension d, with residue field k. Then A is a finitely generated module over a subring $B \subset A$ such that*

$$B \cong k[[x_1, \ldots, x_d]].$$

Proof. Let $a_1, \ldots, a_d$ be a system of parameters for A, and $Q = (a_1, \ldots, a_d)$. Since $\sqrt{Q} = \mathcal{M}$ and A is complete with respect to $\mathcal{M}$, it is also complete with respect to Q. It follows that it is a well-defined homomorphism

$$\phi\colon k[[x_1, \ldots, x_d]] \to A$$

such that $\phi(x_i) = a_i$. Let $B := \operatorname{im} \phi$. Since A/Q is finitely generated over B/Q, by Corollary 7.6.4 A is finitely generated as a B-module.

In particular, $\dim B = d$ by Proposition 9.5.10. Since $k[[x_1, \ldots, x_d]]$ is an integral domain, each chain of primes ends in (0), so ϕ must be injective, otherwise $\dim B < d$. It follows that $B \cong k[[x_1, \ldots, x_d]]$ as desired. □

We can also derive from Cohen's theorem a proof of the multiplicity one criterion of Nagata, albeit with some additional assumptions.

Partial Proof of Theorem 10.2.7. Let A be a local ring with maximal ideal $\mathcal{M}$. If A is regular, then $e(A) = 1$ by Remark 10.2.3. Moreover, the completion $\widehat{A}$ with respect to the $\mathcal{M}$-adic topology is regular as well, hence an integral domain by Theorem 10.1.9. So, the only associated prime of $\widehat{A}$ is 0, and A is unmixed.

Conversely, assume that $e(A) = 1$ and A is unmixed. To prove that A is regular, it is not restrictive to assume that A is $\mathcal{M}$-adically complete. To prove this implication, we make the additional assumption that A is equicharacteristic and the residue field $k = A/\mathcal{M}$ is infinite (for instance, both are true if $\operatorname{char} k = 0$). Since k is infinite, by Proposition 10.5.5 there is an ideal $Q \subset \mathcal{M}$ generated by a system of parameters such that $e(Q, A) = 1$.

If $Q = (a_1, \ldots, a_d)$, where $\dim A = d$, using Corollary 10.6.9, we know that A is finitely generated over the power series ring $B = k[[a_1, \ldots, a_d]]$. Using the additivity formula (Proposition 10.3.3) and the hypothesis that A

is unmixed, we conclude that A has only one associated prime of 0. If P is this associated prime, by primary decomposition it follows that $P = \mathcal{N}(A)$. Moreover, by the additivity formula, $\ell(A_P) = 1$, which means that A_P is a field. Since P is nilpotent, this can only happen if $P = 0$, so in fact A is an integral domain.

We can now use Corollary 10.3.15 to deduce that A has the same fraction field as B. Since B is integrally closed and A is an integral extension of B, we must have $A = B$, so A is regular. □

We now prove Cohen's theorem. We split the proof in various cases, which we treat differently. We recall the notation that A is complete local ring, $\mathcal{M}$ its maximal ideal and $k = A/\mathcal{M}$ its residue field.

Proof of Cohen's theorem when $\operatorname{char}(A) = \operatorname{char}(k) = 0$**.**

By Zorn's lemma, there exists a maximal subfield $K \subset A$. We will prove that the induced map $K \to k$ is surjective, hence an isomorphism. Let

$$\pi\colon A \to k$$

be the projection, and let $\overline{K} = \pi(K)$ be the image of K. If $\overline{K} \subsetneq k$, we find $a \in A$ such that $\overline{a} = \pi(a) \notin \overline{K}$.

If $\overline{a}$ is transcendental over $\overline{K}$, then a is transcendental over K. In this case, $K[a] \cap \mathcal{M} = 0$, so every nonzero element of $K[a]$ has an inverse in A. This means that $K(a) \subset A$, contradicting the maximality of K.

If $\overline{a}$ is algebraic over $\overline{K}$, let $\overline{f}$ be its minimal polynomial over $\overline{K}$, where $f \in K[x]$. Then, since $\operatorname{char}(k) = 0$, $\overline{a}$ is a *simple* root of $\overline{f}$, hence it can be lifted to a simple root of f in A by Hensel's lemma 7.6.2, call it a'. Again we find that $K(a') \subset A$, contradicting the maximality of K. □

We now pass to the equicharacteristic case where $\operatorname{char}(A) = \operatorname{char}(k) = p > 0$. In this case, the last step of the proof can fail, so we need a slightly adapted argument. The argument is similar to the proof of Hensel's lemma, but we need an additional result.

