Chapter 5

Fundamentals of Cooperative Games

5.1. Introduction to Cooperative Games

So far we have been concerned with noncooperative models, where the main focus is on the strategic aspects of the interaction among the players. The approach in cooperative game theory is different. Now, it is assumed that players can commit to behave in a way that is socially optimal. The main issue is how to share the benefits arising from cooperation. Important elements in this approach are the different subgroups of players, referred to as *coalitions*, and the set of outcomes that each coalition can get regardless of what the players outside the coalition do.[1] When discussing the different equilibrium concepts for noncooperative games, we were concerned about whether a given strategy profile was self-enforcing or not, in the sense that no player had incentives to deviate. We now assume that players can make binding agreements and, hence, instead of being worried about issues like self-enforceability, we care about notions like fairness and equity.

In this chapter, as usual, the set of players is denoted by N and its cardinality by n. Unless specified otherwise, we assume that $N = \{1, \ldots, n\}$. Differently from noncooperative games, where most of the analysis was done at the individual level, coalitions are very important in cooperative models. For each $S \subset N$, we refer to S as a *coalition*, with $|S|$ denoting

[1]In Peleg and Sudhölter (2003, Chapter 11), the authors discuss in detail some relations between the two approaches and, in particular, they derive the definition of *cooperative game without transferable utility* (Definition 5.2.1 below) from a strategic game in which the players are allowed to form coalitions and use them to coordinate their strategies through binding agreements.

the number of players in S. Coalition N is often referred to as the *grand coalition*.

We start this chapter by briefly describing the most general class of cooperative games, the so called *nontransferable utility games*. Then, we discuss two important subclasses: *bargaining problems* and *transferable utility games*. For each of these two classes we present the most important solution concepts and some axiomatic characterizations. We conclude this chapter with Section 5.15, devoted to *implementation theory*, where we discuss how some of these solution concepts can be obtained as the equilibrium outcomes of different noncooperative games. Since most literature on cooperative games revolves around transferable utility games, we have devoted to them most of the sections in this chapter.

5.2. Nontransferable Utility Games

In this section we present a brief introduction to the most general class of cooperative games: *nontransferable utility games* or *NTU-game*. The main source of generality comes from the fact that, although binding agreements between the players are implicitly assumed to be possible, utility is not transferable across players. Below, we present the formal definition and then we illustrate it with an example.

Given $S \subset N$ and a set $A \subset \mathbb{R}^S$, we say that A is *comprehensive* if, for each pair $x, y \in \mathbb{R}^S$ such that $x \in A$ and $y \leq x$, we have that $y \in A$. Moreover, the *comprehensive hull* of a set A is the smallest comprehensive set containing A.

Definition 5.2.1. An n-player *nontransferable utility game* (*NTU-games*) is a pair (N, V), where N is the set of players and V is a function that assigns, to each coalition $S \subset N$, a set $V(S) \subset \mathbb{R}^S$. By convention, $V(\emptyset) := \{0\}$. Moreover, for each $S \subset N$, $S \neq \emptyset$:

i) $V(S)$ is a nonempty and closed subset of $\mathbb{R}^S$.

ii) $V(S)$ is *comprehensive*. Moreover, for each $i \in N$, $V(\{i\}) \neq \mathbb{R}$, i.e., there is $v_i \in \mathbb{R}$ such that $V(\{i\}) = (-\infty, v_i]$.

iii) The set $V(S) \cap \{y \in \mathbb{R}^S : \text{ for each } i \in S,\ y_i \geq v_i\}$ is bounded.

Remark 5.2.1. In an NTU-game, the following elements are implicitly involved:

- For each $S \subset N$, R^S is the set of outcomes that the players in coalition S can obtain by themselves.

- For each $S \subset N$, $\{\succeq_i^S\}_{i\in S}$ are the preferences of the players in S over the outcomes in R^S. They are assumed to be complete, transitive, and representable through a utility function.[2]
- For each $S \subset N$, $\{U_i^S\}_{i\in S}$ are the utility functions of the players, which represent their preferences on R^S.

Hence, an NTU-game is a "simplification" in which, for each $S \subset N$ and each $x \in V(S)$, there is an outcome $r \in R^S$ such that, for each $i \in S$, $x_i = U_i^S(r)$.

Remark 5.2.2. It is common to find slightly different (and often equivalent) conditions for the $V(S)$ sets in the definition of an NTU-game. In each case, the authors choose the definition that is more suitable for their specific objectives. Nonemptiness and closedness are two technical requirements, which are also fairly natural. Requiring the $V(S)$ sets to be comprehensive is a convenient assumption, whose basic idea is that the players in coalition S can throw away utility if they want to. Moreover, it is worth mentioning that it is also often assumed that the $V(S)$ sets are convex, which allows to model, in particular, situations where the players inside each coalition S can choose lotteries over the elements of R^S and their utility functions are of the von Neumann and Morgenstern type.

Definition 5.2.2. Let (N, V) be an NTU-game. Then, the vectors in $\mathbb{R}^N$ are called *allocations*. An allocation $x \in \mathbb{R}^N$ is *feasible* if there is a partition $\{S_1, \ldots, S_k\}$ of N satisfying that, for each $l \in \{1, \ldots k\}$, there is $y \in V(S_l)$ such that, for each $i \in S_l$, $y_i = x_i$.

Example 5.2.1. (The banker game (Owen 1972b)). Consider the NTU-game given by:

$$\begin{aligned}
V(\{i\}) &= \{x_i : x_i \leq 0\}, \quad i \in \{1,2,3\},\\
V(\{1,2\}) &= \{(x_1, x_2) : x_1 + 4x_2 \leq 1000 \text{ and } x_1 \leq 1000\},\\
V(\{1,3\}) &= \{(x_1, x_3) : x_1 \leq 0 \text{ and } x_3 \leq 0\},\\
V(\{2,3\}) &= \{(x_2, x_3) : x_2 \leq 0 \text{ and } x_3 \leq 0\},\\
V(N) &= \{(x_1, x_2, x_3) : x_1 + x_2 + x_3 \leq 1000\}.
\end{aligned}$$

One can think of this game in the following way. On its own, no player can get anything. Player 1, with the help of player 2, can get 1000 dollars. Player 1 can reward player 2 by sending him money, but the money sent is lost or stolen with probability 0.75. Player 3 is a banker, so player 1 can ensure his transactions are safely delivered to player 2 by using player 3 as an intermediary. Hence, the question is how much should player 1 pay to player 2 for his help to get the 1000 dollars and how much to player 3 for

[2]There is a countable subset of R^S that is order dense in R^S (see Theorem 1.2.3).

helping him to make costless transactions to player 2. The reason for referring to these games as nontransferable utility games is that some transfers among the players may not be allowed. In this example, for instance, $(1000, 0)$ belongs to $V(\{1,2\})$, but players 1 and 2 cannot agree to the share $(500, 500)$ without the help of player 3. In Section 5.4 we define games with transferable utility, in which all transfers are assumed to be possible. ◇

The main objective of the theoretical analysis in this field is to find appropriate rules for choosing feasible allocations for the general class of NTU-games. These rules are referred to as *solutions* and aim to select allocations that have desirable properties according to different criteria such as equity, fairness, and stability. If a solution selects a single allocation for each game, then it is commonly referred to as an *allocation rule*. The definition of NTU-games allows to model a wide variety of situations and yet, at the same time, because of its generality, the study of NTU-games quickly becomes mathematically involved. This is the reason why the literature has focused more on studying some special cases than on studying the general framework. In this book we follow the same approach and do not cover NTU-games in full generality. For discussions and characterizations of different solution concepts in the general setting, the reader may refer, for instance, to Aumann (1961, 1985), Kalai and Samet (1985), Hart (1985), Borm et al. (1992a), and to the book by Peleg and Sudhölter (2003). We discuss the two most relevant subclasses of NTU-games: *bargaining problems* and *games with transferable utility* (TU-games).

5.3. Bargaining

In this section we study a special class of NTU-games, referred to as *bargaining problems*, originally studied in Nash (1950a). In a bargaining problem, there is a set of possible allocations, the *feasible set* F, and one of them has to be chosen by the players. Importantly, all the players have to agree on the chosen allocation; otherwise, the realized allocation is d, the *disagreement point*. Given a set $F \subset \mathbb{R}^n$ and an allocation $d \in \mathbb{R}^n$, we define the set $F_d := \{x \in F : x \geq d\}$.

Definition 5.3.1. An n-player *bargaining problem* with set of players N is a pair (F, d) whose elements are the following:

Feasible set: F is a closed, convex, and comprehensive subset of $\mathbb{R}^N$ such that F_d is compact.

Disagreement point: d is an allocation in F. It is assumed that there is $x \in F$ such that $x > d$.

Remark 5.3.1. An n-player bargaining problem (F, d) can be seen as an NTU-game (N, V), where $V(N) := F$ and, for each nonempty coalition $S \neq N$, $V(S) := \{y \in \mathbb{R}^S : \text{ for each } i \in S, y_i \leq d_i\}$.

Hence, given a bargaining problem (F, d), the feasible set represents the utilities the players get from the outcomes associated with the available agreements. The disagreement point delivers the utilities in the case in which no agreement is reached. The assumptions on the feasible set are mathematically convenient and, at the same time, natural and not too restrictive. As we already pointed out when introducing NTU-games, the convexity assumption can be interpreted as the ability of the players to choose lotteries over the possible agreements, with the utilities over lotteries being derived by means of von Neumann and Morgenstern utility functions.

We denote the set of n-player bargaining problems by B^N. Given two allocations $x, y \in F$, we say that x is *Pareto dominated by* y or that y *Pareto dominates* x if $x \leq y$ and $x \neq y$, i.e., for each $i \in N$, $x_i \leq y_i$, with strict inequality for at least one player. An allocation $x \in F$ is *Pareto efficient in* F, or just *efficient*, if no allocation in F Pareto dominates x.[3]

We now discuss several solution concepts for n-player bargaining problems. More precisely, we study *allocation rules*.

Definition 5.3.2. An *allocation rule* for n-player bargaining problems is a map $\varphi\colon B^N \to \mathbb{R}^N$ such that, for each $(F, d) \in B^N$, $\varphi(F, d) \in F_d$.

In the seminal paper by Nash (1950a), he gave some appealing properties that an allocation rule should satisfy and then proved that they characterize a unique allocation rule, which is known as the *Nash solution*.[4]

Let φ be an allocation rule and consider the following properties we may impose on it.

Pareto Efficiency (EFF): The allocation rule φ satisfies EFF if, for each $(F, d) \in B^N$, $\varphi(F, d)$ is a Pareto efficient allocation.

Symmetry (SYM): Let π denote a permutation of the elements of N and, given $x \in \mathbb{R}^N$, let x^π be defined, for each $i \in N$, by $x_i^\pi := x_{\pi(i)}$. We say that a bargaining problem $(F, d) \in B^N$ is *symmetric* if, for each permutation π of the elements of N, we have that i) $d^\pi = d$ and ii) for each $x \in F$, $x^\pi \in F$.

[3]Pareto efficiency is also known as Pareto optimality, the term being named after the studies of the economist V. Pareto in the early twentieth century.

[4]Nash introduced his allocation rule for the two-player case. We present a straightforward generalization to the n-player case.

Now, φ satisfies SYM if, for each symmetric bargaining problem $(F,d) \in B^N$, we have that, for each pair $i,j \in N$, $\varphi_i(F,d) = \varphi_j(F,d)$.

Covariance with positive affine transformations (CAT): $f^A\colon \mathbb{R}^N \to \mathbb{R}^N$ is a *positive affine transformation* if, for each $i \in N$, there are $a_i, b_i \in \mathbb{R}$, with $a_i > 0$, such that, for each $x \in \mathbb{R}^N$, $f_i^A(x) = a_i x_i + b_i$. Now, φ satisfies CAT if, for each $(F,d) \in B^N$ and each positive affine transformation f^A,

$$\varphi(f^A(F), f^A(d)) = f^A(\varphi(F,d)).$$

Independence of irrelevant alternatives (IIA): φ satisfies IIA if, for each pair of problems $(F,d), (\hat{F},d) \in B^N$, with $\hat{F} \subset F$, $\varphi(F,d) \in \hat{F}$ implies that $\varphi(\hat{F},d) = \varphi(F,d)$.

The four properties above are certainly appealing. EFF and SYM are very natural and CAT states that the choice of the utility representations should not affect the allocation rule (see Theorem 1.3.3). The most controversial one, as we will see later on, is IIA. However, the interpretation of this property is clear and reasonable: If the feasible set is reduced and the proposal of the allocation rule for the original problem is still feasible in the new one, then the allocation rule has to make the same proposal in the new problem. Before giving the definition of the Nash solution we need an auxiliary result.

Throughout this chapter, given $(F,d) \in B^N$, let $g^d\colon \mathbb{R}^N \to \mathbb{R}$ be defined, for each $x \in \mathbb{R}^N$, by $g^d(x) := \prod_{i\in N}(x_i - d_i)$, i.e., if $x > d$, $g^d(x)$ represents the product of the gains of the players at x with respect to their utilities at d.

Proposition 5.3.1. *Let $(F,d) \in B^N$. Then, there is a unique $z \in F_d$ that maximizes the function g^d over the set F_d.*

Proof. Since g^d is continuous and F_d is compact, g^d has a maximum in F_d. Suppose that there are $z, \hat{z} \in F_d$, with $z \neq \hat{z}$, such that

$$\max_{x\in F_d} g^d(x) = g^d(z) = g^d(\hat{z}).$$

Since there is $x \in F$ such that $x > d$, we have that, for each $i \in N$, $z_i > d_i$ and $\hat{z}_i > d_i$. By the convexity of F_d, $\bar{z} := \frac{z}{2} + \frac{\hat{z}}{2} \in F_d$. We now show that $g^d(\bar{z}) > g^d(z)$, which contradicts the fact that z is a maximum. We have that $\ln(g^d(\bar{z})) = \sum_{i\in N} \ln(\bar{z}_i - d_i) = \sum_{i\in N} \ln(\frac{z_i - d_i}{2} + \frac{\hat{z}_i - d_i}{2})$, which, by the strict concavity of the logarithmic functions, is strictly larger than $\sum_{i\in N}(\frac{1}{2}\ln(z_i - d_i) + \frac{1}{2}\ln(\hat{z}_i - d_i)) = \frac{1}{2}\ln(g^d(z)) + \frac{1}{2}\ln(g^d(\hat{z})) = \ln(g^d(z))$ and, hence, $g^d(\bar{z}) > g^d(z)$. $\square$

Definition 5.3.3. The Nash solution, NA, is defined, for each bargaining problem $(F,d) \in B^N$, by $\text{NA}(F,d) := z$, where $g^d(z) = \max_{x\in F_d} g^d(x) = \max_{x\in F_d} \prod_{i\in N}(x_i - d_i)$.

