Chapter 1

Strings, drums, and the Laplacian

Pierre-Simon, marquis de **Laplace** (1749–1827)

In this chapter, we introduce the Laplacian, both in the Euclidean space and on a Riemannian manifold, and consider the eigenvalue problems with the Dirichlet and Neumann boundary conditions. We discuss the related models of vibrating strings and drums and consider a few examples in which spectral problems can be explicitly solved.

Eugenio **Beltrami** (1835–1900)

1.1. Basic examples

1.1.1. The Laplace operator. In the Euclidean space $\mathbb{R}^d$ of dimension d with Cartesian coordinates $x = (x_1, \ldots, x_d)$, let

$$\Delta f = \sum_{j=1}^{d} \frac{\partial^2 f}{\partial x_j^2}, \tag{1.1.1}$$

where $f = f(x_1, \ldots, x_d)$ is a twice differentiable function.

Definition 1.1.1 (The Laplacian). The operator $-\Delta$ is called the *Laplace operator* (or the *Laplacian*) in $\mathbb{R}^d$.

Remark 1.1.2. There is no unique sign convention for Δ. In this book, we define Δ by (1.1.1), that is, in the analyst's sense; geometers often incorporate the minus sign into the definition of Δ. (The authors have argued long and hard about which notation to adopt.) Additionally, the term *Laplacian* may also be applied to the negative of our Laplacian.

One can rewrite (1.1.1) as

$$\Delta f = \operatorname{div} \nabla f,$$

where div denotes the divergence of a vector field and ∇ is the gradient of a scalar function; see §B.1. We will use this representation later on in order to define the Laplacian on a Riemannian manifold.

The Laplace operator appears in major partial differential equations arising in mathematical physics. Here are some examples; in all of them we set $\Delta := \Delta_x$; i.e., the operator acts only in the x variable.

- *Wave equation:*

$$\frac{\partial^2 U(t,x)}{\partial t^2} = \Delta U(t,x).$$

Here $U(t,x)$ denotes the displacement from the equilibrium of the vibrating object at the point $x \in \mathbb{R}^d$ at time t.

- *Heat (or diffusion) equation:*

$$\frac{\partial U(t,x)}{\partial t} = \Delta U(t,x).$$

Here $U(t,x)$ denotes the temperature of the object (or the density of the matter) at the point x at time t.

- *Laplace equation:*

$$\Delta U(x) = 0.$$

The solutions of the Laplace equation are called *harmonic* functions. In hydrodynamics, the velocity potential $U(x)$ of an incompressible fluid flow is a solution of the Laplace equation.

- *Poisson equation:*

$$-\Delta U(x) = f(x).$$

In electrostatics, $U(x)$ is interpreted as an electric potential corresponding to a given charge distribution f.

- *Schrödinger equation:*

$$\mathrm{i}\,\frac{\partial U(t,x)}{\partial t} = -\Delta U(t,x),$$

where $\mathrm{i}^2 = -1$. In quantum mechanics, the solution $U(t,x)$ of this equation is called the wave function. Note that $U(t,x)$ is complex valued; the quantity $|U(t,x)|^2$ describes the probability density for a particle to be at the position x at time t.

Let us start with two simple real-life examples, which are also among the most relevant ones from the viewpoint of spectral geometry: the vibrating strings and drums.

1.1.2. Vibrating strings. Even if you never played a guitar yourself, you probably know that thicker guitar strings produce lower sounds and that pressing down on a string raises the pitch. These phenomena could be easily explained using a mathematical model of a vibrating string, given by the *one-dimensional wave equation*.

Consider a string of length l and uniform density ρ, fixed at both ends. Let $U : \mathbb{R}_+ \times [0, l] \to \mathbb{R}$ be a function, whose value $U(t, x)$ is equal to the deviation from the equilibrium of a transversally vibrating string at the point $x \in [0, l]$ at the time $t \in \mathbb{R}_+$ (transversal vibrations mean that each point of the string moves along the vertical line orthogonal to the equilibrium position). The function $U(t, x)$ satisfies the one-dimensional wave equation

$$U_{tt} = a^2 \Delta U = a^2 U_{xx}, \tag{1.1.2}$$

where the constant a can be expressed in terms of the tension τ of the string and the density ρ:

$$a = \sqrt{\tau/\rho}.$$

Since the string is attached at both ends, we impose the *Dirichlet* boundary conditions:

$$U(t, 0) = U(t, l) = 0, \qquad t \in \mathbb{R}_+. \tag{1.1.3}$$

In order to find a solution of this equation we use the Fourier method.

Jean-Baptiste Joseph **Fourier**
(1768–1830)

The first step is to separate the variables and to look for a solution in the form

$$U(t, x) = T(t)X(x).$$

This is a so-called *standing wave*. From equation (1.1.2) we get

$$T''(t)X(x) = a^2T(t)X''(x),$$

and, since $X(x)$ and $T(t)$ are not identically zero, we obtain

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{a^2T(t)} = -\lambda,$$

where λ is some constant (the choice of the minus sign will become clear later). Indeed, the left-hand side of the equality does not depend on t, and the middle part is independent of x, so both are equal to a constant.

We now consider the equations for the functions $X(x)$ and $T(t)$ separately.

Taking into account (1.1.3), we obtain a Sturm–Liouville eigenvalue problem for the function $X(x)$ with Dirichlet boundary conditions:

$$\begin{cases} -X''(x) = \lambda X(x), \\ X(0) = X(l) = 0. \end{cases} \tag{1.1.4}$$

Definition 1.1.3. A nontrivial solution $X(x)$ of the Sturm–Liouville problem (1.1.4) is called an *eigenfunction* corresponding to an *eigenvalue* λ.

Exercise 1.1.4. Show that the eigenvalues and eigenfunctions of the Sturm–Liouville problem (1.1.4) are given by

$$\lambda_m = \left(\frac{\pi m}{l}\right)^2, \qquad X_m(x) = \sin\left(\frac{\pi m}{l}x\right), \qquad m = 1, 2, \ldots.$$

Exercise 1.1.5. Show that for all natural numbers $k \neq m$,

$$\int_0^l X_k(x)X_m(x)\mathrm{d}x = 0.$$

Resolving a similar Sturm–Liouville problem for $T(t) = T_m(t)$ we obtain

$$T_m(t) = A_m \cos\left(\frac{a\pi m}{l}t\right) + B_m \sin\left(\frac{a\pi m}{l}t\right),$$

where A_m and B_m are arbitrary constants. Taking a superposition of the standing waves $U_m(t,x) = T_m(x)X_m(x)$, we get a formal solution of the wave equation (1.1.2):

$$U(t,x) = \sum_{m=1}^{\infty} \left(A_m \cos\left(\frac{a\pi m}{l}t\right) + B_m \sin\left(\frac{a\pi m}{l}t\right)\right) \sin\left(\frac{\pi m}{l}x\right). \tag{1.1.5}$$

Exercise 1.1.6. Show that the constants A_m and B_m, $m \in \mathbb{N}$, are uniquely determined by the initial conditions $u(0,x) = \varphi(x)$ (initial position), $u_t(0,x) = \psi(x)$ (initial velocity). Calculate A_m and B_m using the Fourier decompositions of the functions φ and ψ.