Lemma 10.6.10. *Let $A, \mathcal{M}$ be a local ring with residue field k and assume that* $\operatorname{char}(A) = \operatorname{char}(k) = p > 0$*. If $\mathcal{M}^p = 0$, then A admits a coefficient field.*

Proof. Since $\operatorname{char}(A) = p$, the set $A^p := \{a^p \mid a \in A\}$ is a subring of A. Moreover let $b = a^p \in A^p$ be any nonzero element. Then $a \notin \mathcal{M}$ (as $\mathcal{M}^p = 0$), so a has an inverse in A, and b has one in A^p. It follows that $A^p \subset A$ is a *subfield*.

By Zorn's lemma, there exists a maximal subfield $K \subset A$ containing A^p. We want to prove that the induced map $K \to k$ is surjective. Let

$$\pi\colon A \to k$$

be the projection, $\overline{K} = \pi(K)$ the image of K. If this is not the case, we find $a \in A$ such that $\overline{a} = \pi(a) \notin \overline{K}$.

The situation is similar to the previous proof, but this time we know that $\overline{a}^p \in \overline{K}$, so the minimal polynomial of $\overline{a}$ is $x^p - \overline{a}^p$. It follow that a satisfies the same polynomial over K, hence $K(a)$ is a subfield of A, contradicting the maximality of K. □

We can now easily conclude the proof of Cohen's theorem in the equicharacteristic case.

Proof of Cohen's theorem when $\operatorname{char}(A) = \operatorname{char}(k) = p > 0$**.**

Denote

$$\pi_n\colon \frac{A}{\mathcal{M}^{n+1}} \to \frac{A}{\mathcal{M}^n}$$

the projection. We will recursively find a coefficient field $K_n \subset A/\mathcal{M}^n$ such that $\pi_n(K_{n+1}) = K_n$. Then the inverse limit of the fields $\{K_n\}$ is the desired coefficient field.

We start with $n = 2$. In this case, Lemma 10.6.10 applies, as $p \geq 2$, and gives a coefficient field $K_2 \subset A/\mathcal{M}^2$.

For the induction step, assume that we have found K_n and let $B := \pi_n^{-1}(K_n)$. Also, denote $P := \ker \pi_n = \mathcal{M}^n/\mathcal{M}^{n+1}$.

Given $b \in B \setminus P$, let $\overline{b} := \pi_n(b) \neq 0$. Since $\overline{b} \in K_n$, we have $\overline{b} \notin \mathcal{M}/\mathcal{M}^n$, which implies that $b \notin \mathcal{M}/\mathcal{M}^{n+1}$. It follows that b is invertible in $A/\mathcal{M}^{n+1}$, and in fact the inverse of b lies in B. (Why?) This proves that B is local with maximal ideal P.

We can then apply Lemma 10.6.10 to the ring B having residue field $B/P \cong K_n$, since $P^p = 0$. This gives us a coefficient field in B, which we can take as K_{n+1}. □

For the proof of Cohen's theorem in the mixed characteristic case, we are going to rely on the theory of Witt vectors developed in Section 7.7.

Proof of Cohen's theorem when $\operatorname{char}(A) \neq \operatorname{char}(k)$**.**

First, assume that k is a perfect field, and let $p = \operatorname{char} k$. In this case, Theorem 7.7.11 tells us that the ring $W_p(k)$ is a DVR, having maximal ideal $\mathcal{M} = p \cdot W_p(k)$, and complete with respect to the $\mathcal{M}$-adic topology. Moreover, $k \cong W_p(k)/\mathcal{M}$.

By Theorem 7.7.12, there exists a homomorphism $W_p(k) \to A$. The image of $W_p(k)$ inside A is the desired coefficient ring C. Notice that in this case if char $A = 0$, the map is injective, hence C is itself a DVR.

The case where k is not perfect is done by reduction to the previous case, but we are only giving a brief sketch. The steps are:

(i) Construct a DVR V complete with respect to its maximal ideal $\mathcal{M}$ and such that $k \cong V/\mathcal{M}$, even when k is not perfect

(ii) Consider the perfect closure $\overline{k}^{per}$ of k—this is a construction similar to the algebraic closure of k, but done by recusively adding p-th roots of elements of k (see Definition A.3.30).

(iii) By a similar procedure, starting from A, construct another complete local ring $\overline{A}^{per}$ with residue field $\overline{k}^{per}$.

(iv) Apply Theorem 7.7.12 to find a lift $\phi\colon W_p(\overline{k}^{per}) \to \overline{A}^{per}$

(v) Prove that $V \subset W_p(\overline{k}^{per})$, and in fact $\phi(V) \subset A$—this is the desired coefficient ring C.

Some more details can be found in [**Katb**], or the original paper [**Coh46**]. □

10.7. Exercises

1. Prove directly that a regular local ring of dimension 1 is a discrete valuation ring.

2. Let A be a regular ring of dimension 1. Prove that A is a Dedekind domain.

3. Prove the properties stated in Proposition 10.3.1.

4. Compute the multiplicity of n lines meeting at the origin in $\mathbb{A}^2$.

5. Use Example 10.3.7 to compute the multiplicity $e(A)$, where A is the local ring of the node $y^2 = x^2 + x^3$ in a simpler way.