Proposition 5.3.1 ensures that the Nash solution is a well defined allocation rule. Given a bargaining problem $(F,d) \in B^N$, the Nash solution selects the unique allocation in F_d that maximizes the product of the gains of the players with respect to the disagreement point. This allows for a nice geometric interpretation, which, for $n = 2$, says that the Nash solution is the point z in F_d that maximizes the area of the rectangle with vertices z, d, (z_1, d_2), and (d_1, z_2). Below, we present the axiomatic characterization that Nash (1950a) provided for this allocation rule. First, we need an auxiliary lemma.

Lemma 5.3.2. *Let $(F,d) \in B^N$ and let $z := \text{NA}(F,d)$. For each $x \in \mathbb{R}^N$, let $h(x) := \sum_{i\in N}\prod_{j\neq i}(z_j - d_j)x_i$. Then, for each $x \in F$, $h(x) \leq h(z)$.*

Proof. Suppose that there is $x \in F$ with $h(x) > h(z)$. For each $\varepsilon \in (0,1)$, let $x^\varepsilon := \varepsilon x + (1-\varepsilon)z$. By the convexity of F, $x^\varepsilon \in F$. Since $z \in F_d$ and $z > d$ then, for sufficiently small ε, $x^\varepsilon \in F_d$. Moreover,

$$\begin{aligned} g^d(x^\varepsilon) &= \prod_{i\in N}(z_i - d_i + \varepsilon(x_i - z_i)) \\ &= \prod_{i\in N}(z_i - d_i) + \varepsilon \sum_{i\in N}\prod_{j\neq i}(z_j - d_j)(x_i - z_i) + \sum_{i=2}^{n} \varepsilon^i f_i(x,z,d) \\ &= g^d(z) + \varepsilon(h(x) - h(z)) + \sum_{i=2}^{n} \varepsilon^i f_i(x,z,d), \end{aligned}$$

where every $f_i(x,z,d)$ is a function that only depends on x, z, and d. Then, since $h(x) > h(z)$, $g^d(x^\varepsilon)$ is greater than $g^d(z)$ for a sufficiently small ε, which contradicts that $z = \max_{x\in F_d} g^d(x)$. □

Theorem 5.3.3. *The Nash solution is the unique allocation rule for n-player bargaining problems that satisfies* EFF, SYM, CAT, *and* IIA.

Proof. It can be easily checked that NA satisfies EFF, SYM, CAT, and IIA. Let φ be an allocation rule for n-player bargaining problems that satisfies the four properties and let $(F,d) \in B^N$. Let $z := \text{NA}(F,d)$. We now show that $\varphi(F,d) = z$. Let $U := \{x \in \mathbb{R}^N : h(x) \leq h(z)\}$, where h is defined as in Lemma 5.3.2, which, in turn, ensures that $F \subset U$. Let f^A be the positive affine transformation that associates, to each $x \in \mathbb{R}^N$, the vector $(f_1^A(x), \ldots, f_n^A(x))$ where, for each $i \in N$,

$$f_i^A(x) := \frac{1}{z_i - d_i}x_i - \frac{d_i}{z_i - d_i}.$$

Now, we compute $f^A(U)$.

$$\begin{aligned} f^A(U) &= \{y \in \mathbb{R}^N : (f^A)^{-1}(y) \in U\} \\ &= \{y \in \mathbb{R}^N : h((f^A)^{-1}(y)) \leq h(z)\} \\ &= \{y \in \mathbb{R}^N : h((z_1 - d_1)y_1 + d_1, \ldots, (z_n - d_n)y_n + d_n) \leq h(z)\} \\ &= \{y \in \mathbb{R}^N : \sum_{i \in N} \prod_{j \neq i} (z_j - d_j)((z_i - d_i)y_i + d_i) \leq \sum_{i \in N} \prod_{j \neq i} (z_j - d_j) z_i\}, \end{aligned}$$

which, after straightforward algebra, leads to

$$\begin{aligned} f^A(U) &= \{y \in \mathbb{R}^N : \sum_{i \in N} \prod_{j \in N} (z_j - d_j) y_i \leq \sum_{i \in N} \prod_{j \in N} (z_j - d_j)\} \\ &= \{y \in \mathbb{R}^N : \prod_{j \in N} (z_j - d_j) \sum_{i \in N} y_i \leq \prod_{j \in N} (z_j - d_j) \sum_{i \in N} 1\} \\ &= \{y \in \mathbb{R}^N : \sum_{i \in N} y_i \leq n\}. \end{aligned}$$

Note that $f^A(d) = (0, \ldots, 0)$ and, hence, $(f^A(U), f^A(d))$ is a symmetric bargaining problem. Since φ satisfies EFF and SYM, $\varphi(f^A(U), f^A(d)) = (1, \ldots, 1)$. Since φ also satisfies CAT, $\varphi(U, d) = (f^A)^{-1}((1, \ldots, 1)) = z$. Finally, since $z \in F$, $F \subset U$, and φ satisfies IIA, $\varphi(F, d) = z$. □

After the characterization result above, it is natural to wonder if the result is tight or, in other words, if any of the axioms is superfluous.

Proposition 5.3.4. *None of the axioms used in the characterization of the Nash solution given by Theorem 5.3.3 is superfluous.*

Proof. We show that, for each of the axioms in the characterization, there is an allocation rule different from the Nash solution that satisfies the remaining three, at least in the two-player case.

Remove EFF: The allocation rule φ defined, for each bargaining problem (F, d), by $\varphi(F, d) := d$ satisfies SYM, CAT, and IIA.

Remove SYM: Let φ be the allocation rule defined as follows. Let (F, d) be a bargaining problem. Then, $\varphi_1(F, d) := \max_{x \in F_d} x_1$. For each $i > 1$, $\varphi_i(F, d) := \max_{x \in F_d^i} x_i$, where $F_d^i := \{x \in F_d : \text{ for each } j < i, x_j = \varphi_j(F, d)\}$. These kind of solutions are known as *serial dictatorships*, since there is an ordering of the players and each player chooses the allocation he prefers among those that are left when he is given the turn to choose. This allocation rule satisfies EFF, CAT, and IIA.

Remove CAT: Let φ be the allocation rule that, for each bargaining problem (F, d), selects the allocation $\varphi(F, d) := d + \bar{t}(1, \ldots, 1)$, where $\bar{t} := \max\{t \in \mathbb{R} : d + t(1, \ldots, 1) \in F_d\}$ (the compactness of F_d ensures that $\bar{t}$ is well defined). This allocation rule, known as the *egalitarian solution* (Kalai 1977), satisfies SYM and IIA, but, although φ always selects points in the boundary of F, it does not satisfy EFF. Consider now an allocation rule $\hat{\varphi}$ that, for each two-player bargaining problem (F, d), selects the Pareto efficient allocation closest to $\varphi(F, d)$. It is not difficult to check that $\hat{\varphi}$ satisfies EFF, SYM, and IIA for two-player bargaining problems. Thus, Theorem 5.3.3 is not true if we drop CAT.

Remove IIA: The *Kalai-Smorodinsky solution*, which we present below, satisfies EFF, SYM, and CAT for two-player bargaining problems. Thus, Theorem 5.3.3 is not true if we drop IIA. □

As we have already mentioned, the only property in the characterization of the Nash solution that might be controversial is IIA. Now, we show an example that illustrates the main criticism against this property.

Example 5.3.1. Consider the two-player bargaining problem (F, d), where $d = (0,0)$ and F is the comprehensive hull of the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \leq 2\}$. Since NA satisfies SYM and EFF, then $\text{NA}(F, d) = (1,1)$. Consider now the problem $(\tilde{F}, d)$, where $\tilde{F}$ is the intersection of F with the set $\{(x_1, x_2) \in \mathbb{R}^2 : x_2 \leq 1\}$. Since NA satisfies IIA, $\tilde{F} \subset F$, and $\text{NA}(F, d) \in \tilde{F}$, then $\text{NA}(\tilde{F}, d) = (1,1)$. These two bargaining problems and the corresponding proposals made by the Nash solution are depicted in Figure 5.3.1.

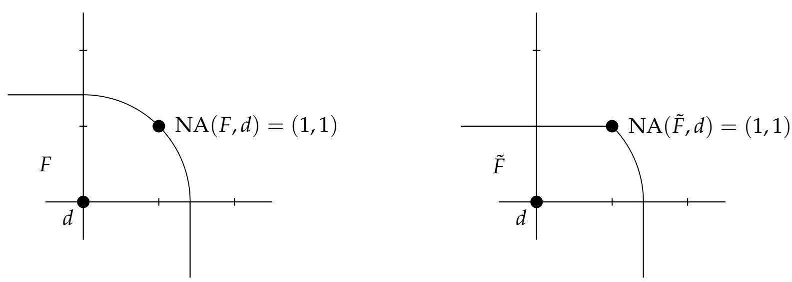


Figure 5.3.1. The Nash solution in Example 5.3.1.

This example suggests that the Nash solution is quite insensitive to the *aspirations* of the players. The maximal aspiration of player 2 in F is larger than in $\tilde{F}$ (meanwhile, the maximal aspiration of player 1 does not change), but the Nash solution proposes the same allocation for player 2 in both problems; indeed, in $(\tilde{F}, d)$, it proposes his maximal aspiration. ◇

In Kalai and Smorodinsky (1975), an alternative allocation rule was introduced for two-player bargaining problems. This new solution strongly depends on the aspiration levels of the players[5]. These levels give rise to

[5] This solution had also been discussed in Raiffa (1953). For this reason, some texts refer to it as the Raiffa-Kalai-Smorodinsky solution.

the so-called *utopia point*, which we define below. Then, we introduce the Kalai-Smorodinsky solution for n-player bargaining problems.

Definition 5.3.4. The *utopia point* of a bargaining problem $(F,d) \in B^N$ is given by the vector $b(F,d) \in \mathbb{R}^N$ where, for each $i \in N$, $b_i(F,d) = \max_{x \in F_d} x_i$.

Therefore, for each $i \in N$, $b_i(F,d)$ denotes the largest utility that player i can get in F_d.

Definition 5.3.5. The Kalai-Smorodinsky solution, KS, is defined, for each $(F,d) \in B^N$, by

$$\text{KS}(F,d) := d + \bar{t}(b(F,d) - d),$$

where $\bar{t} := \max\{t \in \mathbb{R} : d + t(b(F,d) - d) \in F_d\}$.

Note that, in the above definition, the compactness of F_d ensures that $\bar{t}$ is well defined.

Example 5.3.2. Consider again the bargaining problems described in Example 5.3.1. Now, $b(F,d) = (\sqrt{2}, \sqrt{2})$ and $b(\tilde{F},d) = (\sqrt{2},1)$. Then, it is easy to check that $\text{KS}(F,d) = (1,1)$ and $\text{KS}(\tilde{F},d) = (\sqrt{2/3}, \sqrt{1/3})$. Thus, the Kalai-Smorodinsky solution is sensitive to the less favorable situation of player 2 in $(\tilde{F},d)$. The computation of the Kalai-Smorodinsky solution for these two bargaining problems is illustrated in Figure 5.3.2. Note that this example also illustrates that the Kalai-Smorodinsky solution does not satisfy IIA. ◇

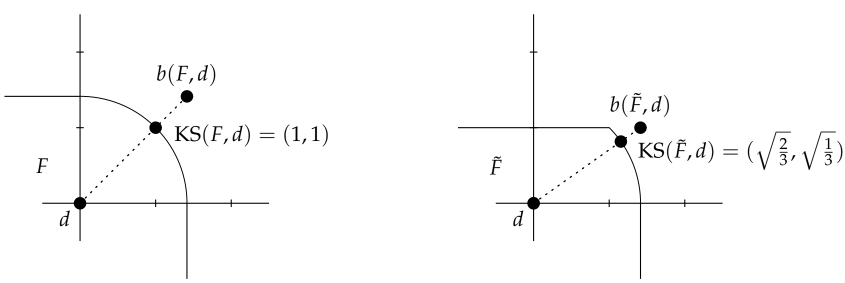


Figure 5.3.2. The Kalai-Smorodinsky solution in Example 5.3.2.

Kalai and Smorodinsky (1975) provided a characterization of their allocation rule for two-player bargaining problems that is based on the following monotonicity property.

Individual Monotonicity (IM): Let $(F,d), (\hat{F},d) \in B^N$ be a pair of bargaining problems such that $\hat{F}_d \subset F_d$. Let $i \in N$ be such that, for each $j \neq i$, $b_j(\hat{F},d) = b_j(F,d)$. If φ is an allocation rule for n-player bargaining problems that satisfies IM, then $\varphi_i(\hat{F},d) \leq \varphi_i(F,d)$.

Theorem 5.3.5. *The Kalai-Smorodinsky solution is the unique allocation rule for two-player bargaining problems that satisfies* EFF, SYM, CAT, *and* IM.

