Preface

Alexandrov spaces are defined via axioms similar to those given by Euclid. The Alexandrov axioms replace certain equalities with inequalities. Depending on the signs of the inequalities, we obtain Alexandrov spaces with curvature bounded above (CBA) and curvature bounded below (CBB). The definitions of the two classes of spaces are similar, but their properties and known applications are quite different.

The goal of this book is to give a comprehensive exposition of the structure theory of Alexandrov spaces with curvature bounded above and below. It includes all the basic material as well as selected topics inspired by considering the two contexts simultaneously. We only consider the intrinsic theory, leaving applications aside. Our presentation is linear, with a few exceptions where topics are deferred to later chapters to streamline the exposition. This book includes material up to the definition of dimension. Another volume still in preparation will cover further topics.

Brief history

The first synthetic description of curvature is due to Abraham Wald [159]; it was given in a lone publication on a “coordinateless description of Gauss surfaces” published in 1936. In 1941, similar definitions were rediscovered by Alexandr Alexandrov [16].

In Alexandrov’s work the first fruitful applications of this approach were given. Mainly, Alexandrov’s embedding theorem [11,12], which describes closed convex surfaces in Euclidean 3-space, and the gluing theorem [13], which gave a flexible tool to modify nonnegatively curved metrics on a sphere. These two
results together gave an intuitive geometric tool to study embeddings and bending of surfaces in Euclidean space and changed the subject dramatically. They formed the foundation of the branch of geometry now called Alexandrov geometry.

**Curvature bounded below.** The theory grew out of studying intrinsic and extrinsic geometry of convex surfaces without the smoothness condition. It was developed by Alexandr Alexandrov and his school. Here is a very incomplete list of contributors to the subject: Yuriy Borisov, Yuriy Burago, Boris Dekster, Iosif Liberman, Sergey Olovyanishnikov, Aleksey Pogorelov, Yuriy Reshetnyak, Yuriy Volkov, Viktor Zalgaller.

The first result in higher dimensional Alexandrov spaces was the splitting theorem. It was proved by Anatoliy Milka [118] and appeared in 1967. Milka used a global definition similar to the one used in this book.

In the 1980s the interest in convergence of Riemannian manifolds spurred by Gromov’s compactness theorem [71] turned attention toward the singular spaces that can occur as limits of Riemannian manifolds. Immediately it was recognized that if the manifolds have a uniform lower sectional curvature bound, then the limit spaces have a lower curvature bound in the sense of Alexandrov. There followed during the 1990s an explosion of work on intrinsic theory of Alexandrov spaces starting with papers of Yuriy Burago, Grigory Perelman, and Michael Gromov [44, 127]. Similar ideas were developed independently by Karsten Grove and Peter Petersen, whose work was not converted into a publication, and also by Conrad Plaut [137].

Around the same time an implicit application of higher-dimensional Alexandrov geometry was given by Michael Gromov in his bound on Betti numbers [75]. Another implicit application, which essentially used Alexandrov geometry before it was actually introduced, given later by Wu-Yi Hsiang and Bruce Kleiner in their paper on nonnegatively curved 4-manifolds with infinite symmetry groups [83]. The work of Hsiang and Kleiner and its extension by Karsten Grove and Burkhard Wilking [78] are some of the most beautiful applications of this branch of Alexandrov geometry.

The above activity was very much related to so-called comparison geometry, a branch of differential geometry that compares Riemannian manifolds to spaces of constant curvature. In addition to the already mentioned Gromov’s compactness theorem, the following results had a big influence on the development of Alexandrov geometry: Toponogov comparison theorem [156], which is a generalization of the theorem of Alexandrov [14]; Toponogov splitting theorem [156], which is a generalization of Cohn-Vossen’s theorem [55]; Finiteness theorems of Cheeger and Grove–Petersen [52, 77]; Gromov’s bound on the number
of generators of the fundamental group [73, 1.5]; and the Yamaguchi fibration theorem [162].

Let us give a list of available introductory texts on Alexandrov spaces with curvature bounded below:

- The first introduction to Alexandrov geometry is given in the original paper of Yuriy Burago, Michael Gromov, and Grigory Perelman [44] and its extension [127] written by Perelman.

- A brief and reader-friendly introduction was written by Katsuhiro Shiohama [150, Sections 1–8].