We are now in a position to address the questions about sounds emitted by a guitar string raised at the beginning of this section. As can be easily seen from (1.1.5), the natural frequencies of the string are given by

$$\omega_m = a\sqrt{\lambda_m} = \frac{a\pi m}{l}, \qquad m \in \mathbb{N}. \tag{1.1.6}$$

The frequency ω_1 is called the *principal frequency*, or the *fundamental tone* of the string, and the higher frequencies are called *overtones*. It follows immediately from (1.1.6) that the frequencies decrease as the length l increases; in other words, shorter strings produce higher notes. This is precisely what we observe when pressing down on a guitar string (pressing down is essentially a way to change the length of the vibrating part of the string). Recall now that the constant a decreases as the density of a string increases. Therefore, the thicker the string is, the lower are the sounds that it emits. Similarly, the higher the tension of the string is, the higher the pitch is.

The eigenfunctions $X_m(x)$ describe the shape of the pure vibration modes. In particular, one may observe that for each $m = 1, 2, \ldots$, the eigenfunction $X_m(x)$ has precisely $m-1$ zeros on the open interval $(0,l)$; see Figure 1.1. This fact has interesting higher-dimensional generalisations that we will discuss later.

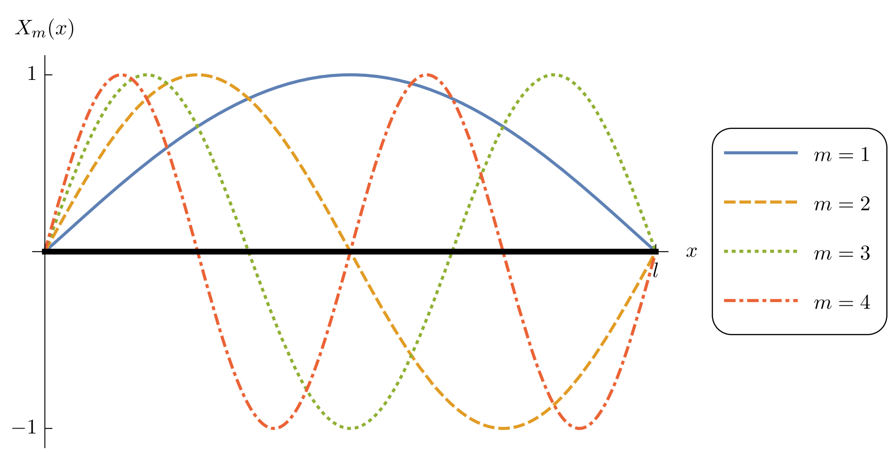

Figure 1.1. First four eigenfunctions of a fixed vibrating string.

Exercise 1.1.7. The vibrations of a free string of length l are modelled by the equation (1.1.2) with Neumann boundary conditions

$$U_x(t,0) = U_x(t,l) = 0, \qquad t \in \mathbb{R}_+. \tag{1.1.7}$$

Find the eigenfrequencies of a free vibrating string and compare them with the (Dirichlet) eigenfrequencies given by (1.1.6).

Let us explain the physical meaning of the Neumann condition (1.1.7). As follows from the model leading to the wave equation (1.1.2), the tension force acting at the point x is equal to τU_x. *Free* vibration means that the endpoints of the string experience no tension, and therefore at these points U_x must vanish.

Example 1.1.8. Consider the vibrations of a string whose ends are neither fixed nor free but joined together in a circular loop. If the length of the string is 2π, we arrive, after the separation of variables, at the spectral problem

$$\begin{cases} -X''(x) = \lambda X(x), \\ X(x) \text{ is } 2\pi\text{-periodic.} \end{cases} \tag{1.1.8}$$

Looking for the values of λ for which (1.1.8) has a nontrivial solution, we obtain

$$\lambda_0 = 0, \qquad X_0(x) = 1,$$

and also eigenvalues m^2, $m \in \mathbb{N}$, for each of which there are two linearly independent eigenfunctions $X_{m,1}(x) = \sin mx$ and $X_{m,2}(x) = \cos mx$.

1.1.3. Vibrating drums.

Consider now a two-dimensional analogue of the problem discussed in the previous section. Imagine a drum with a membrane (drumhead) shaped as a bounded domain $\Omega \subset \mathbb{R}^2$. The function

$$U(t,x,y) : \mathbb{R}_+ \times \Omega \to \mathbb{R}$$

describing the vibration of the drumhead satisfies the wave equation

$$\begin{cases} U_{tt} - a^2 \Delta U = 0, \\ U|_{\partial\Omega} = 0, \end{cases}$$

where the constant a depends on the physical characteristics of the membrane. Again, searching for solutions in the form $U(t,x,y) = T(t)\,u(x,y)$, we get a familiar (ordinary) Sturm–Liouville equation for $T(t)$ and a Dirichlet eigenvalue problem for the function $u(x,y)$, that is the eigenvalue problem for the Laplacian in Ω,

$$-\Delta u = \lambda u, \tag{1.1.9}$$

subject to the Dirichlet condition

$$u|_{\partial\Omega} = 0. \tag{1.1.10}$$

We say, as in Definition 1.1.3, that λ is an eigenvalue of the Dirichlet problem (1.1.9)–(1.1.10) if this problem has a nontrivial solution $u(x, y)$.

Unlike (1.1.4), the problem (1.1.9)–(1.1.10) usually cannot be explicitly solved. However, for certain geometries — for example, for a rectangle or for a disk — that could be done by using once again the separation of variables (in this case, the spatial variables x and y).

Exercise 1.1.9. Let $R_{a,b} = (0, a) \times (0, b)$ be a rectangle with sides a and b. Show that

$$\lambda^{\mathrm{D}}_{k,m} = \pi^2 \left(\frac{k^2}{a^2} + \frac{m^2}{b^2} \right), \quad k, m = 1, 2, \ldots, \tag{1.1.11}$$

are the eigenvalues of the Dirichlet problem (1.1.9)–(1.1.10) on $R_{a,b}$, and the corresponding eigenfunctions are given by

$$u^{\mathrm{D}}_{k,m}(x, y) = \sin \frac{k\pi}{a} x \, \sin \frac{m\pi}{b} y. \tag{1.1.12}$$

Prove that these functions form an orthogonal basis in $L^2(R_{a,b})$.

Remark 1.1.10. The separation of variables does not immediately imply that (1.1.11) and (1.1.12) provide *all* eigenvalues and eigenfunctions of the Dirichlet problem (1.1.9) on a rectangle. This has to be shown separately and, indeed, it follows from the fact that the set (1.1.12) forms a basis in $L^2(R_{a,b})$.

More generally, the fact that eigenfunctions of (1.1.9)–(1.1.10) in a bounded domain Ω can be chosen to form a basis in $L^2(\Omega)$ follows from the *spectral theorems*; see Chapter 2.

Definition 1.1.11. The *multiplicity* of an eigenvalue λ is the dimension of the corresponding eigenspace. If the dimension is equal to one, the eigenvalue is called *simple*.

Exercise 1.1.12. Show that if $\frac{a^2}{b^2}$ is irrational, then all the Dirichlet eigenvalues of a rectangle $R_{a,b}$ are simple.

Note that if $\frac{a^2}{b^2}$ is rational, then the multiplicities of the Dirichlet eigenvalues of $R_{a,b}$ can be arbitrarily large. This follows from number-theoretic results on the representation of integers as binary quadratic forms. In the case of a square, the precise answer could be found using the so-called sum of squares function, see [**HarWri08**, §16.9], and also Remark 1.2.14 below. For example, if $a = b = \pi$, one can check that the eigenvalue $\lambda = 5^{2k-1}$, $k \in \mathbb{N}$, has multiplicity $2k$.

Example 1.1.13. Since for an eigenvalue of multiplicity m we have an m-dimensional linear space of corresponding eigenfunctions, particular eigenfunctions may look quite unlike each other; see Figure 1.2.