Proof. It is easy to check that KS satisfies EFF, CAT, SYM, and IM. Let φ be an allocation rule for two-player bargaining problems that satisfies the four properties and let $(F,d) \in B^2$. We now show that $\varphi(F,d) = \text{KS}(F,d)$. Since φ and KS satisfy CAT, we can assume, without loss of generality, that $d = (0,0)$ and that $b(F,d) = (1,1)$. Then, $\text{KS}(F,d)$ lies in the segment joining $(0,0)$ and $(1,1)$ and, hence, $\text{KS}_1(F,d) = \text{KS}_2(F,d)$. Define $\hat{F}$ by:

$$\hat{F} := \{x \in \mathbb{R}^2 : \text{there is } y \in \text{conv}\{(0,0),(0,1),(1,0),\text{KS}(F,d)\} \text{ with } x \leq y\}.$$

Since φ satisfies IM, $\hat{F} \subset F$, and $b(\hat{F},d) = b(F,d)$, we have that $\varphi(\hat{F},d) \leq \varphi(F,d)$. Since $\hat{F}$ is symmetric, by SYM and EFF, $\varphi(\hat{F},d) = \text{KS}(F,d)$. Then, $\varphi(\hat{F},d) = \text{KS}(F,d) \leq \varphi(F,d)$ but, since $\text{KS}(F,d)$ is Pareto efficient, we have $\text{KS}(F,d) = \varphi(F,d)$. □

A minor modification of the proof of Proposition 5.3.4 can be used to show that none of the axioms in Theorem 5.3.5 is superfluous; Exercise 5.2 asks the reader to formally show this.

Theorem 5.3.5 provides a characterization of the Kalai-Smorodinsky solution for two-player bargaining problems. Unfortunately, this characterization cannot be generalized to $n \geq 3$, as the following proposition, taken from Roth (1979), shows.

Proposition 5.3.6. *Let $n \geq 3$. Then, there is no solution for n-player bargaining problems satisfying* EFF, SYM, *and* IM.

Proof. Let $n \geq 3$ and suppose that φ is a solution for n-player bargaining problems satisfying EFF, SYM, and IM. Let $d = (0,\ldots,0)$ and

$$\hat{F} := \{x \in \mathbb{R}^N : \text{there is } y \in \text{conv}\{(0,1,\ldots,1),(1,0,1,\ldots,1)\} \text{ with } x \leq y\}.$$

By EFF, $\varphi(\hat{F},d)$ belongs to the segment joining $(0,1,\ldots,1)$ and $(1,0,1,\ldots,1)$ and, hence, $\varphi_3(\hat{F},d) = 1$. Let

$$F = \{x \in \mathbb{R}^N : \sum_{i=1}^{N} x_i \leq n-1 \text{ and, for each } i \in N, x_i \leq 1\}.$$

Since φ satisfies EFF and SYM then, for each $i \in N$, $\varphi_i(F,d) = \frac{n-1}{n}$. However, $\hat{F} \subset F$ and $b(\hat{F},d) = b(F,d) = (1,\ldots,1)$. Then, by IM, $\varphi(\hat{F},d) \leq \varphi(F,d)$, which is a contradiction with $\varphi_3(\hat{F},d) = 1 > \frac{n-1}{n} = \varphi_3(F,d)$. □

In spite of this negative result, Thomson (1980) showed that EFF, SYM, and IM characterize the Kalai-Smorodinsky solution if we restrict attention to a certain (large) domain of n-player bargaining problems.

The two allocation rules we have presented here for bargaining problems are the ones that have been most widely studied. There are, however, other proposals in the game theoretical literature. Those willing to study this topic in depth may refer to Peters (1992).

5.4. Transferable Utility Games

We now move to the most widely studied class of cooperative games: those with *transferable utility*, in short, *TU-games*. The situation is very similar to the one described in Section 5.2. The different coalitions that can be formed among the players in N can enforce certain allocations (possibly through binding agreements); the problem is to decide how the benefits generated by the cooperation of the players (formation of coalitions) have to be shared among them. However, there is one important departure from the general NTU-games framework. In a TU-game, given a coalition S and an allocation $x \in V(S) \subset \mathbb{R}^S$ that the players in S can enforce, all the allocations that can be obtained from x by transfers of utility among the players in S also belong to $V(S)$. Hence, $V(S)$ can be characterized by a single number, given by $\max_{x \in V(S)} \sum_{i \in S} x_i$. We denote the last number by $v(S)$, the *worth* of coalition S. The transferable utility assumption has important implications, both conceptually and mathematically. From the conceptual point of view, it implicitly assumes that there is a *numéraire* good (for instance, money) such that the utilities of all the players are linear with respect to it and that this good can be freely transferred among players. From the mathematical point of view, since the description of a game consists of a number for each coalition of players, TU-games are much more tractable than general NTU-games.

Definition 5.4.1. A TU-game is a pair (N, v), where N is the set of players and $v\colon 2^N \to \mathbb{R}$ is the *characteristic function* of the game. By convention, $v(\emptyset) := 0$.

In general, we interpret $v(S)$, the worth of coalition S, as the benefit that S can generate. When no confusion arises, we denote the game (N, v) by v. Also, we denote $v(\{i\})$ and $v(\{i,j\})$ by $v(i)$ and $v(ij)$, respectively. Let G^N be the class of TU-games with n players.

Remark 5.4.1. A TU-game (N, v) can be seen as an NTU-game (N, V) by defining, for each nonempty coalition $S \subset N$, $V(S) := \{y \in \mathbb{R}^S : \sum_{i \in S} y_i \leq v(S)\}$.

Definition 5.4.2. Let $v \in G^N$ and let $S \subset N$. The restriction of (N, v) to the coalition S is the TU-game (S, v_S), where, for each $T \subset S$, $v_S(T) := v(T)$.

Now, we show some examples.

Example 5.4.1. (Divide a million). A wealthy man dies and leaves one million euro to his three nephews, with the condition that at least two of them must agree on how to divide this amount among them; otherwise, the million euro will be burned. This situation can be modeled as the TU-game (N,v), where $N = \{1,2,3\}$, $v(1) = v(2) = v(3) = 0$, and $v(12) = v(13) = v(23) = v(N) = 1$. ◇

Example 5.4.2. (The glove game). Three players are willing to divide the benefits of selling a pair of gloves. Player 1 has a left glove and players 2 and 3 have one right glove each. A left-right pair of gloves can be sold for one euro. This situation can be modeled as the TU-game (N,v), where $N = \{1,2,3\}$, $v(1) = v(2) = v(3) = v(23) = 0$, and $v(12) = v(13) = v(N) = 1$. ◇

Example 5.4.3. (The Parliament of Aragón). This example illustrates that TU-games can also be used to model coalitional bargaining situations in which players negotiate with something more abstract than money. In this case we consider the Parliament of Aragón, one of the regions in which Spain is divided. After the elections which took place in May 1991, its composition was: PSOE (Socialist Party) had 30 seats, PP (Conservative Party) had 17 seats, PAR (Regionalist Party of Aragón) had 17 seats, and IU (a coalition mainly formed by communists) had 3 seats. In a Parliament, the most relevant decisions are made using the simple majority rule. We can use TU-games to measure the power of the different parties in a Parliament. This can be seen as "dividing" the power among them. A coalition is said to have the power if it collects more than half of the seats of the Parliament, 34 seats in this example. Then, this situation can be modeled as the TU-game (N,v), where $N = \{1,2,3,4\}$ (1=PSOE, 2=PP, 3=PAR, 4=IU), $v(S) = 1$ if there is $T \in \{\{1,2\},\{1,3\},\{2,3\}\}$ with $T \subset S$ and $v(S) = 0$ otherwise. In Section 6.4, we discuss a special class of TU-games, known as weighted majority games, that generalize the situation we have just described. The objective when dealing with these kind of games is to define *power indices* that measure how the total power is divided among the players. ◇

Example 5.4.4. (The visiting professor). We now illustrate how TU-games can also model situations that involve costs instead of benefits. Three research groups, from the universities of Milano (group 1), Genova (group 2), and Santiago de Compostela (group 3), plan to invite a Japanese professor to give a course on game theory. To minimize the costs, they coordinate the courses, so that the professor makes a tour visiting Milano, Genova, and Santiago de Compostela. Then, the groups want to allocate the cost of the tour among themselves. For this purpose they have estimated the travel cost (in euro) of the visit for all the possible coalitions of groups: $c(1) = 1500$, $c(2) = 1600$, $c(3) = 1900$, $c(12) = 1600$, $c(13) = 2900$,

$c(23) = 3000$, $c(N) = 3000$ (for each S, $c(S)$ indicates the minimum cost of the tour that the professor should make to visit all the groups in S). Let $N = \{1, 2, 3\}$. Note that (N, c) is a TU-game. However, it is what we usually call a *cost game* since, for each coalition S, $c(S)$ does not represent the benefits that S can generate, but the cost that must be covered. The *saving game* associated to this situation (displaying the benefits generated by each coalition) is (N, v) where, for each $S \subset N$,

$$v(S) = \sum_{i \in S} c(i) - c(S).$$

Thus, $v(1) = v(2) = v(3) = 0$, $v(12) = 1500$, $v(13) = 500$, $v(23) = 500$, and $v(N) = 2000$. ◇

These examples show that a wide range of situations can be modeled as TU-games. Now, we define a class of TU-games that is especially important.

Definition 5.4.3. A TU-game $v \in G^N$ is *superadditive* if, for each pair $S, T \subset N$, with $S \cap T = \emptyset$,

$$v(S \cup T) \geq v(S) + v(T).$$

We denote by SG^N the set of n-player superadditive TU-games. Note that a TU-game is superadditive when the players have real incentives for cooperation, that is, the union of any pair of disjoint coalitions of players never diminishes the total benefits. Hence, when dealing with superadditive games, it is natural to assume that the grand coalition will form and so, the question is how to allocate $v(N)$ among the players. In fact, a relevant part of the theory of TU-games is actually developed having superadditive games as the benchmark situation although, for simplicity, we always deal with the whole class G^N. Note that all the games in the examples above, with the only exception of (N, c) in Example 5.4.4, are superadditive.

We now present some other interesting classes of TU-games.

Definition 5.4.4. A TU-game $v \in G^N$ is *weakly superadditive* if, for each player $i \in N$ and each coalition $S \subset N \backslash \{i\}$, $v(S) + v(i) \leq v(S \cup \{i\})$.

Definition 5.4.5. A TU-game $v \in G^N$ is *additive* if, for each player $i \in N$ and each coalition $S \subset N \backslash \{i\}$, $v(S) + v(i) = v(S \cup \{i\})$. In particular, for each $S \subset N$, $v(S) = \sum_{i \in S} v(i)$.

Definition 5.4.6. A TU-game is *monotonic* if, for each pair $S, T \subset N$ with $S \subset T$, we have $v(S) \leq v(T)$.

Definition 5.4.7. A TU-game $v \in G^N$ is *zero-normalized* if, for each player $i \in N$, $v(i) = 0$.

Given a game $v \in G^N$, the *zero-normalization* of v is the zero-normalized game v^0 defined, for each $S \subset N$, by $v^0(S) := v(S) - \sum_{i \in S} v(i)$.

Definition 5.4.8. A TU-game $v \in G^N$ is *zero-monotonic* if its zero-normalization is a monotonic game.

The following result is straightforward.

Lemma 5.4.1. *Let $v \in G^N$. Then, v is weakly superadditive if and only if it is zero-monotonic.*

Proof. Exercise 5.5. □

The main goal of the theory of TU-games is to define solutions (and, in particular, allocation rules) that select, for each TU-game, sets of allocations (singletons in the case of allocation rules) that are admissible for the players. There are two important approaches in developing the previous task. One approach is based on *stability*, where the objective is to find solutions that pick sets of allocations that are stable according to different criteria. This is the approach underlying, for instance, the *core* (Gillies 1953), the *stable sets* (von Neumann and Morgenstern 1944), and the *bargaining set* (Aumann and Maschler 1964). The second approach is based on *fairness*: it aims to find allocation rules that propose, for each TU-game, an allocation that represents a fair compromise for the players. This is the approach underlying, for instance, the *Shapley value* (Shapley 1953), the *nucleolus* (Schmeidler 1969), the core-center (González-Díaz and Sánchez-Rodríguez 2007), and the *τ-value*, also known as *compromise value* and *Tijs value* (Tijs 1981).

5.5. The Core and Related Concepts

In this section we study the most important concept dealing with stability: the *core*. First, we introduce some properties of the allocations associated with a TU-game. Let $v \in G^N$. Let $x \in \mathbb{R}^N$ be an allocation. Then, x is *efficient* if $\sum_{i \in N} x_i = v(N)$. Hence, provided that v is a superadditive game, efficiency just requires that the total benefit from cooperation is actually shared among the players. The allocation x is *individually rational* if, for each $i \in N$, $x_i \geq v(i)$, that is, no player gets less than what he can get by himself. The *set of imputations* of a TU-game, $I(v)$, consists of all the efficient and individually rational allocations.

Definition 5.5.1. Let $v \in G^N$. The *set of imputations* of v, $I(v)$, is defined by

$$I(v) := \{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{and, for each } i \in N,\ x_i \geq v(i)\}.$$

The set of imputations of a superadditive game is always nonempty. The individual rationality of the allocations in $I(v)$ ensures that no player will individually block an allocation in $I(v)$, since he cannot get anything better by himself. Yet, it can be the case that a coalition of players has incentive to block some of the allocations in $I(v)$. To account for this, the core is defined by imposing a coalitional rationality condition on the allocations of $I(v)$. Given $v \in G^N$, an allocation $x \in \mathbb{R}^N$ is *coalitionally rational* if, for each $S \subset N$, $\sum_{i \in S} x_i \geq v(S)$. The core (Gillies 1953) is the set of efficient and coalitionally rational allocations.

Definition 5.5.2. Let $v \in G^N$. The *core* of v, $C(v)$, is defined by

$$C(v) := \{x \in I(v) : \text{ for each } S \subset N, \sum_{i \in S} x_i \geq v(S)\}.$$

The elements of $C(v)$ are usually called *core allocations*. The core is always a subset of the set of imputations. By definition, in a core allocation no coalition receives less than what it can get on its own (coalitional rationality). Hence, core allocations are stable in the sense that no coalition has incentives to block any of them. We now show that core allocations are also stable in a slightly different sense.

