Preface

Commutative algebra is a discipline that draws inspiration from many different fields of mathematics. When I wrote the book *Commutative algebra* [**Fer20**], I tried to convey this by showing the links to various other topics. Sometimes a concept is clarified by its geometric counterpart; in other situations, a computational point of view is more apt; the structure of some rings appearing in number theory is revealed by looking at them as lattices inside $\mathbb{R}^n$; yet other times, topology is the right lens through which a problem can be attacked.

Since the pioneering work of Serre [**Ser56b**] (and many other mathematicians) on regular rings, it has become clear that homological techniques are a powerful tool for the study of rings and modules. Yet, I felt I could not do justice to homological algebra by just adding a chapter on homological methods to an already long book. The present volume is intended to remedy this absence.

The aim of this book is to teach homological algebra first, and then gradually focus on the study of rings and modules. Instead of just constructing derived functors on categories of modules, I decided to dedicate the first four chapters to developing homological algebra on its own. It is my hope that this will help the student who is not familiar with these techniques to learn this beautiful—but sometimes dry—subject, and bring them to a level of proficiency that will allow them to pursue further topics in topology, algebraic geometry, and so on. While the presentation is self contained and reasonably complete, it is quite old style: I resisted the temptation to add at least a chapter on derived categories, which, while fundamental in any modern application of the subject, is tangential to the text.

The second part of the text is dedicated to the application of these techniques in commutative algebra, through the study of projective, injective, and flat modules; the construction of explicit resolutions via the Koszul complex; the properties of regular sequences and their application to the study of regular rings and various other classes of rings that are most easily characterized by their homological properties.

While of course [**Fer20**] and the present volume are naturally complementary, it is hoped that each of them makes sense on its own. The reader of this book is assumed to have a general background in elementary commutative algebra: of course, [**Fer20**] is suitable for the purpose, but familiarity with other books, such as [**AM69**] or [**AK13**] is also helpful.

Since there are many books on commutative algebra, it may be worthwhile to highlight a few of the author's favorite results, which are not always found in comparable texts, such as

(1) the Freyd–Mitchell theorem on embedding small Abelian categories into categories of modules;
(2) the Ischebeck spectral sequences and their application to a common proof of the Auslander–Buchsbaum formula and the equality between depth and injective dimension (these appear as exercises in Chapter 9);
(3) the proof of the Quillen–Suslin theorem on the freeness of projective modules in polynomial rings;
(4) the construction of the Macaulay resultant and its computation using Koszul complexes;
(5) Kunz's characterization of regular local rings in characteristic p, a prime;
(6) the Northcott–Rees theory of irreducible decomposition of ideals and its link with the Cohen–Macaulay and Gorenstein properties; and
(7) the Grothendieck duality for local cohomology.

Of course, this is just a selection of topics according to the taste of the author. We next review the contents of each chapter.

Category theory was originally developed—at least in part—to give a foundation to homological algebra. This is our starting point as well. Chapter 1 introduces the language and tools of category theory. Categories are introduced both as an organizing gadget for other structures, and as algebraic objects themselves. One of the puzzling points when learning about categories is the set theoretic distinction between sets and classes. For this reason, Section 1.2 is dedicated to some background on set theory, both with the ZFC and the NBG formalizations, in order to explain how classes can be introduced as a sound mathematical object. We then move to some standard construction with categories: natural transformation, limits, universal objects, and adjoint pairs, elucidating the various connections between them.

In Chapter 2, we begin the study of Abelian categories. These have formal properties that resemble those of the module categories, and are a suitable place to develop the machinery of derived functors. After setting up the basic definitions, we develop some sheaf theory, both on topological spaces and on categories endowed with a Grothendieck topology. In fact, one of the motivations to set up homological algebra in the context of Abelian categories was to unify the treatment of derived functors for modules and of sheaf cohomology. We study sheaves to give some motivation for the abstract approach, and to provide an interesting example of an Abelian category other than the category of modules on a ring. We then come back to the general settings, where we develop some results on subobjects which allow us to prove standard results such as the snake lemma. The chapter culminates in a proof of the celebrated Freyd–Mitchell theorem, which ensures that (small) Abelian categories can be embedded as full subcategories of the category of left modules over a possibly noncommutative ring.

This result simplifies a lot working with Abelian categories, by enabling proofs by diagram chasing. It is quite common to cite the result in order to simplify the presentation of general Abelian categories, seldom with proof. I decided to include it for various reasons. First, I try to avoid relying on black box results, to a reasonable extent. More importantly, it provides a beautiful example of a fairly nontrivial result on Abelian categories, which provides some substance to a chapter that could otherwise feel rather dull. Finally, it crucially relies on the construction on sheaves on a nontopological site, giving a very nice application of the general theory of sheaves developed before.