- Another reader-friendly introduction, written by Dmitri Burago, Yuriy Burago, and Sergei Ivanov, is given in [37, Chapter 10].


In addition, let us mention two surveys, one by Conrad Plaut [139] and the other by the third author [132].

Curvature bounded above. The study of spaces with curvature bounded above started later, inspired by analogy with the theory of curvature bounded below. The first paper on the subject was written by Alexandrov [18], appearing in 1951. An analogous weaker definition was considered earlier by Herbert Busemann [45].

Contributions to the subject were made by Valerii Berestovskii, Arne Beurling, Igor Nikolaev, Dmitry Sokolov, Yuriy Reshetnyak, Samuel Shefel; this list is not complete as well. The most fundamental results were obtained by Yuriy Reshetnyak. They include his majorization theorem and gluing theorem. The gluing theorem states that if two nonpositively curved spaces have isometric convex sets, then the space obtained by gluing these sets along an isometry is also nonpositively curved.

The development of Alexandrov geometry was greatly influenced by the Hadamard–Cartan theorem. Its original formulation states that the exponential map at any point of a complete Riemannian manifold with nonpositive sectional curvature is a covering. In particular, it implies that the universal cover is diffeomorphic to Euclidean space of the same dimension. See further discussion below 9.65.

An influential implicit application of Alexandrov spaces with curvature bounded above can be seen in Euclidean buildings, introduced by Jacques Tits as a means to study algebraic groups.
Here is a list of available texts covering the basics of Alexandrov spaces with curvature bounded above:

- The book of Martin Bridson and André Haefliger [34] gives the most comprehensive introduction available today.
- The lecture notes of Werner Ballmann [21, 22] include a brief and clear introduction.
- The volume [37, Chapter 9] gives another reader-friendly introduction by Yuriy Burago, Dmitry Burago, and Sergei Ivanov.
- The book [10] by the three authors of the present volume gives an introduction aiming at reaching interesting applications and theorems with a minimum of preparation.
- The book of Jürgen Jost [90] gives a more analytic viewpoint to the subject.

One of the most striking applications of CAT(0) spaces was given by Dmitry Burago, Sergei Ferleger, and Alexey Kononenko [38], who used them to study billiards; this idea was developed further in [39–43]. Another beautiful application is the construction of exotic aspherical manifolds by Michael Davis [59]; related results are surveyed in [50, 60]. Both of these topics are discussed in [10]. The study of group actions on CAT(0) spaces and CAT(0) cube complexes played a key role in the proof of the virtually fibered conjecture that a finite cover of every closed hyperbolic 3-manifold fibers over the circle.

**Satellites and successors**

Surfaces with *bounded integral curvature* were studied by Alexandrov’s school. An excellent book on the subject was written by Alexandr Alexandrov and Viktor Zalgaller [15]; see also a more up-to-date survey by Yuriy Reshetnyak [141].

Spaces with *two-sided bounded curvature* is another subject already studied by Alexandrov’s school; a good survey is written by Valerij Berestovskij and Igor Nikolaev [24].

A spin-off of the idea of synthetically defining upper curvature bounds was given by Michael Gromov [76]. He defined so-called *δ-hyperbolic spaces*, which satisfy a coarse version of the negative curvature condition, applying in particular to discrete metric spaces. This notion and its various generalizations such as semihyperbolicity (a coarse version of nonpositive curvature) and relative hyperbolicity have led to the emergence of the subject of *geometric group theory*, which relates geometric properties of groups to their algebraic ones. This is a well-developed subject with a large number of subfields and applications, such as the theory of small cancellation groups, automatic groups, mapping class
groups, automorphisms of free groups, isoperimetric inequalities on groups, actions on $\mathbb{R}$-trees, and Gromov’s boundaries of groups.

The so-called curvature dimension condition introduced by John Lott, Cédric Villani, and Karl-Theodor Sturm gives a synthetic description of Ricci curvature bounded below; see the book of Villani [158] and references therein. A striking application of this theory to geodesic flow in CBB spaces was found recently by Elia Bruè, Andrea Mondino, and Daniele Semola [35].

Alexandrov geometry influenced the development of analysis on metric spaces. An excellent book on the subject was written by Juha Heinonen, Pekka Koskela, Nageswari Shanmugalingam, and Jeremy Tyson [82].

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