Definition 5.5.3. Let $v \in G^N$. Let $S \subset N$, $S \neq \emptyset$, and let $x, y \in I(v)$. We say that *y dominates x through S* if

i) for each $i \in S$, $y_i > x_i$, and
ii) $\sum_{i \in S} y_i \leq v(S)$.

We say that *y dominates x* if there is a nonempty coalition $S \subset N$ such that y dominates x through S. Finally, x is an *undominated* imputation of v if there is no $y \in I(v)$ such that y dominates x.

Observe that y dominates x through S if players in S prefer y to x and, moreover, y is a reasonable claim for S. Besides, if y dominates x, then there are coalitions willing and able to block x. Thus, a stable allocation should be undominated. This is in fact the case for core allocations.[6]

Proposition 5.5.1. *Let $v \in G^N$. Then,*

i) *If $x \in C(v)$, x is undominated.*
ii) *If $v \in SG^N$, $C(v) = \{x \in I(v) : x \text{ is undominated}\}$.*

Proof. i) Let $x \in C(v)$ and suppose there is $y \in I(v)$ and $S \subset N$, $S \neq \emptyset$, such that y dominates x through S. Then, $v(S) \geq \sum_{i \in S} y_i > \sum_{i \in S} x_i \geq v(S)$, which is a contradiction.

[6]The set of undominated imputations of a TU-game v is often called the D-core of v.

ii) Let $x \in I(v) \backslash C(v)$. Then, there is $S \subset N$ such that $\sum_{i \in S} x_i < v(S)$. Let $y \in \mathbb{R}^N$ be defined, for each $i \in N$, by

$$y_i := \begin{cases} x_i + \frac{v(S) - \sum_{j \in S} x_j}{|S|} & i \in S \\ v(i) + \frac{v(N) - v(S) - \sum_{j \in N \backslash S} v(j)}{|N \backslash S|} & i \notin S. \end{cases}$$

Since v is superadditive, $v(N) - v(S) - \sum_{j \in N \backslash S} v(j) \geq 0$ and, hence, $y \in I(v)$. Therefore, y dominates x through S. □

Both the set of imputations and the core of a TU-game are bounded sets. Moreover, since they are defined as the intersection of a series of half-spaces, they are convex polytopes. Indeed, the set of imputations of a superadditive game is either a point (if the TU-game is additive) or a simplex with n extreme points.

Next, we provide several examples. The first one illustrates that non-superadditive games may have undominated imputations outside the core. The other examples study the cores of the games introduced in Examples 5.4.1, 5.4.2, 5.4.3, and 5.4.4.

Example 5.5.1. Consider the TU-game (N, v), where $N = \{1, 2, 3\}$, $v(1) = v(2) = 0$, $v(3) = 1$, $v(12) = 2$, $v(13) = v(23) = 1$, and $v(N) = 2$. Since $v(N) < v(12) + v(3)$, this is not a superadditive game and, moreover, it has an empty core. However, among many others, $(1, 0, 1)$ is an undominated imputation. ◇

Example 5.5.2. It is easy to check that the core of the divide a million game is empty. This shows that the bargaining situation modeled by this game is strongly unstable. ◇

Example 5.5.3. The core of the glove game is $\{(1, 0, 0)\}$. It may seem strange that the unique core allocation of this game consists of giving all the benefits to the player who has the unique left glove. However, this is the unique undominated imputation of this game. An interpretation of this is that the price of the right gloves becomes zero because there are too many right gloves available in the market. ◇

Before moving to the game in Example 5.4.3, we introduce a new class of games.

Definition 5.5.4. A TU-game $v \in G^N$ is a *simple game* if i) it is monotonic, ii) for each $S \subset N$, $v(S) \in \{0, 1\}$, and iii) $v(N) = 1$.

We denote by S^N the class of simple games with n players. Note that, to characterize a simple game v, it is enough to specify the collection W of its *winning coalitions* $W := \{S \subset N : v(S) = 1\}$ or, equivalently, the collection W^m of its *minimal winning coalitions* $W^m := \{S \in W : \text{ for each } T \in$

W, if $T \subset S$, then $T = S\}$. For each $i \in N$, W_i^m denotes the set of minimal coalitions to which i belongs.

Definition 5.5.5. Let $v \in S^N$. Then, a player $i \in N$ is a *veto player* in v if $v(N \backslash \{i\}) = 0$.

Proposition 5.5.2. *Let $v \in S^N$ be a simple game. Then, $C(v) \neq \emptyset$ if and only if there is at least one veto player in v. Moreover, if $C(v) \neq \emptyset$, then*

$$C(v) = \{x \in I(v) : \text{ for each nonveto player } i \in N,\ x_i = 0\}.$$

Proof. Let $v \in S^N$. Let $x \in C(v)$ and let A be the set of veto players. Suppose that $A = \emptyset$. Then, for each $i \in N$, $v(N \backslash \{i\}) = 1$ and, hence,

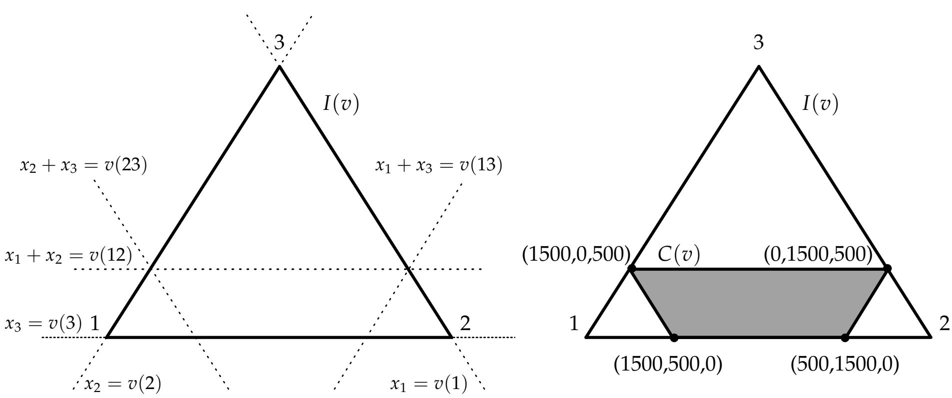
$$0 = v(N) - v(N \backslash \{i\}) \geq \sum_{j \in N} x_j - \sum_{j \in N \backslash \{i\}} x_j = x_i \geq 0,$$

which is incompatible with the efficiency of x. The second part of the statement is straightforward. □

Example 5.5.4. In view of the results above, since the game of Example 5.4.3 is a simple game with an empty set of veto players, it has an empty core. Note that the games in Examples 5.4.1 and 5.4.2 are also simple games. ◇

Example 5.5.5. The core of the saving game v associated with the visiting professor allocation problem is given by the nonempty set $\{x \in I(v) : x_1 + x_2 \geq 1500, x_1 + x_3 \geq 500, x_2 + x_3 \geq 500\}$. Figure 5.5.1 (a) depicts the hyperplanes that bound $I(v)$ and $C(v)$. Then, it is easy to check that

$$C(v) = \text{conv}\{(1500, 0, 500), (1500, 500, 0), (500, 1500, 0), (0, 1500, 500)\}.$$



(a) Bounding hyperplanes for $I(v)$ and $C(v)$.

(b) The set $C(v)$ and its extreme points.

Figure 5.5.1. The core of the visiting professor game.

In Figure 5.5.1 (b) we represent $C(v)$ and its extreme points. Actually, what is represented in Figure 5.5.1 are $I(v)$ and $C(v)$ as subsets of the efficiency hyperplane, which is given by $\{x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N)\}$. The vertex of $I(v)$ with label i represents the allocation x such that, for each $j \neq i$, $x_j = v(j) = 0$ and $x_i = v(N) - \sum_{j \neq i} v(j) = 2000$. ◇

As we have seen, there are coalitional bargaining situations that are highly unstable and, hence, the corresponding games have an empty core. Next, we provide a necessary and sufficient condition for a game to have a nonempty core. The corresponding theorem was independently (although not simultaneously) proved in Bondareva (1963) and Shapley (1967), and is known as the Bondareva–Shapley theorem.

Definition 5.5.6. A family of coalitions $\mathcal{F} \subset 2^N \backslash \{\emptyset\}$ is *balanced* if there are positive real numbers $\{\alpha_S : S \in \mathcal{F}\}$ such that, for each $i \in N$,

$$\sum_{\substack{S \in \mathcal{F} \\ i \in S}} \alpha_S = 1.$$

The numbers $\{\alpha_S : S \in \mathcal{F}\}$ are called *balancing coefficients*.

Definition 5.5.7. A TU-game $v \in G^N$ is *balanced* if, for each balanced family $\mathcal{F}$, with balancing coefficients $\{\alpha_S : S \in \mathcal{F}\}$,

$$\sum_{S \in \mathcal{F}} \alpha_S v(S) \leq v(N).$$

A TU-game $v \in G^N$ is *totally balanced* if, for each $S \subset N$, the TU-game (S, v_S) is balanced.

Remark 5.5.1. Each partition of N is a balanced family of coalitions, with all the balancing coefficients being equal to 1. Then, given a TU-game $v \in G^N$, a necessary condition for v to be a balanced game is that $\sum_{i \in N} v(i) \leq v(N)$.

Given a balanced family of coalitions $\mathcal{F}$, the balancing coefficients can be interpreted in the following way. For each $S \in \mathcal{F}$, the coefficient α_S represents the time the players in S are allocating to S; the balancing condition then requires that each player has a unit of time to allocate among the different coalitions in $\mathcal{F}$. Now, the balancing condition of a game can be interpreted as the impossibility of the players to allocate their time among the different coalitions in a way that yields an aggregate payoff higher than $v(N)$. This suggests that, for a TU-game to be balanced, the worth of the coalitions different from N has to be relatively small when compared to the worth of the grand coalition. The last observation is, in some sense, related to the stability of the coalitional bargaining situation modeled by the TU-game. This connection is precisely established by the Bondareva–Shapley theorem. In practice, this result is especially useful as a tool to prove that a given game or class of games has an nonempty core.

Theorem 5.5.3 (Bondareva–Shapley theorem)**.** *Let $v \in G^N$. Then, $C(v) \neq \emptyset$ if and only if v is balanced.*

Proof.[7] Let $v \in G^N$ be such that $C(v) \neq \emptyset$. Let $x \in C(v)$ and let $\mathcal{F}$ be a balanced family with balancing coefficients $\{\alpha_S : S \in \mathcal{F}\}$. Then,

$$\sum_{S\in\mathcal{F}} \alpha_S v(S) \leq \sum_{S\in\mathcal{F}} \sum_{i\in S} \alpha_S x_i = \sum_{i\in N} \Big(x_i \sum_{\substack{S\in\mathcal{F}\\ i\in S}} \alpha_S \Big) = \sum_{i\in N} x_i = v(N).$$

Conversely, suppose that v is balanced and consider the following linear programming problem (P):

$$\begin{aligned} &\text{Minimize} && \sum_{i\in N} x_i \\ &\text{subject to} && \sum_{i\in S} x_i \geq v(S), \quad \forall S \in 2^N\backslash\{\emptyset\}. \end{aligned}$$

Clearly, $C(v) \neq \emptyset$ if and only if there is $\bar{x}$ an optimal solution of (P) with $\sum_{i\in N} \bar{x}_i = v(N)$. The dual of (P) is the following linear programming problem (D):[8]

$$\begin{aligned} &\text{Maximize} && \sum_{S\in 2^N\backslash\{\emptyset\}} \alpha_S v(S) \\ &\text{subject to} && \sum_{\substack{S\in 2^N\backslash\{\emptyset\}\\ i\in S}} \alpha_S = 1, \quad \forall i \in N, \\ & && \alpha_S \geq 0, \quad \forall S \in 2^N\backslash\{\emptyset\}. \end{aligned}$$

Since the set of feasible solutions of (D) is nonempty and compact and the objective function is continuous, (D) has at least one optimal solution $\bar{\alpha}$. Let $\bar{\mathcal{F}} := \{S \subset N : \bar{\alpha}_S > 0\}$. Then, $\bar{\mathcal{F}}$ is a balanced family with balancing coefficients $\{\bar{\alpha}_S : S \in \bar{\mathcal{F}}\}$. Since (D) has an optimal solution, then, by the duality theorem, (P) also has an optimal solution $\bar{x}$ and, moreover,

$$\sum_{i\in N} \bar{x}_i = \sum_{S\in 2^N\backslash\{\emptyset\}} \bar{\alpha}_S v(S).$$

Hence, since v is balanced and $\bar{x}$ is an optimal solution of (P), $v(N) = \sum_{i\in N} \bar{x}_i$. Therefore, $C(v) \neq \emptyset$. □

[7] The "if" part of this proof uses some linear programming results. We refer the reader to Section 2.8 for the basics of linear programming.

[8] See footnote 20 in Section 2.8 for a derivation of the dual problem when the primal has no nonnegativity constraints.

Different from the proof of Bondareva–Shapley theorem above, which relies on the duality theorem, there is an alternative proof for the "if" part that uses the minimax theorem, i.e., that uses that every matrix game is strictly determined (von Neumann (1928)). This argument was developed by Aumann and nicely connects a classic result in noncooperative game theory and a classic result in cooperative game theory. Before formally presenting it, we need some definitions and auxiliary lemmas.

Definition 5.5.8. A TU-game $v \in G^N$ is a 0-1-*normalized game* if and only if $v(N) = 1$ and, for each $i \in N$, $v(i) = 0$.

Definition 5.5.9. Let $v, \hat{v} \in G^N$. The games v and $\hat{v}$ are *S-equivalent* if there are $k > 0$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that, for each $S \subset N$,

$$\hat{v}(S) = kv(S) + \sum_{i \in S} a_i.$$

The next result is straightforward and the proof is left to the reader.