6. Verify the final computation (10.2.3) in Proposition 10.2.14.

7. Let A be an integral domain of dimension 1, and assume that the order function v_A is a valuation. Prove that A is a discrete valuation ring.

8. Let $B = k[x, y]/(x^2, xy)$ and A the localization of B at 0. Compute $\mu_A(y)$ and $\mu_A(y^2)$ and show that $\mu_A(y^2) \neq 2\mu_A(y)$—in particular the hypothesis that a is a regular element is necessary in Proposition 10.4.6.

9. Let A be a Noetherian ring, $a \in A$ a nonzero divisor. Assume that for $a \in \mathcal{M} \setminus \mathcal{M}^2$ for every maximal ideal $\mathcal{M} \subset A$. If $A/(a)$ is regular, prove that A is regular.

10. Let A be a regular local ring. Prove that the ring $A[[x]]$ is regular (use the previous exercise).

11. Let $A, \mathcal{M}$ be a complete local ring of dimension d, of mixed characteristic, and let $p = \operatorname{char} A/\mathcal{M}$. Assume that $\operatorname{ht}(p \cdot A) = 1$. Prove that A is a finitely generated module over a subring $B \subset A$ such that

$$B \cong C[[x_1, \ldots, x_{d-1}]],$$

where C is a DVR.

12. Prove Corollary 10.4.14—that is, $e(A[[x]]) = e(A)$ for a local Noetherian ring A—by a direct computation (it is easier to write the Hilbert polynomial of A as a sum of binomial coefficients, instead of powers of n).

The following exercises discuss the notion of *reduction* of ideals as a means to compute multiplicities. Given a ring A with two ideals $J \subset I$, we say that J is a *reduction* of I if $J \cdot I^n = I^{n+1}$ for some $n \in \mathbb{N}$.

13. Let A be a semilocal Noetherian ring, $J \subset I$ two ideals of definition. Prove that if J is a reduction of I, then $e(J) = e(I)$.

A sort of converse was proved by Rees in [**Ree61**]:

Theorem (Rees). *Let $A, \mathcal{M}$ be a local Noetherian ring. Assume that A is unmixed, and let $J \subset I$ be two $\mathcal{M}$-primary ideals such that $e(J) = e(I)$. Then J is a reduction of I.*

14. Let A be a Noetherian ring, $J \subset I \subset A$ two ideals. Let B be the integral closure of A in its total fraction ring. Prove that J is a reduction of I if and only if I and J have the same integral closure inside B.

15. Let $A = \mathbb{C}[[x, y]]$ and $I = (x^3, x^2y, y^2)$. Find a reduction of I and use it to compute $e(I)$.

16. Use Corollary 10.3.15 to give an alternative proof of the inertia-ramification formula (Theorem 6.3.2).

The following exercises, up to Exercise 21, discuss a structure result for principal ideal *rings*, due to Hungerford [**Hun68**], building on Cohen's theorem and [**ZS76a**, Theorem 33, Part IV]. A *principal ideal ring* is just a ring (not necessarily an integral domain) whose ideals are all principal. Such a ring is called *special* if it has a single prime ideal.

17. Let A be a principal ideal ring, $P_1, P_2 \subset A$ prime ideals. Prove that either P_1 and P_2 are coprime, or $P_1 \subset P_2$, or $P_2 \subset P_1$.

18. Let A be a principal ideal ring, $P_1 \subset P_2 \subset A$ prime ideals. If Q is a P_2-primary ideal, then $P_1 \subset Q$.

19. Let A be a principal ideal ring. Prove that A is a finite direct sum of principal ideal *domains* and special principal ideal rings. (Use primary decomposition for 0 and the previous exercises.)

20. Let A be a special principal ideal ring. Prove that A is an image of a PID. Deduce the theorem of Hungerford: every principal ideal ring is a finite direct sum of images of principal ideal domains. (A is a complete local ring, so we can apply the structure theorem of Cohen. If A has a coefficient field, we are done. Assume that A is the image of $C[[x]]$, where C is a coefficient ring. Find a quotient of $C[[x]]$ which is a PID and still surjects onto A.)

21. One may want a stronger form of Hungerford's theorem, but it is not the case that every principal ideal ring is a quotient of a principal ideal domain. To see this, take $A = \mathbb{R} \oplus \mathbb{Z}$, and show that $\mathbb{Q}$, seen as an A-module with a trivial $\mathbb{R}$-action, is not a direct sum of cyclic A-modules. Conclude by the classification of modules over a PID. (Compare this with Exercise 32 in Chapter 7.)