Chapter 3 begins the study of derived functors of an additive functor between Abelian categories, assuming that the source has enough injectives or enough projectives. After checking that the notion is well defined, the bulk of the chapter is taken by studying some derived functors in more detail: sheaf cohomology, the Ext and Tor functors and the $\lim^n$ functors. In particular, we show that Ext and Tor can be computed by resolving either of the arguments, and we also develop the construction of the Ext functors as Yoneda extensions.

We end the presentation of homological algebra with Chapter 4 on spectral sequences. These are a fundamental computational tool, with plenty of applications that we try to showcase in the rest of the book. We start from the general to the particular, in the hope that this makes the theory less tricky. After the reader has accepted the definition of an exact pair, which is admittedly not obvious, the construction of the derived exact pair immediately leads to spectral sequences. All examples of spectral sequences in the text are special cases of this construction. By increasing specialization, we review the spectral sequences associated to fibered complexes, double complexes, and finally the Grothendieck spectral sequence for the composition of two derived functors. We end the chapter by including some spectral sequences that link the Ext and Tor functors. Of these, the base change spectral sequences can be obtained from the Grothendieck formalism, while the Ischebeck spectral sequences are derived from scratch.

With Chapter 5, we begin to enter the world of commutative algebra proper. This chapter is dedicated to the Ext functor, and in particular to the study modules that are acyclic for Ext. These are projective or injective modules, according to the side, and have a very different flavor. Projective modules, which are the topic of Section 5.1, are not too far from being locally free, and resemble modules of sections of vector bundles on topological spaces. In constrast, injective modules are divisible, and we have a rich theory of them due to Matlis, based on the notion of injective hull, which we present in Section 5.3. Both can be used to develop a theory of dimension for modules, called the projective (resp., injective) dimension. We study how this quantity changes under various constructions, in particular in the case of quotients, where we provide some change of rings theorems. These are crucial for the study of global dimension of a ring, which is the supremum of either of these quantities. One of the main results of the chapter is Hilbert's syzygy theorem, which in its modern form states that $\operatorname{gl.dim} A[x] = \operatorname{gl.dim} A + 1$ for a Noetherian ring A. We end the chapter by connecting this abstract form to the more classical formulation by Hilbert, in particular proving the Quillen and Suslin theorem stating that projective modules over polynomial rings are in fact free.

In Chapter 6, we focus on the Tor functor, again by characterizing modules that are acyclic for Tor—namely, flat modules. A case of particular interest in geometry is the flatness of B as an A-module when $f\colon A \to B$ is a map of rings. In this case, we say that f is a flat morphism. After showing that localizations and completions are flat, we give various criteria for flatness, and we show various connections between the notions of flat, projective and locally free modules. We end the chapter by showing that flat modules are in fact the direct limit of free modules.

In Chapter 7, we start making things more concrete by borrowing an idea from differential geometry. Namely, by mimicking the construction of the De Rham complex, we construct a complex associated to a sequence of elements in a ring A. We dub this the Koszul complex. By studying the question of when this complex is actually acyclic, one is naturally led to consider regular sequences. These are fundamental to connect the homological and ring-theoretic properties of A, and give rise to yet another invariant of ideals and modules, called the depth. We end the chapter by giving two applications of these ideas: the first one is a formula for multiplicities due to Auslander and Buchsbaum, and the second one is a construction of the Macaulay resultant, a key tool in computational algebra.

Regular rings were introduced in Chapter 10 of [**Fer20**], where we studied their most elementary properties. We tackle them again in Chapter 8, but this time we are better equipped. The key result of the chapter is Serre's theorem, which characterizes regular local rings as local Noetherian rings of finite global dimension. Using this, we are able to analyze the behavior of regularity under many operations, such as localizations, quotient and polynomial extensions. We then turn to the celebrated theorem of Auslander and Buchsbaum stating that local regular rings have unique factorization, and we end the chapter by characterizing regular rings in characteristic p.

Chapter 9 introduces many other classes of rings, that are best understood via their homological properties. Under the assumption of locality, these form a well-known hierarchy:

$$\{\text{regular}\} \subset \{\text{complete intersections}\} \subset \{\text{Gorenstein}\} \subset \{\text{CM}\}.$$

All of the above classes are introduced in the chapter, starting from CM (Cohen–Macaulay) rings (and modules). These are characterized by having depth equal to their dimension. This simple definition is actually equivalent to a myriad of other characterizations—for instance, CM rings are those where the notions of (maximal) regular sequences and systems of parameters are the same. This fact makes CM rings appear essentially everywhere in commutative algebra. Among other things, we prove that CM rings have a particularly simple structure of the lattice of prime ideals. Section 9.4 characterizes them by the number of factors in irreducible decompositions of primary ideals. When this number is 1, we obtain an even more special class, that of Gorenstein rings, which we study in Section 9.7. These are also characterized by having finite injective dimension.