Lemma 5.5.4. *Let $v \in G^N$ be such that $v(N) > \sum_{i \in N} v(i)$. Then, there is a unique* 0-1-*normalized game $\hat{v}$ such that v and $\hat{v}$ are S-equivalent.*

Proof. Exercise 5.6. □

Lemma 5.5.5. *Let $v, \hat{v} \in G^N$ be S-equivalent and such that $v(N) > \sum_{i \in N} v(i)$. Then, v is balanced if and only if $\hat{v}$ is balanced.*

Proof. Let $k > 0$ and $a_1, \ldots, a_n \in \mathbb{R}$ be such that, for each $S \subset N$, $\hat{v}(S) = kv(S) + \sum_{i \in S} a_i$. Suppose that v is balanced. Let $\mathcal{F}$ be a balanced family of coalitions with balancing coefficients $\{\alpha_S : S \in \mathcal{F}\}$. Then,

$$\begin{aligned}\sum_{S \in \mathcal{F}} \alpha_S \hat{v}(S) &= \sum_{S \in \mathcal{F}} \alpha_S \big(kv(S) + \sum_{i \in S} a_i\big) = k \sum_{S \in \mathcal{F}} \alpha_S v(S) + \sum_{S \in \mathcal{F}} \Big(\alpha_S \sum_{i \in S} a_i\Big) \\ &= k \sum_{S \in \mathcal{F}} \alpha_S v(S) + \sum_{i \in S} \Big(a_i \sum_{\substack{S \in \mathcal{F} \\ i \in S}} \alpha_S\Big) = k \sum_{S \in \mathcal{F}} \alpha_S v(S) + \sum_{i \in S} a_i.\end{aligned}$$

Since $k > 0$ and v is balanced,

$$\sum_{S \in \mathcal{F}} \alpha_S \hat{v}(S) \le kv(N) + \sum_{i \in N} a_i = \hat{v}(N).$$

Then, $\hat{v}$ is balanced. The converse is analogous. □

Lemma 5.5.6. *Let $v, \hat{v} \in G^N$ be S-equivalent and such that $v(N) > \sum_{i \in N} v(i)$. Then, $C(v) \neq \emptyset$ if and only if $C(\hat{v}) \neq \emptyset$.*

Proof. It immediately follows from definitions of core and S-equivalence (definitions 5.5.2 and 5.5.9). □

Alternative proof of the "if" part in Bondareva–Shapley theorem: i.e.,we again prove that, for each balanced game $v \in G^N$, $C(v) \neq \emptyset$. Let $v \in G^N$ be balanced. Then, $\sum_{i \in N} v(i) \leq v(N)$. We distinguish two cases.

Case 1: $\boldsymbol{v(N) = \sum_{i \in N} v(i)}$**.** For each $S \subsetneq N$, the partition of N given by S and the singletons of the players outside S is a balanced family of coalitions (all the balancing coefficients being 1); then, since v is balanced, $v(S) + \sum_{i \notin S} v(i) \leq v(N) = \sum_{i \in N} v(i)$ and, hence, $v(S) \leq \sum_{i \in S} v(i)$. Therefore, $(v(1), \ldots, v(n)) \in C(v)$.

Case 2: $\boldsymbol{v(N) > \sum_{i \in N} v(i)}$**.** By Lemmas 5.5.4, 5.5.5, and 5.5.6, we can restrict attention to the unique 0-1-normalized game that is S-equivalent to v. For the sake of notation, suppose that v itself is a 0-1-normalized game. Suppose that $C(v) = \emptyset$. Let $\mathcal{F}_1 := \{S \subset N : v(S) > 0\}$. Let $\mathcal{A}$ be the matrix game having entries a_{iS}, with $i \in N$ and $S \in \mathcal{F}_1$, where

$$a_{iS} := \begin{cases} 1/v(S) & i \in S \\ 0 & i \notin S. \end{cases}$$

By the minimax theorem, the game $\mathcal{A}$ is strictly determined. Let V be its value. For the sake of notation, let X and Y denote the sets of mixed strategies of the row player and the column player, respectively, in $\mathcal{A}$. Recall that, for each $x \in X$ and each $y \in Y$, the payoff of player 1 in $\mathcal{A}$ is given by $u_1(x, y) = x\mathcal{A}y^t$. Let $x_0 := (\frac{1}{n}, \ldots, \frac{1}{n}) \in X$. Now, $\min_{y \in Y} u_1(x_0, y) > 0$ and, hence, $V > 0$. Next, we show that $V < 1$. Let $x \in X$. Then, $\sum_{i \in N} x_i = 1 = v(N)$ and, for each $i \in N$, $x_i \geq 0 = v(i)$. Since $C(v) = \emptyset$, there is $S_x \subset N$ such that $0 \leq \sum_{i \in S_x} x_i < v(S_x)$. Let $y^x \in Y$ be defined by $y^x_S = 0$ if $S \neq S_x$ and $y^x_S = 1$ if $S = S_x$. Then, for each $x \in X$, $\min_{y \in Y} u_1(x, y) = \min_{y \in Y} x\mathcal{A}y^t \leq x\mathcal{A}(y^x)^t = \sum_{i \in S_x} x_i / v(S_x) < 1$. Hence, $V < 1$.

Let $\bar{y} \in Y$ be an optimal strategy of player 2 and let $\mathcal{F}_2$ be the family of coalitions given by $\mathcal{F}_2 := \mathcal{F}_1 \cup \{\{1\}, \ldots, \{n\}\}$. For each $S \in \mathcal{F}_2$, we define $\alpha_S \in \mathbb{R}$ as follows

$$\alpha_S := \begin{cases} \dfrac{\bar{y}_S}{Vv(S)} & S \in \mathcal{F}_1 \\ 1 - \displaystyle\sum_{\substack{S \in \mathcal{F}_1 \\ i \in S}} \alpha_S & S = \{i\}. \end{cases}$$

Since $V > 0$, v is a 0-1-normalized game, and $\bar{y}_S \in Y$, we have that, for each $S \in \mathcal{F}_1$, $\alpha_S \geq 0$. For each $i \in N$, let $x^i \in X$ be defined by $x^i_i := 1$ and, for each $j \neq i$, $x^i_j := 0$. Note that

$$\frac{1}{V} u_1(x^i, \bar{y}) = \frac{1}{V} \sum_{\substack{S \in \mathcal{F}_1 \\ i \in S}} \frac{\bar{y}_S}{v(S)} = \sum_{\substack{S \in \mathcal{F}_1 \\ i \in S}} \alpha_S.$$

Since $\bar{y}$ is an optimal strategy for player 2, we have that, for each $i \in N$, $V = \sup_{x \in X} u_1(x, \bar{y}) \geq u_1(x^i, \bar{y})$. Hence, for each $i \in N$, $\alpha_i \geq 0$.

Let $\mathcal{F} := \{S \in \mathcal{F}_2 : \alpha_S > 0\}$. By definition, the $\{\alpha_S\}_{S \in \mathcal{F}}$ are balanced coefficients for $\mathcal{F}$. Since $V < 1$ and $\bar{y} \in Y$,

$$\sum_{S \in \mathcal{F}} \alpha_S v(S) = \sum_{S \in \mathcal{F}_1} \alpha_S v(S) = \sum_{S \in \mathcal{F}_1} \frac{\bar{y}_S}{V v(S)} v(S) = \frac{1}{V} > 1 = v(N).$$

Therefore, we have that v is not a balanced game and we reach a contradiction. □

To conclude this section, we present a result, taken from Kalai and Zemel (1982b), that relates the class of additive games and the class of totally balanced games. First, we need to introduce two more concepts.

Definition 5.5.10. Let $v, \hat{v} \in G^N$. The *maximum game of v and $\hat{v}$, $v \vee \hat{v}$*, is defined, for each $S \subset N$, by $v \vee \hat{v}(S) := \max\{v(S), \hat{v}(S)\}$. Analogously, the *minimum game of v and $\hat{v}$, $v \wedge \hat{v}$*, is defined, for each $S \subset N$, by $v \wedge \hat{v}(S) := \min\{v(S), \hat{v}(S)\}$.

Lemma 5.5.7. *Let $v, \hat{v} \in G^N$. If v and $\hat{v}$ are totally balanced, then $v \wedge \hat{v}$ is totally balanced.*

Proof. Let $S \subset N$. Assume, without loss of generality, that $v(S) \leq \hat{v}(S)$. Since v is totally balanced, $C(v_S) \neq \emptyset$. Let $x \in C(v_S)$. Then, it is straightforward to see that $x \in C((v \wedge \hat{v})_S)$. □

The result below shows that the class of nonnegative additive games, together with the minimum operator, spans the class of nonnegative totally balanced games.

Theorem 5.5.8. *A nonnegative TU-game $v \in G^N$ is totally balanced if and only if it is the minimum game of a finite collection of nonnegative additive games.*

Proof. A nonnegative additive game is totally balanced and, hence, the "if" part follows from Lemma 5.5.7. Conversely, let $v \in G^N$ be totally balanced. For each nonempty coalition $S \subset N$, we define $v^S \in G^N$ as follows. Let $x^S \in C(v_S)$. For each $i \in N \backslash S$, let $x_i^S \in \mathbb{R}$ be such that $x_i^S > v(N)$. Let v^S be the additive game such that, for each $i \in N$, $v^S(i) = x_i^S$. For each $S \subset N$ and each $T \subset N$, $v^S(T) \geq v(T)$ and $v^S(S) = v(S)$. Hence, for each $T \subset N$,

$$\min_{S \in 2^N \backslash \{\emptyset\}} v^S(T) = v(T).$$

Therefore, v is the minimum game of the finite collection $\{v^S\}_{S \in 2^N \backslash \{\emptyset\}}$. □

5.6. The Shapley Value

In the previous section we studied the core of a TU-game, which is the most important set-valued solution concept for TU-games. Now, we present the most important allocation rule: the *Shapley value* (Shapley 1953). Formally, an allocation rule is defined as follows.

Definition 5.6.1. An *allocation rule* for n-player TU-games is just a map $\varphi\colon G^N \to \mathbb{R}^N$.

Shapley (1953), following an approach similar to the one taken by Nash when studying the Nash solution for bargaining problems, gave some appealing properties that an allocation rule should satisfy and proved that they characterize a unique allocation rule. First, we need to introduce two other concepts.

Definition 5.6.2. Let $v \in G^N$.

i) A player $i \in N$ is a *null player* if, for each $S \subset N$, $v(S \cup \{i\}) - v(S) = 0$.

ii) Two players i and j are *symmetric* if, for each coalition $S \subset N \backslash \{i, j\}$, $v(S \cup \{i\}) = v(S \cup \{j\})$.

Let φ be an allocation rule and consider the following properties we might impose on it.

Efficiency (EFF): The allocation rule φ satisfies EFF if, for each $v \in G^N$, $\sum_{i \in N} \varphi_i(v) = v(N)$.

Null Player (NPP): The allocation rule φ satisfies NPP if, for each $v \in G^N$ and each null player $i \in N$, $\varphi_i(v) = 0$.

Symmetry (SYM): The allocation rule φ satisfies SYM if, for each $v \in G^N$ and each pair $i, j \in N$ of symmetric players, $\varphi_i(v) = \varphi_j(v)$.[9]

Additivity (ADD): The allocation rule φ satisfies ADD if, for each pair $v, w \in G^N$, $\varphi(v + w) = \varphi(v) + \varphi(w)$.

Property EFF requires that φ allocates the total worth of the grand coalition, $v(N)$, among the players. Property NPP says that players that contribute zero to every coalition, i.e., that do not generate any benefit, should

[9]Note that this symmetry property is slightly different from the one we introduced for bargaining problems. Roughly speaking, the property for bargaining problems said that, if all the players are symmetric, then they get the same utility. The symmetry property says that, for each pair of symmetric players, both of them get the same utility.

receive nothing. Property SYM asks φ to treat equal players equally.[10] Finally, ADD is the unique controversial of these axioms. Despite being a natural requirement, ADD is not motivated by any fairness notion.

We now present the definition of the Shapley value as originally introduced in Shapley (1953).

Definition 5.6.3. The *Shapley value*, Φ^S, is defined, for each $v \in G^N$ and each $i \in N$, by

$$\Phi_i^S(v) := \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!}(v(S \cup \{i\}) - v(S)). \tag{5.6.1}$$

Therefore, in the Shapley value, each player gets a weighted average of the contributions he makes to the different coalitions. Actually, the formula in Eq. (5.6.1) can be interpreted as follows. The grand coalition is to be formed inside a room, but the players have to enter the room sequentially, one at a time. When a player i enters, he gets his contribution to the coalition of players that are already inside (i.e., if this coalition is S, he gets $v(S \cup \{i\}) - v(S)$). The order of the players is decided randomly, with all the $n!$ possible orderings being equally likely. It is easy to check that the Shapley value assigns to each player his expected value under this randomly ordered entry process.

The above discussion suggests an alternative definition of the Shapley value, based on the so-called *vectors of marginal contributions*. Let $\Pi(N)$ denote the set of all permutations of the elements in N and, for each $\pi \in \Pi(N)$, let $P^\pi(i)$ denote the set of predecessors of i under the ordering given by π, i.e., $j \in P^\pi(i)$ if and only if $\pi(j) < \pi(i)$.

Definition 5.6.4. Let $v \in G^N$ be a TU-game. Let $\pi \in \Pi(N)$. The *vector of marginal contributions* associated with π, $m^\pi(v) \in \mathbb{R}^N$, is defined, for each $i \in N$, by $m_i^\pi(v) := v(P^\pi(i) \cup \{i\}) - v(P^\pi(i))$.

The convex hull of the set of vectors of marginal contributions is commonly known as the *Weber set*, formally introduced as a solution concept by Weber (1988). It is clear from the random order story that the formula of the Shapley value is equivalent to

$$\Phi_i^S(v) := \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^\pi(v). \tag{5.6.2}$$

[10]Actually, this property is often referred to as *equal treatment of equals*. Sometimes the symmetry property is replaced by a similar property called anonymity, which requires that a relabeling of the players has to induce the same relabeling in the proposed allocation.

Exercise 5.7 asks the reader to formally show the equivalence between the two formulas. Before presenting the classic characterization of the Shapley value we introduce a class of TU-games that plays an important role in its proof.

Definition 5.6.5. Inside the class G^N, given $S \subset N$, the *unanimity game of coalition* S, w^S, is defined as follows. For each $T \subset N$, $w^S(T) := 1$ if $S \subset T$ and $w^S(T) := 0$ otherwise.