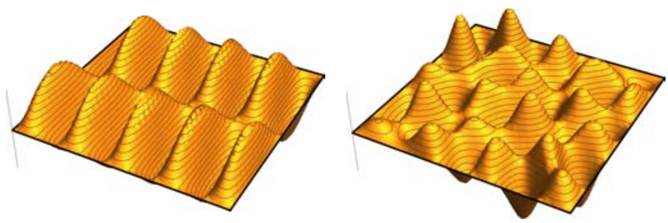

Figure 1.2. Two eigenfunctions corresponding to the same Dirichlet eigenvalue $85\pi^2$ of the unit square $[0,1]^2$: on the left, the eigenfunction $\sin(2\pi x)\sin(9\pi y)$, and on the right, the eigenfunction $\frac{1}{\sqrt{5}}(\sin(2\pi x)\sin(9\pi y) - \sin(9\pi x)\sin(2\pi y) - \sin(6\pi x)\sin(7\pi y) + 2\sin(7\pi x)\sin(6\pi y))$.

Along with the Dirichlet boundary condition $u|_{\partial\Omega} = 0$ corresponding to a membrane with a fixed boundary, one may consider the vibration of a free membrane. This problem gives rise to the Neumann boundary condition, which can be viewed as an appropriate generalisation of (1.1.7):

$$\partial_n u = 0, \tag{1.1.13}$$

where from now on we set

$$\partial_n u := \langle (\nabla u)|_{\partial\Omega}, n\rangle$$

to denote the *normal derivative* of u. Here n is the exterior unit normal to the boundary $\partial\Omega$, and $\langle\cdot,\cdot\rangle$ stands for the standard vector inner product in $\mathbb{R}^d$ (or $\mathbb{C}^d$); see §B.1. It is clear that in order for the Neumann condition (1.1.13) to be well-defined, certain regularity of the boundary has to be assumed. For instance, if one assumes the boundary to be Lipschitz (i.e., locally representable as a graph of a Lipschitz function; see §B.3 for the definition), the normal derivative is well-defined at almost every point of the boundary. More general conditions under which the Neumann problem is well-defined will be discussed later.

Exercise 1.1.14. Show that

$$\lambda^{\mathrm{N}}_{k,m} = \pi^2\left(\frac{k^2}{a^2} + \frac{m^2}{b^2}\right), \qquad k, m \in \mathbb{N}_0 \ldots, \tag{1.1.14}$$

are the eigenvalues of the Neumann problem (1.1.9), (1.1.13) in the rectangle $R_{a,b}$, with the corresponding eigenfunctions

$$u^{\mathrm{N}}_{k,m}(x,y) = \cos\frac{k\pi}{a}x\,\cos\frac{m\pi}{b}y.$$

Note that the indices k, m of the Neumann eigenvalues may take the value zero, while in the Dirichlet case they start with one. In particular, the lowest Neumann eigenvalue is zero and the corresponding eigenfunction is a constant. In fact, this is true for any bounded domain Ω on which the Neumann problem is well-defined.

Exercise 1.1.15. Using the formula (1.1.11) for the eigenvalues of the Laplacian in an arbitrary rectangle with Dirichlet boundary conditions, find which rectangle minimises the first Dirichlet eigenvalue among all rectangles of fixed area. Similarly, using (1.1.14), find which rectangle of a fixed area maximises the first nonzero Neumann eigenvalue. What happens if we interchange minimisation and maximisation in these questions?

Exercise 1.1.16. Compute the Dirichlet and Neumann eigenvalues and eigenfunctions of a rectangular box in $\mathbb{R}^d$.

Example 1.1.17. Let us describe the eigenvalues and eigenfunctions of the Dirichlet and Neumann problems in the unit disk $\mathbb{D}$. Switching to polar coordinates (r, φ), using the standard expression

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}$$

for the Laplacian in planar polar coordinates, and looking for solutions of (1.1.9) in the form

$$u(r,\varphi) = \sum_{m=-\infty}^{+\infty} u_m(r)\mathrm{e}^{\mathrm{i}m\varphi},$$

we arrive at the equations

$$u_m''(r) + \frac{1}{r}u_m'(r) + \left(\lambda - \frac{m^2}{r^2}\right)u_m(r) = 0 \tag{1.1.15}$$

for unknown functions u_m.

The equations (1.1.15) are closely related to the *Bessel equation*

$$y''(r) + \frac{1}{r}y'(r) + \left(1 - \frac{m^2}{r^2}\right)y(r) = 0. \tag{1.1.16}$$

For $m \in \mathbb{N}_0$, equation (1.1.16) possesses, up to a multiplicative constant, only one solution regular at $r = 0$. A specific choice of that constant corresponds to the solution defined via a power series

$$J_m(r) = \left(\frac{r}{2}\right)^m \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(m+k+1)} \left(\frac{r}{2}\right)^{2k}, \tag{1.1.17}$$

which is called the *Bessel function of the first kind* of order m. In fact, Bessel functions $J_\nu(r)$ can be defined in a similar manner for $\nu \in \mathbb{R}$ by taking $m = \nu$ in (1.1.17), see [**Wat95**, Chapter 3] for details, and it follows that $J_{-m}(r) = (-1)^m J_m(r)$ for $m \in \mathbb{N}$. We refer to [**Wat95**] for a complete treatment of the theory of Bessel functions and recall only some facts which we will use in the sequel. One can show that Bessel functions have infinitely many real zeros. Denote by $j_{m,k}$ the kth positive zero of the mth Bessel function $J_m(r)$ and by $j'_{m,k}$ the kth positive zero of the derivative $J'_m(r)$ (with the exception $j'_{0,1} = 0$ for the first zero of $J'_0(r)$; see [**DLMF22**, §10.21(i)]); cf. Figure 1.3.

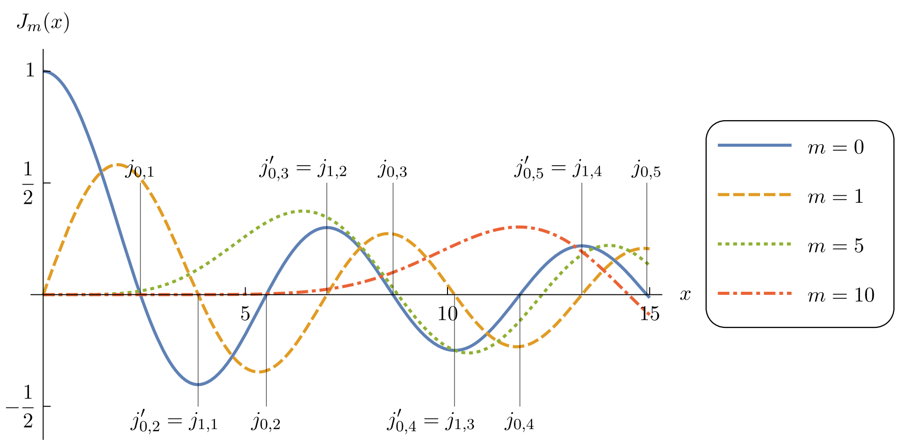

Figure 1.3. The graphs of some Bessel functions, with zeros of $J_0(x)$ and $J_1(x)$ marked. Note that $j_{1,k} = j'_{0,k+1}$ for $k \in \mathbb{N}$.

Returning now to the equations (1.1.15) and comparing to (1.1.16), one can easily see that the regular solutions of (1.1.15) are given, modulo a multiplicative constant, by $u_m(r) = J_m\left(\sqrt{\lambda}r\right)$.