Chapter 10 introduces various types of dualities for modules over a local Noetherian ring. The most elementary, Matlis duality, has a more algebraic flavor, and generalizes the duality for finitely generated vector spaces over a field. A more sophisticated duality, Grothendieck local duality, is the topic of Section 10.7. This is the local analogue of the celebrated Serre duality for sheaves on projective varieties. In order to state the result, we introduce the notion of local cohomology in Sections 10.5 and 10.6, and the canonical module in Section 10.4. By using these notions, we give another characterization of Cohen–Macaulay and Gorenstein rings by their duality properties.

Apart from a general mathematical maturity, the prerequisites for the book are an introduction to commutative algebra, such as [**Fer20**], [**AM69**] or [**AK13**]. Also some familiarity with algebraic geometry, at the level of algebraic varieties, is assumed. The main results that are used in the text are collected in Appendix A for ease of reference. Notice, though, that the appendix is no substitute for a general familiarity with the treated arguments, and should only be used as a reference. Each result stated in the appendix appears in [**Fer20**], and is mentioned with an appropriate reference. We also include a reference to a different source—where possible [**AM69**]—to avoid a strict dependency between the volumes. A few results from [**Fer20**] are not listed in the appendix, but are actually proved again in the book. This is either because a new method allows for a different proof, or because the result logically fits the scope of the text and it does not make sense to omit it.

In the text, we also make some references to some basic algebraic topology or differential geometry—for instance, we mention simplicial and De Rham cohomology. In particular, the very first section draws examples of categories from many branches of mathematics. None of these is a strict requirement: generally these topics are mentioned only as examples or as motivation for a particular construction. The reader that is not familiar with them should not lose much by skipping them, but they can be useful aids for the knowledgeable reader.

Other than cover to cover, there are various ways to read the text, of which we suggest a few. First, Chapter 1 contains a short, self-contained introduction to the most basic ideas in category theory. There are many such introductions, but this is mine. Chapters 1 and 2 together provide a proof of the Freyd–Mitchell theorem, developed from scratch. The short book [**Fre64**] says in its introduction that it provides a geodesic way to the proof of Freyd–Mitchell. It is probably no coincidence that the page count of that book and the first two chapters are more or less the same.

The reader interested in a general introduction to homological algebra can read Chapters 1 to 4. This should give a reasonable introduction to the classical approach, after which one may want to learn about derived and triangulated categories, for instance, from [**GM03**]. Chapter 4 alone makes sense for someone who wants to learn about spectral sequences, and already has some familiarity with derived functors, even if only on categories of modules.

The reader who already has some familiarity with homological algebra and is interested in learning about commutative algebra can start from

Chapter 5, refering back to earlier chapters as needed. The dependency graph among Chapters 5 to 10 is rather linear, but on a first read one can safely skip Sections 5.8, 6.5, 7.5, 7.6, 7.7, 8.6, 9.3, 9.8, and 10.3. This should leave a book about half the size that introduces all relevant ideas, such as depth, regularity, Koszul complex, local duality, Cohen–Macaulay rings, and Gorenstein rings.

Examples in the text usually require only trivial verifications. They are part of the core of the text and should not be skipped. Occasionally, an example requires more work, and it should be considered as an exercise. I frequently punctuate the book with remarks such as “(Prove it!)” to underline the fact that I am skipping some necessary verification, left to the reader.

Exercises vary from simple to hard, and there is (intentionally) no indication to distiguish the level. I suggest, as always, that you try at as many exercises as you can. If some of them look too hard, they can be postponed and tried again after some rumination.

In general, I have tried to avoid depending on exercises for the main body of the text. The cases where I have done so should be easy verifications. On the other hand, many important and subtle counterexamples are presented as series of exercises.

No contribution in this book is original, except of course the usual amount of errors, that should be attributed only to the author. If you spot some of them, you can send an email to ferrettiandrea@gmail.com.

I hope that you will enjoy reading this book as much as I enjoyed writing it!

If I was able to write this book, it is because Massimo Gobbino and Paolo Tilli, when I was young and did not know better, believed in me and persuaded me to undertake the study of mathematics. This turned out to be one of the best choices I made, and I have to really thank them for this. Thanks to Roberto Dvornicich, who instilled in me a lasting love for algebra. I take the opportunity to thank the AMS for their editorial support, especially Ina Mette, who believed in the project and followed it with great patience through many years. Most of all, I want to thank my wife Sbambi, who with her love shows me every day what is really important, and with her patience and understanding has given me the time and peace of mind to do mathematics and finish this book. $\ddot\smile$