Theorem 5.6.1. *The Shapley value is the unique allocation rule in* G^N *that satisfies* EFF, NPP, SYM, *and* ADD.[11]

Proof. First, Φ^S satisfies both NPP and ADD. Moreover, it should be clear at this point that it also satisfies EFF and SYM. Each vector of marginal contributions is an efficient allocation and, hence, EFF follows from Eq. (5.6.2). Also, SYM can be easily derived from Eq. (5.6.2).

Now, let φ be an allocation rule satisfying EFF, NPP, SYM, and ADD. Recall that each $v \in G^N$ can be seen as the vector $\{v(S)\}_{S \in 2^N \backslash \{\emptyset\}} \in \mathbb{R}^{2^n - 1}$. Then, G^N can be identified with a vector space of $2^n - 1$ dimensions. Now, we show that the unanimity games $U(N) := \{w^S : S \in 2^N \backslash \{\emptyset\}\}$ are a basis of such a vector space, i.e., we show that $U(N)$ is a set of linearly independent vectors. Let $\{\alpha_S\}_{S \in 2^N \backslash \{\emptyset\}} \subset \mathbb{R}$ be such that $\sum_{S \in 2^N \backslash \{\emptyset\}} \alpha_S w^S = 0$ and suppose that there is $T \in 2^N \backslash \{\emptyset\}$ with $\alpha_T \neq 0$. We can assume, without loss of generality, that there is no $\hat{T} \subsetneq T$ such that $\alpha_{\hat{T}} \neq 0$. Then, $0 = \sum_{S \in 2^N \backslash \{\emptyset\}} \alpha_S w^S(T) = \alpha_T \neq 0$ and we have a contradiction. Since φ satisfies EFF, NPP, and SYM, we have that, for each $i \in N$, each $\emptyset \neq S \subset N$, and each $\alpha_S \in \mathbb{R}$,

$$\varphi_i(\alpha_S w^S) = \begin{cases} \frac{\alpha_S}{|S|} & i \in S, \\ 0 & \text{otherwise.} \end{cases}$$

Then, if φ also satisfies ADD, φ is uniquely determined, because $U(N)$ is a basis of G^N. □

The proof above relies on the fact that a TU-game can be expressed as a linear combination of unanimity games. In Exercise 5.8 the reader is asked to obtain an explicit formula for the corresponding coefficients. As we did when characterizing the Nash solution for bargaining problems, we show that there is no superfluous axiom in the above characterization.

Proposition 5.6.2. *None of the axioms used in the characterization of the Shapley value given by Theorem 5.6.1 is superfluous.*

[11] These are not exactly the properties used by Shapley to characterize his value. He considered a property called *support property* instead of EFF and NPP. The support property is equivalent to EFF plus NPP.

Proof. We show that, for each of the axioms in the characterization, there is an allocation rule different from the Shapley value that satisfies the remaining three.

Remove EFF: The allocation rule φ defined, for each $v \in G^N$, by $\varphi(v) := 2\Phi^S(v)$ satisfies NPP, SYM, and ADD.

Remove NPP: Let φ be the allocation rule defined as follows. For each $v \in G^N$ and each $i \in N$, $\varphi_i(v) = v(N)/n$. This allocation rule is known as the *equal division value* and it satisfies EFF, SYM, and ADD.[12]

Remove SYM: Let φ be the allocation rule defined as follows. Let $v \in G^N$ and let $\Pi^1(N)$ be the set of orderings of the players in N in which player 1 is in the first position, i.e., $\pi \in \Pi^1(N)$ if and only if $\pi(1) = 1$. Then, for each $i \in N$, $\varphi_i(v) := \frac{1}{(n-1)!} \sum_{\pi \in \Pi^1(N)} m_i^\pi(v)$. This allocation rule satisfies EFF, NPP, and ADD.

Remove ADD: Let φ be the allocation rule defined as follows. Let $v \in G^N$ and let d be the number of null players in game v. Then, for each $i \in N$, $\varphi_i(v) = 0$ if i is a null player and $\varphi_i(v) = \frac{v(N)}{n-d}$ otherwise. This allocation rule satisfies EFF, NPP, and SYM. □

As we have already pointed out, the additivity axiom does not come from any fairness consideration and, because of this, it has received some criticisms in the literature. These criticisms motivated the appearance of alternative characterizations of the Shapley value. The most important ones are due to Young (1985), Hart and Mas-Colell (1989), and Chun (1989).

Remark 5.6.1. Let $v \in SG^N$. Then, for each $i \in N$ and each $\pi \in \Pi(N)$, we have $m_i^\pi(v) \geq v(i)$. Then, if $v \in SG^N$, for each $i \in N$,

$$\Phi_i^S(v) = \frac{1}{n!} \sum_{\pi \in \Pi(N)} m_i^\pi(v) \geq \frac{1}{n!} \sum_{\pi \in \Pi(N)} v(i) = v(i).$$

So, for the class of superadditive games, the Shapley value belongs to the set of imputations.

Below we compute the Shapley value for the games discussed in Examples 5.4.1, 5.4.2, 5.4.3, and 5.4.4. We omit the detailed computations, which can be easily done using the vectors of marginal contributions and Eq. (5.6.2).

Example 5.6.1. The Shapley value for the divide a million game is the allocation $(1/3, 1/3, 1/3)$. Note that, although the core of this game is empty,

[12] In Section 5.9 we present a characterization of the equal division value, taken from van den Brink (2007), through the same axioms of the above characterization of the Shapley value with the exception that the null player property is replaced by a property called *nullifying player property*.

the Shapley value proposes an allocation for it. The core looks for allocations satisfying some stability requirements and, hence, it can be empty. ◇

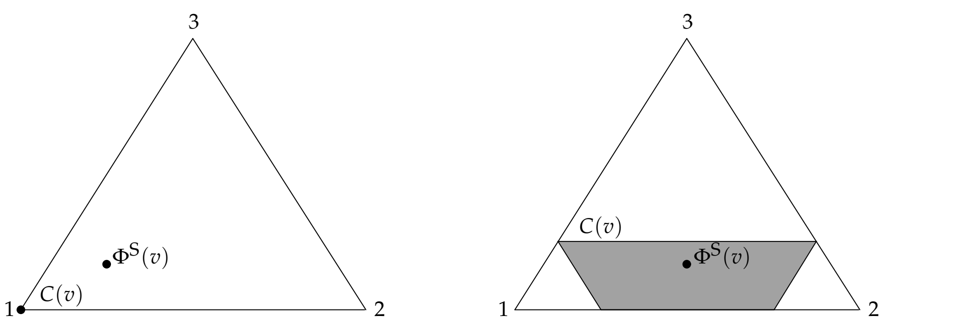
Example 5.6.2. The Shapley value for the glove game is $(2/3, 1/6, 1/6)$. Remember that the core of this game is $\{(1, 0, 0)\}$. The core and the Shapley value of this game are represented in Figure 5.6.1(a). Hence, even when the core is nonempty, the Shapley value may not be a core allocation. ◇

Example 5.6.3. The Shapley value for the Parliament of Aragón is the allocation $(1/3, 1/3, 1/3, 0)$. This is a measure of the power of the four political parties in this Parliament. Note that IU is a null player and that the other three parties are symmetric in the simple game, which only takes into account their voting power. ◇

Example 5.6.4. The Shapley value in the visiting professor game discussed in Example 5.4.4 is $\Phi^S(v) = (5000/6, 5000/6, 2000/6)$. These are the savings for the players. According to this allocation of the savings, the players have to pay $(4000/6, 4600/6, 9400/6)$. Note that the last vector is precisely $\Phi^S(c)$. This relationship holds for any pair of cost/benefit games:

$$\begin{aligned}\Phi_i^S(v) &= \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!}(v(S \cup \{i\}) - v(S)) \\ &= \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n-|S|-1)!}{n!}(c(S) - c(S \cup \{i\}) + c(i)) \\ &= c(i) - \Phi_i^S(c).\end{aligned}$$

Finally, observe that, for this game, $\Phi^S(v) \in C(v)$. In Figure 5.6.1(b) we represent the core and the Shapley value of this game. ◇



(a) Core and Shapley value for the glove game.

(b) Core and Shapley value for the visiting professor game.

Figure 5.6.1. Two Examples of the Shapley value.

Example 5.6.2 shows that the Shapley value for a game can lie outside its core, even when the latter is nonempty. In Shapley (1971), a class of games that satisfy that the Shapley value is always a core allocation is discussed. This is the class of *convex games*, which has received a lot of attention because of its important properties. We study this class in Section 5.12.

The Shapley value is arguably the most widely studied concept in cooperative game theory. Winter (2002) and Moretti and Patrone (2008) are two surveys on the Shapley value, with the latter being especially devoted to its applications. The book Algaba et al. (2019) is a a compilation of recent methodological contributions and applications around the Shapley value.

5.7. Computation of the Shapley Value

It is well known that the Shapley value is difficult to compute as the number of players increases, since the number of required operations grows exponentially fast in the number of players. Because of this, different approaches have been discussed in the literature to ease the computation of the Shapley value. In this section we discuss two of them, one exact and one approximate.

5.7.1. Computation through the multilinear extension. Owen (1972a) introduced a tool that helps to obtain the Shapley value. This tool is based on the *multilinear extension* of a TU-game, which, later on, Owen (1975a) used again to facilitate the computation of the Banzhaf value (see Section 5.8.1).

Definition 5.7.1. Let $v \in G^N$. The *multilinear extension of* v is the function $f_v^{\text{ml}} : [0,1]^n \longrightarrow \mathbb{R}$ such that

$$f_v^{\text{ml}}(x_1, \dots, x_n) := \sum_{T \subset N} \prod_{i \in T} x_i \prod_{j \in N \setminus T} (1 - x_j) v(T). \tag{5.7.1}$$

Note that f_v^{ml} is a linear function in each coordinate, i.e., a multilinear function. Given $v \in G^N$, a coalition $S \subset N$ can be identified with the vector $e^S \in \mathbb{R}^n$ with $e_i^S = 1$ if $i \in S$ and $e_i^S = 0$, otherwise. Then, if we compute $f_v^{\text{ml}}(e^S)$ using Eq. (5.7.1), we have that $\prod_{i \in T} x_i \prod_{j \in N \setminus T}(1 - x_j) = 0$ if $T \neq S$ and $\prod_{i \in T} x_i \prod_{j \in N \setminus T}(1 - x_j) = 1$ if $T = S$. Thus, $f_v^{\text{ml}}(e^S) = v(S)$. In this sense f_v^{ml} is an extension of v.

The multilinear extension admits the following interpretation. Suppose that a coalition has to be formed and that each $x_i \in [0,1]$ represents the probability that player i joins it. Then, $f_v^{\text{ml}}(x_1, \dots, x_n)$ is the expectation across the values of the different coalitions. In particular, given e^S, coalition S will form with probability one, and so $f_v^{\text{ml}}(e^S) = v(S)$.

Proposition 5.7.1. *Let $v \in G^N$. There is a unique multilinear function g defined on $[0,1]^n$ such that, for each $S \subset N$, $g(e^S) = v(S)$.*

Proof. The existence of such multilinear function is given by f_v^{ml}. Next, we prove uniqueness. Let g be a multilinear function such that, for each $S \subset N$, $g(e^S) = v(S)$. Then, it is easy to see that g can be expressed as

$$g(x_1, \ldots, x_n) = \sum_{S \subset N} c_S \prod_{i \in S} x_i.$$

Thus, we have to obtain the family of coefficients $\{c_S : S \subset N\}$. Note that, for each $S \subset N$,

$$g(e^S) = \sum_{T \subset S} c_T,$$

and from the condition $g(e^S) = v(S)$, we have $\sum_{T \subset S} c_T = v(S)$. Then, in order to obtain the coefficients $\{c_T : T \subset N\}$, we need to solve the system of linear equations

$$\text{(5.7.2)} \qquad \sum_{T \subset S} c_T = v(S), \quad \text{for each } S \subset N.$$

Expression 5.7.1 provides a solution of this linear system. Since this system has as many equations as unknowns, a sufficient condition for uniqueness of the solution is that the coefficient matrix is nonsingular. This can be shown by studying the system in the case in which, for each $S \subset N$, $v(S) = 0$, and showing that its unique solution is given, for each $S \subset N$, $c_S = 0$. We proceed by contradiction. Assume that the system given in Eq. (5.7.2) has a nonzero solution. Take $S \subset N$ a minimal coalition with $c_S \neq 0$ and $c_T = 0$, for each $T \subsetneq S$. Then,

$$0 = v(S) = \sum_{T \subset S} c_T = c_S \neq 0,$$

which is a contradiction. Then, the system in Eq. (5.7.2) has a unique solution. □

As noted in the proof of Proposition 5.7.1, f_v^{ml} can be written as

$$f_v^{\mathrm{ml}}(x_1, \ldots, x_n) = \sum_{S \subset N} c_S \prod_{i \in S} x_i.$$

Moreover, for each $S \subset N$, we have $c_S = \sum_{T \subset S} (-1)^{|S|-|T|} v(T)$, which is known as the *Harsanyi dividend* for coalition S (see Exercise 5.8).

Example 5.7.1. Take the TU-game in Example 5.4.1, where $N = \{1,2,3\}$, $v(1) = v(2) = v(3) = 0$, and $v(12) = v(13) = v(23) = v(N) = 1$. Then, its multilinear extension is given by $f_v^{\mathrm{ml}}(x_1, x_2, x_3) = x_1x_2(1-x_3) + x_1x_3(1-x_2) + x_2x_3(1-x_1) + x_1x_2x_3$, which reduces to

$$\text{(5.7.3)} \qquad f_v^{\mathrm{ml}}(x_1, x_2, x_3) = x_1x_2 + x_1x_3 + x_2x_3 - 2x_1x_2x_3,$$

a multilinear function on variables x_1, x_2, and x_3. ◇

Next proposition provides a method to obtain the Shapley value using the multilinear extension.