Imposing the Dirichlet condition (1.1.10) now implies $u_m(1) = J_m\left(\sqrt{\lambda}\right) = 0$, and therefore the Dirichlet eigenvalues of the unit disk $\mathbb{D}$ are given by

$$j_{m,k}^2, \qquad m \in \mathbb{N}_0, \quad k \in \mathbb{N}.$$

For $m > 0$, the eigenvalues should be repeated with multiplicity two and the corresponding linearly independent eigenfunctions can be chosen either as

$$J_m(j_{m,k}r)\sin m\varphi \quad \text{or} \quad J_m(j_{m,k}r)\cos m\varphi. \tag{1.1.18}$$

For $m = 0$ each eigenvalue is simple, with the corresponding eigenfunction $J_0(j_{0,k}r)$ being radially symmetric. To ensure that we have found *all* the eigenfunctions we also need to prove that they form a basis in $L^2(\mathbb{D})$ as discussed in Remark 1.1.10; this is not entirely trivial and follows from the Sturm–Liouville theory; see [**CouHil89**, §V.5.5].

Similarly, imposing the Neumann condition (1.1.13) implies $u'_m(1) = \sqrt{\lambda}J'_m\left(\sqrt{\lambda}\right) = 0$, and therefore the Neumann eigenvalues of the unit disk $\mathbb{D}$ are given by $j'^{\,2}_{m,k}$, $m \in \mathbb{N}_0$, $k \in \mathbb{N}$, where for $m > 0$ the eigenvalues should be repeated with multiplicity two. The eigenfunctions corresponding to $j'^{\,2}_{m,k}$ are given by either

$$J_m\left(j'_{m,k}r\right)\sin m\varphi \quad \text{or} \quad J_m\left(j'_{m,k}r\right)\cos m\varphi \tag{1.1.19}$$

(as before we have only one eigenfunction for $m = 0$).

Finally, let us also note that the zeros of Bessel functions of different orders (respectively, of their derivatives) never coincide, and therefore there are no "accidental" multiplicities in the Dirichlet (respectively, Neumann) spectrum. In the Dirichlet case this follows from the proof of the celebrated Bourget hypothesis (1866) found by C. L. Siegel back in 1929; see [**Sie29**] and also [**Wat95**, pp. 484–485]. Essentially, Siegel proved a rather deep number-theoretic result: if $x \neq 0$ is an algebraic number, $J_m(x)$ is transcendental. At the same time, using relations between Bessel functions of different orders, one can show that if J_m and J_k share a common zero, it has to be an algebraic number. Therefore, the only possible common zero may be $x = 0$. The Neumann analogue of this result is also known; see [**HelSun16**].

Exercise 1.1.18. Using integrals [**DLMF22**, formulas 10.22.37–10.22.38], check the orthogonality in $L^2(\mathbb{D})$ of the eigenfunctions (1.1.18) or (1.1.19) in either the Dirichlet or Neumann case. This is just an illustration of a much more general phenomenon which we will encounter later in Theorems 2.1.20, 2.1.36, and 2.2.21: the eigenfunctions of the Dirichlet or Neumann Laplacian can always be chosen to form an orthonormal basis in L^2.

Remark 1.1.19. In the same manner, the eigenvalues of the Dirichlet and Neumann Laplacians on circular sectors and annuli can be expressed in terms of zeros of some Bessel functions or their combinations, or of their derivatives. Similarly, the variables separate for ellipses, and the eigenvalues can be expressed in terms of zeros of some special functions; see [**GreNgu13**] and [**KutSig84**].

Remark 1.1.20. Apart from the Dirichlet and Neumann boundary conditions, there exist other types of selfadjoint boundary conditions, for example the Robin ones or Zaremba (mixed) ones, which we discuss later in §3.1.3. The Robin conditions arise, for example, when the boundary is neither free nor fixed, but attached by a spring or some elastic material. Dirichlet, Neumann, and Robin conditions also have other physical interpretations, notably in terms of the heat equation; see [**Str08**] for further details.

1.2. The Laplacian on a Riemannian manifold

1.2.1. The Laplace–Beltrami operator. In this section we use various basic notions from Riemannian geometry which can be found in standard textbooks. In particular, lecture notes [**Bur98**] contain a concise and clear exposition of essentially everything that is needed.

Consider a smooth closed (that is, compact without boundary) manifold M of dimension $\dim M = d$ endowed with the Riemannian metric $g = \{g_{ij}\}$, $i, j = 1, \ldots, d$.

For any differentiable function f on M one can define the gradient ∇f; it is a vector field such that for any $p \in M$ and for any vector $\xi \in T_pM$ the following identity holds:

$$\langle \nabla f, \xi \rangle_g = \mathrm{d}f_p(\xi) =: \xi f, \tag{1.2.1}$$

where $\langle \cdot, \cdot \rangle_g$ is a scalar product on T_pM defined by the Riemannian metric; we will usually omit the subscript g. We say that ξf is the *directional derivative* of the function f in the direction of the vector ξ at the point p. It is easy to check that for the Euclidean space (1.2.1) yields the usual definition of the gradient.

Let us now introduce the *divergence* $\operatorname{div} X$ of a vector field X on a Riemannian manifold. Let $\mathrm{d}V_g$ be the volume density on (M, g). In local coordinates $x_1, \ldots, x_d$ it takes the form

$$\mathrm{d}V_g = \sqrt{\det g}\, \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_d.$$

We will sometimes write this as

$$\mathrm{d}V = \mathrm{d}V_g = \sqrt{\det g}\,\mathrm{d}x$$

for brevity. Given a smooth vector field X, one can define $\operatorname{div} X$ as a smooth function on M satisfying the identity

$$\int_M f \operatorname{div} X \,\mathrm{d}V_g = -\int_M \langle \nabla f, X \rangle \,\mathrm{d}V_g \tag{1.2.2}$$

for all $f \in C^1(M)$. To verify that the divergence exists, we note that using a partition of unity it suffices to check (1.2.2) for functions f supported in a coordinate chart, which is done below. We refer to [**Ros97**, §1.2.3] for a discussion concerning this approach.

Let us calculate the gradient and the divergence in local coordinates $(x_1, \ldots, x_d)$. The corresponding basis in the tangent bundle TM is given by $\left(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_d}\right)$ satisfying

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle = g_{ij}. \tag{1.2.3}$$

The gradient ∇f in this basis is given by

$$\nabla f = \sum_{j=1}^{d} c^j(x) \frac{\partial}{\partial x_j} \tag{1.2.4}$$

for some coefficients $c^j(x)$. Applying formula (1.2.1) we get

$$\sum_{j=1}^{d} c^j(x) g_{ji} = \left\langle \sum_{j=1}^{d} c^j \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \right\rangle = \mathrm{d}f\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial f}{\partial x_i}.$$

Applying the inverse matrix $\{g^{ij}\}$ and substituting the values of c^j into (1.2.4) we obtain

$$\nabla f = \sum_{i,j=1}^{d} g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{1.2.5}$$

Let us now calculate the divergence. Let f be a differentiable function compactly supported in a coordinate chart. Applying formula (1.2.2) to a vector field $X = \left(a^1(x), \ldots, a^d(x)\right)$ and substituting (1.2.5) in the right-hand side we obtain

$$
(1.2.6)\qquad
\begin{aligned}
&\int_M f \ \operatorname{div} X \sqrt{\det g}\, \mathrm{d}x_1 \dots \mathrm{d}x_d \\
&\qquad = -\int_M \left\langle \sum_{i,j=1}^d g^{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}, \sum_{i=1}^d a^i \frac{\partial}{\partial x_i} \right\rangle \sqrt{\det g}\, \mathrm{d}x_1 \dots \mathrm{d}x_d \\
&\qquad = -\int_M \sum_{i=1}^d \frac{\partial f}{\partial x_i} a^i \sqrt{\det g}\, \mathrm{d}x_1 \dots \mathrm{d}x_d \\
&\qquad = \int_M f \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(a^i \sqrt{\det g} \right) \mathrm{d}x_1 \dots \mathrm{d}x_d.
\end{aligned}
$$

The second equality follows from (1.2.3), and the last equality is a result of the integration by parts. Since formula (1.2.6) holds for any such function f, comparing its left- and right-hand sides we get

$$
(1.2.7)\qquad \operatorname{div} X = \frac{1}{\sqrt{\det g}} \sum_{i=1}^d \frac{\partial}{\partial x_i} \left(a^i \sqrt{\det g} \right).
$$

Recall that for a vector field X in the Euclidean space $\mathbb{R}^d$,

$$
(1.2.8)\qquad \operatorname{div} X = \sum_{i=1}^d \frac{\partial a^i}{\partial x_i}.
$$

It is easy to check that (1.2.7) agrees with (1.2.8) in this case.