Proposition 5.7.2. *Let $v \in G^N$ and $i \in N$. Then, for each $i \in N$,*

$$\Phi_i^S(v) = \int_0^1 \frac{\partial f_v^{ml}}{\partial x_i}(t, \ldots, t)dt.$$

Proof. For the sake of notation, throughout this proof we let $s = |S|$. Let $v \in G^N$ and $i \in N$. For each $(x_1, \ldots, x_n) \in [0,1]^n$,

$$\begin{aligned}\frac{\partial f_v^{\mathrm{ml}}(x_1, \ldots, x_n)}{\partial x_i} &= \sum_{S \subset N,\, i \in S} \prod_{j \in S \setminus \{i\}} x_j \prod_{j \in N \setminus S} (1 - x_j) v(S) \\ &\quad - \sum_{S \subset N,\, i \notin S} \prod_{j \in S} x_j \prod_{j \in N \setminus (S \cup \{i\})} (1 - x_j) v(S) \\ &= \sum_{S \subset N,\, i \notin S} \prod_{j \in S} x_j \prod_{j \in N \setminus (S \cup \{i\})} (1 - x_j)(v(S \cup \{i\}) - v(S)).\end{aligned}$$

Then, for each $t \in [0,1]$,

$$\frac{\partial f_v^{\mathrm{ml}}}{\partial x_i}(t, \ldots, t) = \sum_{S \subset N,\, i \notin S} t^s (1-t)^{n-s-1} (v(S \cup \{i\}) - v(S)). \tag{5.7.4}$$

Thus,

$$\int_0^1 \frac{\partial f_v^{\mathrm{ml}}}{\partial x_i}(t, \ldots, t)dt = \sum_{S \subset N,\, i \notin S} (v(S \cup \{i\}) - v(S)) \int_0^1 t^s (1-t)^{n-s-1} dt.$$

Since

$$\int_0^1 t^s (1-t)^{n-s-1} dt = \frac{\Gamma(s+1)\Gamma(n-s)}{\Gamma(s+1+n-s)} = \frac{s!(n-s-1)!}{n!},$$

we obtain

$$\int_0^1 \frac{\partial f_v^{\mathrm{ml}}}{\partial x_i}(t, \ldots, t)dt = \sum_{S \subset N,\, i \notin S} \frac{s!(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S)) = \Phi_i^S(v).$$

□

We apply Proposition 5.7.2 to Example 5.4.1 and Example 5.4.4 in order to compute the Shapley value.

Example 5.7.2. In Eq. (5.7.3) we have the expression for the multilinear extension of the divide a million game. Take player 1. We have, for each $(x_1, x_2, x_3) \in [0,1]^3$,

$$\frac{\partial f_v^{\mathrm{ml}}(x_1, x_2, x_3)}{\partial x_1} = x_2 + x_3 - 2x_2x_3.$$

Then, for each $t \in [0,1]$, $\frac{\partial f_v^{\text{ml}}}{\partial x_1}(t,t,t) = 2t - 2t^2$ and

$$\int_0^1 \frac{\partial f_v^{\text{ml}}}{\partial x_1}(t,t,t)dt = \int_0^1 (2t - 2t^2)dt = (t^2 - \frac{2t^3}{3})\Big|_0^1 = \frac{1}{3}.$$

Similarly, one can compute the Shapley value for players 2 and 3. ◇

Example 5.7.3. The cost game associated to Example 5.4.4 is $c(1) = 1500$, $c(2) = 1600$, $c(3) = 1900$, $c(12) = 1600$, $c(13) = 2900$, $c(23) = 3000$, $c(N) = 3000$. Its multilinear extension is given by

$$\begin{aligned} f_c^{\text{ml}}(x_1,x_2,x_3) &= 1500x_1(1-x_2)(1-x_3) + 1600x_2(1-x_1)(1-x_3) \\ &\quad +1900x_3(1-x_1)(1-x_2) + 1600x_1x_2(1-x_3) \\ &\quad +2900x_1x_3(1-x_2) + 3000x_2x_3(1-x_1) + 3000x_1x_2x_3 \\ &= 1500x_1 + 1600x_2 + 1900x_3 - 1500x_1x_2 - 500x_1x_3 \\ &\quad -500x_2x_3 + 500x_1x_2x_3. \end{aligned}$$

For player 2, we have

$$\frac{\partial f_c^{\text{ml}}(x_1,x_2,x_3)}{\partial x_2} = 1600 - 1500x_1 - 500x_3 + 500x_1x_3, \text{ and}$$

$$\frac{\partial f_c^{\text{ml}}}{\partial x_2}(t,t,t) = 1600 - 2000t + 500t^2.$$

Then, $\int_0^1 \frac{\partial f_c^{\text{ml}}}{\partial x_2}(t,t,t)dt$ reduces to

$$\int_0^1 (1600 - 2000t + 500t^2)dt = (1600t - 1000t^2 + \frac{500t^3}{3})\Big|_0^1 = \frac{2300}{3}.$$

Similarly, one can compute the Shapley value for players 1 and 3. ◇

Despite the usefulness of the multilinear extension, both theoretically and practically, the calculation of the Shapley value is still a computationally hard task as the number of player grows. Thus, approximation schemes like the one discussed below are essential in many applications.

5.7.2. Estimating the Shapley value. Mann and Shapley (1960) proposed and analyzed several sampling methods to estimate the Shapley value of a simple game. Castro et al. (2009) developed and studied a sampling strategy to estimate the Shapley value of a general TU-game. In fact, they used the method named as *Type 0* in Mann and Shapley (1960). The proposal is based on Expression 5.6.2 for the Shapley value, which indicates that the Shapley value is obtained as the average of the marginal contributions. The procedure is described next.

Input: $v \in G^N$, maximum error, ε, and confidence level, $1-\alpha$.
Output: Sample size, K, and an estimation of the Shapley value, $\hat{\Phi}^S(v)$.

1. Compute the sample size K associated with ε and $1-\alpha$.
2. Set the population as the set of permutations of the set of players, $\Pi(N)$.
3. For each $i \in N$, initialize $\hat{\Phi}^S_i$ to 0.
4. For each $l \in \{1, \ldots, K\}$,
   - Randomly choose a permutation $\pi_l \in \Pi(N)$ (each has probability $1/n!$).
   - For each $i \in N$, set $\hat{\Phi}^S_i(v) = \hat{\Phi}^S_i(v) + m_i^{\pi_l}(v)$.
5. The estimation of the Shapley value of v is $\hat{\Phi}^S(v)/K$.

This procedure delivers an unbiased estimator[13], i.e., for each $i \in N$, the expectation of $\hat{\Phi}^S_i(v)$ is $\Phi^S_i(v)$. Its variance is given by

$$\text{var}(\hat{\Phi}^S_i(v)) = \frac{\sigma_i^2}{K}, \quad \text{with} \quad \sigma_i^2 = \frac{1}{n!} \sum_{\pi \in \Pi(N)} (m_i^\pi(v) - \Phi^S_i(v))^2.$$

Proposition 5.7.3. *Let $v \in G^N$ and $\hat{\Phi}^S(v)$ be the estimation given by the above procedure.*

i) *If $i \in N$ is a null player, then $\hat{\Phi}^S_i(v) = 0$.*

ii) *The allocation $\hat{\Phi}^S$ is efficient, i.e., $\sum_{i\in N} \hat{\Phi}^S_i(v) = v(N)$.*

Proof. Let $v \in G^N$ and $\hat{\Phi}^S(v)$ be the estimation of the Shapley value of v given by the above procedure.

i) Take $i \in N$ a null player. Then, for each $S \subset N\backslash\{i\}$, $v(S \cup i) - v(S) = 0$. In particular, for each $\pi \in \Pi(N)$, $m_i^\pi(v) = 0$. Hence, $\hat{\Phi}^S_i(v) = 0$.

ii) For each $\pi \in \Pi(N)$, the vector m^π is efficient. Then, $\hat{\Phi}^S(v)$ is efficient too. □

The sample size for getting an estimation with the accuracy ε with probability $1-\alpha$ can be obtained using the central limit theorem. This assures that each $\hat{\Phi}^S_i$ approximately follows a normal distribution with mean Φ^S_i and variance σ_i^2/K, when K is large enough. Thus, if we want to guarantee that the error in the estimation is less than $\varepsilon > 0$ with a probability greater than $1-\alpha$ ($\alpha \in (0,1)$), i.e.,

$$P(|\hat{\Phi}^S_i(v) - \Phi^S_i(v)| < \varepsilon) \geq 1 - \alpha,$$

[13]See Cochran (1977) for details.

we take K so that, for each $i \in N$,

$$K \geq z^2_{1-\alpha/2}\frac{\sigma_i^2}{\varepsilon^2}, \tag{5.7.5}$$

with $z_{1-\alpha/2} \in \mathbb{R}$ such that $P(Z \geq z_{1-\alpha/2}) = \alpha/2$, where Z is a random variable following a normal distribution with expectation 0 and variance 1.

Alternatively, for each $i \in N$, we can use *Chebyshev's inequality*. In such case, we have

$$P(|\hat{\Phi}^S_i(v) - \Phi^S_i(v)| \geq k_i\frac{\sigma_i}{\sqrt{K}}) \leq \frac{1}{k_i^2}.$$

Taking k_i such that $\varepsilon = k_i\frac{\sigma_i}{\sqrt{K}}$, we have

$$P(|\hat{\Phi}^S_i(v) - \Phi^S_i(v)| \geq \varepsilon) \leq \frac{\sigma_i^2}{\varepsilon^2 K}.$$

Then, taking $\alpha \in (0,1)$ and making $\alpha \geq \sigma^2/(\varepsilon^2 K)$, we have

$$K \geq \frac{\sigma_i^2}{\alpha\varepsilon^2}. \tag{5.7.6}$$

Note that, in both cases, we need to know the variance σ_i^2 in order to obtain the sample size, but σ_i^2 is also hard to compute. We can bound its value if we know the maximum and the minimum value of each coordinate of the vector of marginal contributions. Take player i and $a^i_{\max} = \max\{m^\pi_i(v) : \pi \in \Pi(N)\}$ and $a^i_{\min} = \min\{m^\pi_i(v) : \pi \in \Pi(N)\}$. These values are also the maximum and the minimum that random variable $\hat{\Phi}^S_i(v)$ can get. In such a case, the maximum variance is achieved when this variable takes these two values with equal probability. Then,

$$\begin{aligned}\sigma_i^2 &\leq \tfrac{1}{2}[(a^i_{\max} - \tfrac{a^i_{\max}+a^i_{\min}}{2})^2 + (a^i_{\min} - \tfrac{a^i_{\max}+a^i_{\min}}{2})^2] \\ &= \tfrac{(a^i_{\max}-a^i_{\min})}{4}.\end{aligned}$$

This bound, known as *Popoviciu's inequality* on variances, can be loose for some classes of games.

Finally, suppose that, for each $i \in N$ and each $l \in \{1, \ldots, K\}$, $m^{\pi_l}_i(v) \in [a_l, b_l]$. Then, given the independence of the sampling procedure, we can apply the *Hoeffding's inequality*[14] to relate the sample size, K, the confidence level, $1-\alpha$, and the maximum error, ε, as follows

$$P(|\frac{1}{K}\sum_{l=1}^{K} m^{\pi_l}_i(v) - \Phi^S_i(v)| \geq \varepsilon) \leq 2\exp(-\frac{2K^2\varepsilon^2}{\sum_{l=1}^K (b_l - a_l)^2}).$$

[14]Hoeffding's inequality (Hoeffding 1963): Let $Y_1, \ldots, Y_K$ be independent random variables such that, for each $l = 1, \ldots, K$, $Y_l \in [a_l, b_l]$. Then, $P(|\sum_{l=1}^K Y_l - \mathbb{E}[\sum_{l=1}^K Y_l]| \geq K\varepsilon) \leq 2exp(-\frac{2K^2\varepsilon^2}{\sum_{l=1}^K (b_l-a_l)^2})$.

Solving the inequality

$$2exp(-\frac{2K^2\varepsilon^2}{\sum_{l=1}^{K}(b_l-a_l)^2}) \leq \alpha,$$

we obtain

$$K \geq \sqrt{\frac{ln(2/\alpha)\sum_{l=1}^{K}(b_l-a_l)^2}{2\varepsilon^2}}. \tag{5.7.7}$$

Maleki (2015) studied the complexity of the algorithm and also studied the stratified sampling method in the class of simple games. Other sampling approaches have recently appeared in the literature. Liben-Nowell et al. (2012) proposed an algorithm to compute the Shapley value for the class of convex games. Castro et al. (2017) used stratified random sampling defining, for each player, the strata on the set of permutations according to its position. Benati et al. (2019) proposed to obtain a sample of the characteristic function and then estimate different allocating rules, in particular the Shapley value. The recently rising use of the Shapley value in the context of machine learning is also contributing to the literature on its estimation (see the survey paper Rozemberczki et al. (2022)). For instance, Ancona et al. (2019) proposed a method based on uncertainty propagation to approximate the Shapley value in deep neural networks, Okhrati and Lipani (2020) studied a sampling method based on the multilinear extension of the Shapley value (Owen 1972a), and Mitchell et al. (2021) proposed kernel-based sampling methods.

5.8. The Banzhaf Value

In this section we present another allocation rule for TU-games: the *Banzhaf value*. This allocation rule was initially proposed in the context of simple games, Definition 5.5.4, by Banzhaf (1965). Later on, Owen (1975a) extended its definition to the family of TU-games where its main drawback is that, in general, it is not efficient. We refer the reader to Section 6.4 for a discussion of the Banzhaf value in its original context.