Remark 1.2.1 (Definitions of the divergence). There are several equivalent ways to define the divergence. Note that the right-hand side of (1.2.8) can be represented as the trace of the operator $\xi \mapsto \xi X := (\xi a^1, \dots, \xi a^d)$ acting on vector fields. On a Riemannian manifold, the analogue of the directional derivative ξX is the covariant derivative $\boldsymbol{\nabla}_\xi X$, where $\boldsymbol{\nabla}$ denotes the Levi–Civita connection. Thus, a standard way to define the divergence in Riemannian geometry is

$$
(1.2.9)\qquad \operatorname{div} X = \operatorname{trace} [\xi \mapsto \boldsymbol{\nabla}_\xi X];
$$

see, for example, [**Bur98**, §2.2] or [**Cha84**, §I.1].

On an orientable manifold one can also define the divergence in a coordinate-free way using differential forms; see [**BerGauMaz71**, §II.G.I]. Let $\omega_g = \sqrt{\det g}\, \mathrm{d}x_1 \mathrm{d}x_2 \dots \mathrm{d}x_d = \mathrm{d}V_g$ be the volume form corresponding to the Riemannian metric g on M. One can show (see, for instance, [**Pet06**, Corollary 46]) that the Lie derivative of ω_g in the direction of a vector field X is given by

$$\mathcal{L}_X(\omega_g) = (\operatorname{div} X)\omega_g. \tag{1.2.10}$$

This formula explains the meaning of the term divergence; it measures the rate of expansion of the volume element as it flows along the vector field X.

Exercise 1.2.2. Show that formulas (1.2.9) and (1.2.10) yield the same expression (1.2.7) for the divergence in local coordinates. See [**Cha84**, §I.1] and [**Ros97**, §1.2.3] for a solution.

Let us now state the main definition of this subsection.

Definition 1.2.3. The operator $-\Delta := -\operatorname{div}\nabla$ defined on smooth functions is called the *Laplacian* (or the *Laplace–Beltrami operator*) on the manifold (M, g). We will sometimes write it as $-\Delta_g = -\Delta_M$ to distinguish a particular manifold or metric.

Combining the formulas (1.2.5) and (1.2.7) we obtain the following expression for the Laplacian:

$$-\Delta f = -\frac{1}{\sqrt{\det g}} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(g^{ij}\sqrt{\det g}\,\frac{\partial f}{\partial x_j}\right). \tag{1.2.11}$$

Example 1.2.4. Let $g_{ij} = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Then the metric is flat and the Laplacian takes the form

$$-\Delta f = -\operatorname{div}\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = -\sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2},$$

and we recover the usual definition (1.1.1) of the Laplace operator in the Euclidean space.

Exercise 1.2.5. Recall that given two Riemannian manifolds (M, g) and (N, h), a diffeomorphism $F : (M, g) \to (N, h)$ is called an *isometry* if it preserves the Riemannian metric, i.e., $F^*h = g$, where F^*h denotes the pullback metric; see, for example [**BerGauMaz71**, Definition A.2]. Using the invariance properties of the divergence and the gradient, show that the Laplace operator commutes with isometries: $-\Delta_g(u \circ F) = (-\Delta_h u) \circ F$ for any function $u \in C^\infty(N)$.

Exercise 1.2.6. Given $u, v \in C^\infty(M)$, show that

$$\Delta(uv) = v\Delta u + 2\langle \nabla u, \nabla v\rangle_g + u\Delta v.$$

Example 1.2.7. Suppose that the Riemannian metric in local coordinates (x, y) on a surface is given by $\mathrm{d}s^2 = h(x, y)(\mathrm{d}x^2 + \mathrm{d}y^2)$, where $h(x, y) > 0$. Such coordinates are called *isothermal* and they locally exist on any surface; see [**Spi88**, Addendum 1, Chapter 9]. Show that the Laplacian in isothermal coordinates has the form

$$-\Delta = -\frac{1}{h(x,y)}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

Remark 1.2.8 (Manifolds with boundary). In what follows, we will also consider compact Riemannian manifolds M with boundary $\partial M \neq \emptyset$. Note that in contrast to domains, which are open sets, by definition $\partial M \subset M$. Somewhat abusing notation, when talking about differential expressions or function spaces on a Riemannian manifold M with boundary, we always have in mind the *interior* of M, that is, $M \setminus \partial M$, without indicating this explicitly. Let us also mention that the definition of the divergence given above has to be adjusted accordingly in case of a manifold with boundary; the equality (1.2.2) should hold for all $f \in C_0^1(M)$, where by our convention $C_0^1(M) := C_0^1(M \setminus \partial M)$.

1.2.2. The Laplacian on a flat torus. Consider a two-dimensional flat square torus $\mathbb{T}_a^2 = \mathbb{R}^2/(a\mathbb{Z})^2$. Separating variables, and using Example 1.1.8, we can find its eigenfunctions using complex notation; they are of the form $\mathrm{e}^{\frac{2\pi\,\mathrm{i}\,\langle x,m\rangle}{a}}$, where $x = (x_1, x_2) \in \mathbb{T}_a^2$, and $m = (m_1, m_2) \in \mathbb{Z}^2$ is a vector with integer coordinates. The eigenvalues are given by $\lambda_{m_1,m_2} = \frac{4\pi^2}{a^2}(m_1^2 + m_2^2)$. In particular, we have a constant eigenfunction coming from the vector $m = (0, 0)$ and corresponding to the eigenvalue zero. The first nonzero eigenvalue $\lambda_1 = \frac{4\pi^2}{a^2}$ is of multiplicity four and comes from the eigenfunctions with $m = (\pm 1, 0)$ and $m = (0, \pm 1)$. The corresponding eigenfunctions may be chosen to be real as

$$\cos\frac{2\pi x_1}{a},\quad \sin\frac{2\pi x_1}{a},\quad \cos\frac{2\pi x_2}{a},\quad \sin\frac{2\pi x_2}{a}.$$

Numerical Exercise 1.2.9. Show that the multiplicity of an eigenvalue $\lambda \in \mathbb{N}$ of the torus $\mathbb{T}_{2\pi}^2$ is equal to the sum of squares function

$$r_2(\lambda) := \#\left\{(m_1, m_2) \in \mathbb{Z}^2 : \lambda = m_1^2 + m_2^2\right\}; \tag{1.2.12}$$

cf. Exercise 1.1.12. Use this to compile a table of all the distinct eigenvalues of $\mathbb{T}^2_{2\pi}$ less than 2,500 together with their multiplicities.

Exercise 1.2.10. Calculate the eigenvalues of the Laplacian on a flat rectangular d-dimensional torus

$$\mathbb{T}^d_{(a_1,\ldots,a_d)} = \mathbb{R}/(a_1\mathbb{Z}) \times \cdots \times \mathbb{R}/(a_d\mathbb{Z}),$$

using separation of variables and the spectrum of the Laplacian on a circle from Example 1.1.8.

Exercise 1.2.11. Find the eigenvalues and eigenfunctions of an arbitrary flat d-dimensional torus $\mathbb{T}^d_\Gamma = \mathbb{R}^d/\Gamma$, where Γ is an arbitrary lattice in $\mathbb{R}^d$. (You can find the answer in [**Cha84**, §II.2], [**BerGauMaz71**, §III.B.1], and [**Can13**, §5.2].)

A flat torus is a rare example of a manifold for which the eigenvalues and eigenfunctions can be calculated explicitly. However, even in this case, some basic questions regarding the properties of eigenvalues turn out to be very difficult.

Let us introduce, for a closed manifold M, the *counting function* of the eigenvalues of the Laplace–Beltrami operator on M,

$$\mathcal{N}_M(\lambda) = \mathcal{N}(\lambda) := \#\{j : \lambda_j(M) \le \lambda\}.$$

Each eigenvalue is counted with its multiplicity. The behaviour of the function $\mathcal{N}_M(\lambda)$ for large values of λ describes the asymptotic distribution of eigenvalues as $\lambda \to +\infty$. Understanding the properties of the counting function is one of the fundamental questions in spectral geometry.

Let us estimate $\mathcal{N}(\lambda) := \mathcal{N}_{\mathbb{T}^2_a}(\lambda)$ for a flat square torus. Each eigenvalue

$$\lambda_{m_1,m_2} = \frac{4\pi^2}{a^2}\left(m_1^2 + m_2^2\right)$$

corresponds to a point with integer coordinates (m_1, m_2) on the plane, and we are counting the number

$$\mathcal{G}(\rho) := \#\left\{(m_1, m_2) \in \mathbb{Z}^2 : m_1^2 + m_2^2 \le \rho^2\right\}$$

of such points inside a circle of radius $\rho := \frac{a\sqrt{\lambda}}{2\pi}$; we have

$$\mathcal{N}_{\mathbb{T}^2_a}(\lambda) = \mathcal{G}\left(\frac{a\sqrt{\lambda}}{2\pi}\right). \tag{1.2.13}$$

Clearly, an approximate number of integer points inside the circle is given by the area of the circle. Therefore, in this case

$$\mathcal{N}(\lambda) = \mathcal{G}\left(\frac{a\sqrt{\lambda}}{2\pi}\right) = \frac{a^2\lambda}{4\pi} + R(\lambda) = \frac{\mathrm{Area}(\mathbb{T}_a^2)\lambda}{4\pi} + R(\lambda), \tag{1.2.14}$$

where $R(\lambda) = o(\lambda)$ as $\lambda \to \infty$. Note the appearance of area in this asymptotic formula — as we will see later, this is not a coincidence. The asymptotic formula (1.2.14) for the counting function of the torus is known as *Weyl's law*; see §3.3.1.

What more can be said about the size of the remainder $R(\lambda)$?

Lemma 1.2.12. The remainder in Weyl's law (1.2.14) on a square torus satisfies the estimate

$$R(\lambda) = O(\sqrt{\lambda}) \qquad \text{as } \lambda \to +\infty.$$

Proof. For simplicity, set $a = 2\pi$; the result would follow for an arbitrary a by rescaling; see Exercise 2.1.42. Let us identify each unit square with integer coordinates in the plane with its left bottom corner (m, n). Then if $m^2 + n^2 < \lambda$, the whole square (corresponding to that corner) is contained inside the disk of radius $\sqrt{\lambda} + \sqrt{2}$; see Figure 1.4.

Therefore, $\mathcal{N}(\lambda) < \pi(\sqrt{\lambda}+\sqrt{2})^2$. Similarly, if the square has a nontrivial intersection with the open disk of radius $\sqrt{\lambda} - \sqrt{2}$, then $m^2 + n^2 < \lambda$. Note that the union of such squares fully covers the disk of radius $\sqrt{\lambda} - \sqrt{2}$, and therefore $\mathcal{N}(\lambda) > \pi(\sqrt{\lambda} - \sqrt{2})^2$. Combining the two bounds on $\mathcal{N}(\lambda)$ we get

$$|\mathcal{N}(\lambda) - \pi\lambda| \le 2\pi\sqrt{2\,\lambda} + 2\pi,$$

which implies the statement of the lemma. □

This result was known to C. F. Gauss, and the problem of counting the number $\mathcal{G}(\rho)$ of integer points inside a disk of radius ρ is called *Gauss's circle problem*. However, the estimate given by Lemma 1.2.12 is quite far from the optimal one.

Conjecture 1.2.13. For any $\varepsilon > 0$, we have $R(\lambda) = O(\lambda^{1/4+\varepsilon})$ as $\lambda \to +\infty$.

This conjecture is due to G. H. Hardy (1916) and has remained wide open for more than a century. It is one of the most famous open problems in analytic number theory. It is known that without ε in the exponent the conjecture is false — this follows from a quite nontrivial lower bound due to Hardy and E. Landau. It was shown by G. Voronoi (1903), W. Sierpiński (1906), and J. G. van der Corput (1923) that the upper bound holds with the

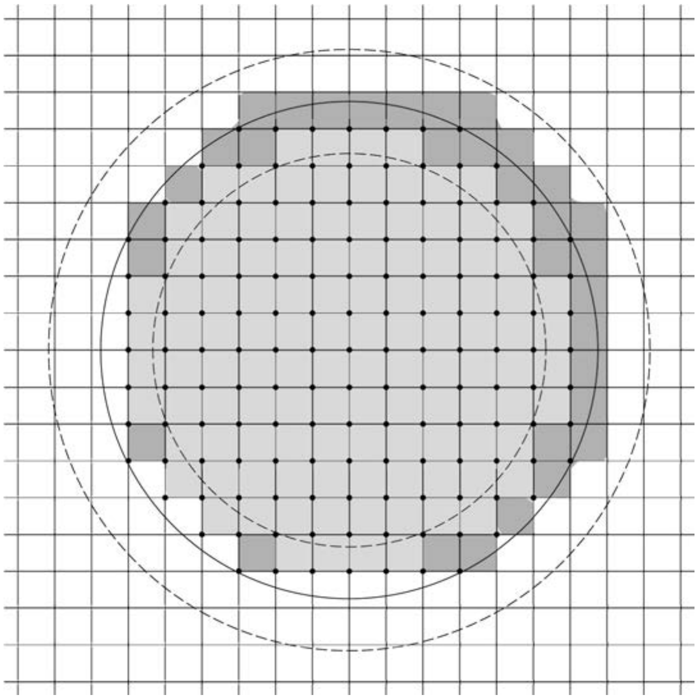

Figure 1.4. Estimating the number of integer points in a disk. The radii of the three concentric circles are $\sqrt{\lambda} - \sqrt{2}$, $\sqrt{\lambda}$, and $\sqrt{\lambda} + \sqrt{2}$.

exponent $\frac{1}{3}$. At present, the best upper bound for $R(\lambda)$ is due to J. Bourgain and N. Watt [**BouWat17**] with the exponent approximately equal to 0.3137.

Remark 1.2.14. There is a surprising link between the eigenvalue counting function for a flat square torus and the Bessel functions which appear in the spectral problems in the disk; see Example 1.1.17. Consider, once more, the torus $\mathbb{T}^2_{2\pi}$. Its eigenvalue counting function $\mathcal{N}_{\mathbb{T}^2_{2\pi}}(\lambda)$ coincides with the disk lattice point counting function $\mathcal{G}(\lambda)$ by (1.2.13). Consider, for an integer $m \geq 0$, the sum of squares function defined by (1.2.12). Then

$$\mathcal{G}(\rho) = \sum_{m=0}^{\lfloor \rho^2 \rfloor} r_2(m),$$

where $\lfloor \cdot \rfloor$ denotes the integer part. The function $\mathcal{G}(\rho)$ experiences a jump whenever ρ^2 is an integer with $r_2(\rho^2) > 0$. The identity due to Hardy [**Har15**] (in some form suggested by S. Ramanujan) is then

$$\mathcal{G}(\rho) - \frac{r_2(\rho^2)}{2} = \pi\rho^2 + \rho \sum_{n=1}^{\infty} \frac{r_2(n)}{\sqrt{n}} J_1\left(2\pi\rho\sqrt{n}\right),$$

thus bringing the Bessel function J_1 into play; see also [**BerDKZ18**] for some historical remarks and generalisations involving other Bessel functions.

1.2.3. The Laplace operator on spheres. This section is based on the material that can be found in [**BerGauMaz71**, §III.C.1], [**Shu01**, §III.22], [**Cha84**, §II.4], and [**AxlBouWad01**, Chapter 5].

Let $(\xi_1, \dots, \xi_d)$ be local coordinates on the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$ centred at the origin. Consider the corresponding spherical coordinates $(r, \xi_1, \dots, \xi_d)$ defined in some open cone in $\mathbb{R}^{d+1}$, where $r > 0$ is the radial variable. The standard Euclidean coordinates can be expressed as $x_i = r\varphi_i(\xi_1, \dots, \xi_d)$, $i = 1, \dots, d+1$, where φ_i, $i = 1, \dots, d+1$, are smooth functions parametrising the unit sphere. Given a function $f \in C^\infty(\mathbb{R}^{d+1})$, we obtain using the chain rule

$$\begin{aligned} \frac{\partial f}{\partial r} &= \sum_{i=1}^{d+1} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial r} = \sum_{i=1}^{d+1} \varphi_i \frac{\partial f}{\partial x_i}, \\ \frac{\partial f}{\partial \xi_j} &= \sum_{i=1}^{d+1} \frac{\partial f}{\partial x_i} \frac{\partial x_i}{\partial \xi_j} = r \sum_{i=1}^{d+1} \frac{\partial \varphi_i}{\partial \xi_j} \frac{\partial f}{\partial x_i}, \qquad j = 1, \dots, d. \end{aligned} \tag{1.2.15}$$

Consider the basis $\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial \xi_1}, \dots, \frac{\partial}{\partial \xi_d}\right)$ in the tangent space $T_x\mathbb{R}^d$. Then formulas (1.2.15) imply

$$\begin{aligned} \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial r} \right\rangle_{g_{\mathbb{R}^{d+1}}} &= \sum_{k=1}^{d+1} \varphi_k^2 = 1, \\ \left\langle \frac{\partial}{\partial r}, \frac{\partial}{\partial \xi_j} \right\rangle_{g_{\mathbb{R}^{d+1}}} &= r \sum_{k=1}^{d+1} \varphi_k \frac{\partial \varphi_k}{\partial \xi_j} = 0, \\ \left\langle \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j} \right\rangle_{g_{\mathbb{R}^{d+1}}} &= r^2 \sum_{k=1}^{d+1} \frac{\partial \varphi_k}{\partial \xi_i} \frac{\partial \varphi_k}{\partial \xi_j} = r^2 \left\langle \frac{\partial}{\partial \xi_i}, \frac{\partial}{\partial \xi_j} \right\rangle_{g_{\mathbb{S}^d}}, \end{aligned}$$

where $g_{\mathbb{S}^d}$ denotes the standard round metric on the sphere $\mathbb{S}^d$, that is, the metric induced by the Euclidean metric $g_{\mathbb{R}^{d+1}}$. Note that the last equality on the first line is simply the equation of the unit sphere; differentiating it with respect to ξ_j we obtain the last equality on the second line.

In view of the formulas above, the Euclidean metric in spherical coordinates $(r, \xi_1, \dots, \xi_d)$ is given by

$$g_{\mathbb{R}^{d+1}} = \begin{pmatrix} 1 & 0 \\ 0 & r^2\, g_{\mathbb{S}^d} \end{pmatrix}.$$

Therefore, applying formula (1.2.11) for the Laplace operator we obtain

$$\Delta_{g_{\mathbb{R}^{d+1}}} = \frac{\partial^2}{\partial r^2} + \frac{d}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{g_{\mathbb{S}^d}}. \tag{1.2.16}$$

Let $\mathcal{P}_m$ be the space of homogeneous polynomials in $\mathbb{R}^{d+1}$ of degree m. By definition, $P \in \mathcal{P}_m$ if and only if $P = r^m \cdot P|_{\mathbb{S}^d}$. In particular,

$$\frac{\partial P}{\partial r} = mr^{m-1} \cdot P|_{\mathbb{S}^d}, \qquad \frac{\partial^2 P}{\partial r^2} = m(m-1)r^{m-2} \cdot P|_{\mathbb{S}^d}. \tag{1.2.17}$$

We will denote by

$$\widetilde{\mathcal{P}}_m := \mathcal{P}_m|_{\mathbb{S}^d} = \{P|_{\mathbb{S}^d} : P \in \mathcal{P}_m\}$$

the restriction of $\mathcal{P}_m$ to the sphere $\mathbb{S}^d$.

Let

$$\mathcal{H}_m := \left\{P \in \mathcal{P}_m : \Delta_{g_{\mathbb{R}^{d+1}}} P = 0\right\}$$

be the space of all harmonic homogeneous polynomials of degree m, and let

$$\widetilde{\mathcal{H}}_m := \mathcal{H}_m|_{\mathbb{S}^d} = \{P|_{\mathbb{S}^d} : P \in \mathcal{H}_m\}$$

be the space of their restrictions to the sphere $\mathbb{S}^d$. It is easy to check that the spaces $\mathcal{H}_m$ and $\widetilde{\mathcal{H}}_m$ are isomorphic; indeed, the restriction map $\mathcal{H}_m \to \widetilde{\mathcal{H}}_m$ has an inverse given by

$$\widetilde{P} \mapsto r^m \widetilde{P}. \tag{1.2.18}$$

Moreover, applying the left- and the right-hand sides of (1.2.16) to $r^m \widetilde{P}$ and taking into account (1.2.17) we obtain

$$0 = r^{m-2}\left(-\Delta_{g_{\mathbb{S}^d}} \widetilde{P} - m(d+m-1)\widetilde{P}\right),$$

which immediately implies that $\widetilde{P}$ is an eigenfunction of the Laplacian on the sphere with the eigenvalue $m(d+m-1)$. In other words, we have proved the following.

Proposition 1.2.15. Any element of the space $\widetilde{\mathcal{H}}_m$ is an eigenfunction of the Laplacian on the sphere corresponding to the eigenvalue $\lambda = m(d+m-1)$.

The space $\widetilde{\mathcal{H}}_m$ of such eigenfunctions is called the space of *spherical harmonics* of degree m. Let us now calculate the multiplicities of the eigenvalues $m(d+m-1)$, $m \in \mathbb{N}_0$, and show that there are no other eigenvalues of the Laplacian on the sphere.

Theorem 1.2.16. The eigenvalues of the Laplace operator on the standard sphere $\mathbb{S}^d$ are given by $m(d+m-1)$, $m \in \mathbb{N}_0$, and the corresponding eigenspaces coincide with $\widetilde{\mathcal{H}}_m$. The multiplicity of the eigenvalue $\lambda = m(d+m-1)$ is equal to

$$\kappa_{d,m} := \dim \widetilde{\mathcal{H}}_m = \binom{d+m}{d} - \binom{d+m-2}{d}. \tag{1.2.19}$$

In order to prove this theorem we use the following proposition.

Proposition 1.2.17. For any $m \geq 0$, the following decomposition of $\mathcal{P}_m$ into a direct sum holds:

$$\mathcal{P}_m = \mathcal{H}_m \oplus \; r^2\mathcal{P}_{m-2}.$$

Here and further on we assume that $\mathcal{P}_m = \{0\}$ if $m < 0$.

Proof. We prove the statement by induction in m. For $m = 0, 1$ the result is trivially true. Assume that it is true for all $l < m$ and let us show that it holds for $l = m$. First, let us show that

$$\mathcal{H}_m \cap r^2\mathcal{P}_{m-2} = \{0\}. \tag{1.2.20}$$

Indeed, suppose there exists $P \in \mathcal{H}_m \cap r^2\mathcal{P}_{m-2}$. Consider its restriction on the sphere $\widetilde{P} \in \widetilde{\mathcal{H}}_m \cap \widetilde{\mathcal{P}}_{m-2}$. Note that $\widetilde{\mathcal{P}_m}$ is isomorphic to $\mathcal{P}_m$, with the inverse to the restriction map given by the same formula as (1.2.18).

As we have already shown, the space $\widetilde{\mathcal{H}}_m$ is contained in the eigenspace of the Laplacian corresponding to the eigenvalue $\lambda = m(m+d-1)$. At the same time, by the induction hypothesis, the space $\widetilde{\mathcal{P}}_{m-2}$ could be represented as a direct sum of certain spaces $\widetilde{\mathcal{H}}_j$, and for all of them $j < m$. Using integration by parts it is easy to show that Laplace eigenfunctions corresponding to distinct eigenvalues are orthogonal in $L^2(\mathbb{S}^d)$. Therefore, we conclude that $\widetilde{P} \equiv 0$. Since $P = r^m\widetilde{P}$ by (1.2.18), we obtain $P \equiv 0$, which implies (1.2.20).

We have thus shown that $\mathcal{P}_m \supset \mathcal{H}_m \oplus r^2\mathcal{P}_{m-2}$, and therefore

$$\dim \mathcal{H}_m \leq \dim \mathcal{P}_m - \dim \mathcal{P}_{m-2}. \tag{1.2.21}$$

At the same time, consider the Laplace operator as a map $\Delta : \mathcal{P}_m \to \mathcal{P}_{m-2}$. Its kernel is precisely $\mathcal{H}_m$, and therefore

$$\dim \mathcal{H}_m \geq \dim \mathcal{P}_m - \dim \mathcal{P}_{m-2}. \tag{1.2.22}$$

Combining (1.2.21) and (1.2.22) we conclude that

$$\dim \mathcal{H}_m = \dim \mathcal{P}_m - \dim \mathcal{P}_{m-2}. \tag{1.2.23}$$

It then follows that the map $\Delta : \mathcal{P}_m \to \mathcal{P}_{m-2}$ is surjective, and by the dimension count $\mathcal{P}_m = \mathcal{H}_m \oplus r^2\mathcal{P}_{m-2}$, which completes the proof of the proposition. □

Proof of Theorem 1.2.16. Let us show first that

$$L^2(\mathbb{S}^d) = \bigoplus_{m=1}^{\infty} \widetilde{\mathcal{H}}_m. \tag{1.2.24}$$

Indeed, applying inductively (1.2.20) and taking restriction to the sphere, we get

$$\bigoplus_{m=1}^{\infty} \widetilde{\mathcal{H}}_m = \bigoplus_{m=1}^{\infty} \widetilde{\mathcal{P}}_m.$$

Note that the direct sum on the right is isomorphic to the space of all polynomials in $\mathbb{R}^{d+1}$ restricted to $\mathbb{S}^d$. Formula (1.2.24) then holds since polynomials are dense in $L^2(\mathbb{R}^{d+1})$. Hence, the first assertion of Theorem 1.2.16 follows from Proposition 1.2.15.

It remains to note that (1.2.19) follows from (1.2.23), taking into account the isomorphism $\mathcal{H}_m \cong \widetilde{\mathcal{H}}_m$ and Lemma 1.2.18 below. □

Lemma 1.2.18. The dimension of the space $\mathcal{P}_m$ of homogeneous polynomials of order m in $\mathbb{R}^{d+1}$ is given by

$$\dim \mathcal{P}_m = \binom{d+m}{d} = \frac{(m+d)(m+d-1)\cdots(m+1)}{d!}. \tag{1.2.25}$$

Proof. The basis in $\mathcal{P}_m$ is given by monomials $x_1^{m_1} \dots x_{d+1}^{m_{d+1}}$, such that $m_1 + \cdots + m_{d+1} = m$. Therefore, the dimension of $\mathcal{P}_m$ is the number of ordered partitions of m into a sum of $d+1$ nonnegative integers. Finding it is equivalent to finding the number of sequences of zeros and ones of length $d+m$ with exactly d zeros (summing up the ones between the neighbouring zeros we get precisely the required partition of m), which is clearly given by (1.2.25). □

■ **Exercise 1.2.19.** Show that the coordinate functions $x_1, \dots, x_{d+1}$ restricted to the sphere $\mathbb{S}^d$ form a basis of the first eigenspace on $\mathbb{S}^d$.

■ **Exercise 1.2.20.** Show that the eigenvalue counting function of the Laplacian on the sphere $\mathbb{S}^d$ satisfies the asymptotics

$$\mathcal{N}_{\mathbb{S}^d}(\lambda) = \frac{2}{d!}\lambda^{\frac{d}{2}} + O\left(\lambda^{\frac{d-1}{2}}\right), \tag{1.2.26}$$

and the power in the remainder estimate cannot be improved. *Hint:* Find the asymptotic behaviour of multiplicities. A complete solution to this exercise can be found in [**Shu01**, §III.22].

Exercise 1.2.21. Using formula (1.2.16) and separation of variables, find eigenvalues and eigenfunctions of the Dirichlet and Neumann Laplacians for Euclidean balls in $\mathbb{R}^d$. In particular, show that for the d-dimensional unit ball $\mathbb{B}^d$, the Dirichlet eigenvalues are

$$\lambda^{\mathrm{D}}_{m,k}(\mathbb{B}^d) = \left(j_{m+\frac{d}{2}-1,k}\right)^2, \qquad m \in \mathbb{N}_0, \quad k \in \mathbb{N},$$

with multiplicity $\kappa_{d-1,m}$ given by (1.2.19), where $j_{m+\frac{d}{2}-1,k}$ is the kth positive zero of the Bessel function $J_{m+\frac{d}{2}-1}(x)$; see Example 1.1.17. Show also that the Neumann eigenvalues are

$$\lambda^{\mathrm{N}}_{m,k}(\mathbb{B}^d) = \left(p'_{d,m,k}\right)^2, \qquad m \in \mathbb{N}_0, \quad k \in \mathbb{N},$$

with the same multiplicity $\kappa_{d-1,m}$, where $p'_{d,m,k}$ is the kth positive zero of the derivative $U'_{d,m}(x)$ of the *ultraspherical Bessel function*

$$U_{d,m}(x) := x^{1-\frac{d}{2}} J_{m+\frac{d}{2}-1}(x),$$

with the exception $p'_{d,0,1} := 0$. For $d = 2$, compare your results with those given in Example 1.1.17.