Definition 5.8.1. The *Banzhaf value*, Φ^{B}, is defined, for each $v \in G^N$ and each $i \in N$, by

$$\Phi_i^{\mathrm{B}}(v) := \frac{1}{2^{n-1}} \sum_{S\subset N\setminus\{i\}} (v(S\cup\{i\}) - v(S)). \tag{5.8.1}$$

For each $i \in N$, let $m_i^S(v) := v(S\cup\{i\}) - v(S)$. Then, for each $S \subset N \setminus \{i\}$, let $\bar{m}_i(v)$ be the sum of the contributions of player i to every coalition he can join, i.e., $\bar{m}_i(v) := \sum_{S \subset N \setminus \{i\}} (v(S \cup \{i\}) - v(S))$. Finally, let $\bar{m}(v)$ be the total amount of the contributions of all players, i.e.,

$$\bar{m}(v) := \sum_{i \in N} \bar{m}_i(v).$$

By definition, the Banzhaf value assigns to each player the average of the contributions he makes to any coalition, giving equal probability to each one of them, i.e.,

$$\Phi_i^{\mathrm{B}}(v) := \frac{1}{2^{n-1}} \sum_{S \subset N \setminus \{i\}} m_i^S(v) = \frac{1}{2^{n-1}} \bar{m}_i(v).$$

The Banzhaf value, in general, does not satisfies EFF, but it satisfies NPP, SYM, ADD. Theorem 5.8.1 provides a characterization that illustrates the difference with the Shapley value.

Theorem 5.8.1. *The Banzhaf value is the unique allocation rule φ in G^N that satisfies* NPP, SYM, ADD, *and that, for each $v \in G^N$,*

(5.8.2)
$$\sum_{i \in N} \varphi_i(v) = \frac{1}{2^{n-1}} \bar{m}(v).$$

Proof. It follows similar lines to the proof in Theorem 5.6.1 and it is left to the reader as Exercise 5.9. □

The proof of Proposition 5.6.2 with slight changes can be used to show that none of the axioms in Theorem 5.8.1 is superfluous. Exercise 5.10 asks the reader to formally show this.

Below we compute the Banzhaf value for the games discussed in Examples 5.4.1, 5.4.2, and 5.4.4. We omit the detailed computations.

Example 5.8.1. The Banzhaf value for the divide a million game is the allocation $(1/2, 1/2, 1/2)$. Note that, in this case all players are symmetric and the Banzhaf value is not efficient. ◇

Example 5.8.2. The Banzhaf value for the glove game is $(1/2, 1/4, 1/4)$. In this case the allocation is efficient, but it does not belong to the core. ◇

Example 5.8.3. The Banzhaf value in the visiting professor game discussed in Example 5.4.4 is $\Phi^{\mathrm{B}}(v) = (875, 875, 375)$. Note that the sum is greater than $v(N)$. According to this allocation of the savings, the players have to pay $(625, 725, 1525)$. This vector is precisely $\Phi^{\mathrm{B}}(c)$. This relationship holds in an analogous way to the case of the Shapley value. ◇

In the literature the characterization given in Theorem 5.8.1 receives some criticisms not only by the use of the additivity axiom but also due to the constraint given by the total amount that players receive. These comments encouraged the search for alternative characterizations of the Banzhaf value, such as the ones in Lehrer (1988), Haller (1994), and Nowak (1997).

5.8.1. Computation of the Banzhaf value. The result below shows that the multilinear extension can also be used to compute the Banzhaf value.

Proposition 5.8.2. *Let $v \in G^N$ and $i \in N$. Then, for each $i \in N$,*

$$\Phi_i^B(v) = \frac{\partial f_v^{ml}}{\partial x_i}(\frac{1}{2}, \ldots, \frac{1}{2}).$$

Proof. Let $v \in G^N$ and $i \in N$. Taking $t = 1/2$ in Eq. (5.7.4), we get

$$\begin{aligned} \frac{\partial f_v^{\mathrm{ml}}}{\partial x_i}(t, \ldots, t) &= \sum_{S \subset N,\, i \notin S} \frac{1}{2^s}(1 - \frac{1}{2})^{n-s-1}(v(S \cup \{i\}) - v(S)) \\ &= \frac{1}{2^{n-1}} \sum_{S \subset N,\, i \notin S} (v(S \cup \{i\}) - v(S)) = \Phi_i^{\mathrm{B}}(v), \end{aligned}$$

where the last equality follows from Eq. (5.8.1). □

The computation of the Banzhaf value in Examples 5.4.1, 5.4.2, and 5.4.4 using the multilinear extension are left to the reader (Exercise 5.11).

Like the Shapley value, also the Banzhaf value is computationally hard. Bachrach et al. (2010) proposed a sampling method to estimate the Banzhaf value of a simple game that can be used for a general TU-game. The sampling procedure is described next.

Input: $v \in G^N$, maximum error, ε, confidence level, $1 - \alpha$.
Output: Sample size, K, and an estimation of the Banzhaf value, $\hat{\Phi}^{\mathrm{B}}(v)$.

1. Compute the sample size K associated with ε and $1 - \alpha$.
2. Set the population as the set of coalitions $S \subset N$.
3. For each $i \in N$, initialize $\hat{\Phi}^{\mathrm{B}}{}_i$ to 0.
4. For each $i \in N$ repeat the following procedure. For each $l \in \{1, \ldots, K\}$:
   - Choose a coalition $S_l \subset N$ such that $i \in S_l$ (each has probability $1/2^{n-1}$).
   - Set $\hat{\Phi}_i^{\mathrm{B}}(v) = \hat{\Phi}_i^{\mathrm{B}}(v) + v(S_l) - v(S_l \backslash \{i\})$.
5. The estimation of the Banzhaf value of v is $\hat{\Phi}^{\mathrm{B}}(v)/K$.

This procedure delivers an unbiased estimator, i.e., for each $i \in N$, the expectation of $\hat{\Phi}_i^{\mathrm{B}}(v)$ is $\Phi_i^{\mathrm{B}}(v)$. Its variance is given by

$$\mathrm{var}(\hat{\Phi}_i^{\mathrm{B}}(v)) = \frac{\sigma_i^2}{K}, \quad \text{with} \quad \sigma_i^2 = \frac{1}{2^{n-1}} \sum_{S \subset N \setminus \{i\}} (v(S) - v(S \setminus \{i\}) - \Phi_i^{\mathrm{B}}(v))^2.$$

Proposition 5.8.3. *Let $v \in G^N$ and $\hat{\Phi}^B(v)$ be the estimation given by the above procedure. If $i \in N$ is a null player, then $\hat{\Phi}_i^B(v) = 0$.*

Proof. Analogous to the proof of statement i) in Proposition 5.7.3. □

The sample size K can be obtained along similar lines to the case of the Shapley value, through Eq. (5.7.5), Eq. (5.7.6), and Eq. (5.7.7), with the appropriate changes.

The sampling procedure studied in Benati et al. (2019) can also be used to estimate the Banzhaf value. Saavedra-Nieves (2021) provides some hints to use *R* software for estimating the Banzhaf value without replacement. The Banzhaf value is employed also in machine learning (see Rozemberczki et al. (2022)). For instance, Karczmarz et al. (2022) proposed an algorithm to compute the Banzhaf value of a tree ensemble model and Wang and Jia (2022) propose a method to estimate the Banzhaf value in the context of data valuation.

5.9. Equal Division Values

In this section we deal with two values for TU-games inspired by egalitarian considerations: the *e*qual division value and the *e*qual surplus division value. The first simply divides the benefits of cooperation equally among all players. The second allocates first to each player the benefits he generates by himself and then divides the remaining benefits equally among all players. Below we present the formal definitions of these values.

Definition 5.9.1. The *equal division value*, Φ^{E}, is defined, for each $v \in G^N$ and each $i \in N$, by

$$\Phi_i^{\mathrm{E}}(v) := \frac{v(N)}{n}. \tag{5.9.1}$$

Definition 5.9.2. The *equal surplus division value*, Φ^{ES}, is defined, for each $v \in G^N$ and each $i \in N$, by

$$\Phi_i^{\mathrm{ES}}(v) := v(i) + \frac{v^0(N)}{n}, \tag{5.9.2}$$

where v^0 denotes the zero-normalization of v, i.e., for each $S \subset N$, $v^0(S) := v(S) - \sum_{j \in S} v(j)$.

Despite being quite simple and unsophisticated, these two values are widely used in practice because, on many occasions, agents who cooperate do so on the basis of a principle of equality, which brings clarity and simplicity and helps to avoid conflicts. Both values are based on the same principle and their fundamental difference is that Φ^{E} does not satisfy the covariance to translations while Φ^{ES} does. Formally, for each $v \in G^N$ and each $x \in \mathbb{R}^N$, let $v + x$ be the game defined, for each $S \subset N$ by

$$(v + x)(S) := v(S) + \sum_{i \in S} x_i.$$

Then, we have that $\Phi^{\mathrm{ES}}(v + x) = \Phi^{\mathrm{ES}}(v) + x$. Yet, this same equality does not hold for Φ^{E}. An example is given below to illustrate these two solutions in practice and compare them with the Shapley value.

Example 5.9.1. Three companies with different fields of expertise agree to jointly carry out a project that generates a profit of 900 thousand euro. Each company could also carry out the project alone; in such a case, they would have to subcontract some services, with costs of 720, 810, and 870 thousand for companies 1, 2, and 3, respectively. Also any pair of companies could carry out the project; in such a case, the cost of subcontracting would be 500 thousand euro for all pairs. This situation gives rise to the following TU-game (N, v): $N = \{1, 2, 3\}$, $v(1) = 180$, $v(2) = 90$, $v(3) = 30$, $v(12) = v(13) = v(23) = 400$, $v(N) = 900$. After some easy algebra,

- $\Phi^{\mathrm{E}}(v) = (300, 300, 300)$,
- $\Phi^{\mathrm{ES}}(v) = (180, 90, 30) + (200, 200, 200) = (380, 290, 230)$,
- $\Phi^{\mathrm{S}}(v) = (340, 295, 265)$.

In this example, Φ^{S} is somewhere in between Φ^{E} and Φ^{ES}. ◇

To conclude this section we present an axiomatic characterization of the equal division value due to van den Brink (2007) and another one of the equal surplus division value due to Casajus and Huettner (2014). Both are based on the characterization of the Shapley value, Theorem 5.6.1, but changing property NPP. First, we need some preliminary definitions.

Definition 5.9.3. Let $v \in G^N$.

i) A player $i \in N$ is a *nullifying player* if, for each $S \subset N$ with $i \in S$, $v(S) = 0$.

ii) A player $i \in N$ is a *dummifying player* if, for each $S \subset N$ with $i \in S$, $v(S) = \sum_{j \in S} v(j)$.

We now build upon these two notions to define two new properties for allocation rules for n-player TU-games.

Nullifying Player (NULG): The allocation rule φ satisfies NULG if, for each $v \in G^N$ and each nullifying player $i \in N$, $\varphi_i(v) = 0$.

Dummifying Player (DUMG): The allocation rule φ satisfies DUMG if, for each $v \in G^N$ and each dummifying player $i \in N$, $\varphi_i(v) = v(i)$.

Before presenting the characterizations we introduce a special class of TU-games that plays an important role in the proofs.

Definition 5.9.4. Inside the class G^N, given $S \subset N$, the *canonical game of coalition* S, e_c^S, is defined as follows. For each $T \subset N$, $e_c^S(T) := 1$ if $T = S$ and $e_c^S(T) := 0$ otherwise.

Theorem 5.9.1. *The equal division value is the unique allocation rule in* G^N *that satisfies* EFF, NULG, SYM, *and* ADD.

Proof. It is an easy exercise to check that Φ^{E} satisfies EFF, NULG, SYM, and ADD. Now, let φ be an allocation rule satisfying these properties and take $v \in G^N$. Clearly,

$$v = \sum_{S \in 2^N \setminus \{\emptyset\}} v(S) e_c^S = v(N) e_c^N + \sum_{S \in 2^N \setminus \{\emptyset\}, S \neq N} v(S) e_c^S. \tag{5.9.3}$$

Take $S \subsetneq N$. Then, all $i \in N \setminus S$ are nullifying players in $v(S)e_c^S$ and all $i \in S$ are symmetric players in $v(S)e_c^S$. Thus, since φ satisfies EFF, NULG, and SYM, we have that, for each $i \in N$, $\varphi_i(v(S)e_c^S) = 0$. Now, all $i \in N$ are symmetric players in $v(N)e_c^N$. Thus, since φ satisfies EFF and SYM, we have that, for each $i \in N$, $\varphi_i(v(N)e_c^N) = v(N)/n$. Therefore, since φ satisfies ADD, Eq. (5.9.3) implies that, for each $i \in N$,

$$\varphi_i(v) = \varphi_i(v(N)e_c^N) = \frac{v(N)}{n} = \Phi_i^{\mathrm{E}}(v),$$

which concludes the proof. □

Theorem 5.9.2. *The equal surplus division value is the unique allocation rule in* G^N *that satisfies* EFF, DUMG, SYM, *and* ADD.

Proof. It is an easy exercise to check that Φ^{ES} satisfies EFF, DUMG, SYM, and ADD. Now, let φ be an allocation rule satisfying these properties and take $v \in G^N$. Denote by v^a the game in G^N given, for each $S \subset N$, by $v^a(S) = \sum_{i \in S} v(i)$. Clearly,

$$\begin{aligned} v = v^a + v^0 &= v^a + \sum_{S \in 2^N \setminus \{\emptyset\}} v^0(S) e_c^S \\ &= v^a + v^0(N) e_c^N + \sum_{S \in 2^N \setminus \{\emptyset\}, S \neq N} v^0(S) e_c^S. \end{aligned} \tag{5.9.4}$$

Take $S \subsetneq N$. Then, all $i \in N \setminus S$ are dummifying players in $v^0(S)e_c^S$ for which, moreover, $v^0(i) = 0$. Also, all $i \in S$ are symmetric players in $v^0(S)e_c^S$. Thus, since φ satisfies EFF, DUMG, and SYM, we have that, for each $i \in N$, $\varphi_i(v^0(S)e_c^S) = 0$. Now, all $i \in N$ are symmetric players in $v^0(N)e_c^N$. Thus, since φ satisfies EFF and SYM, we have that, for each $i \in N$, $\varphi_i(v^0(N)e_c^N) = v^0(N)/n$. Finally, all $i \in N$ are dummifying players in v^a and, since φ satisfies DUMG, for each $i \in N$ we have that $\varphi_i(v^a) = v(i)$. Therefore, since φ satisfies ADD, Eq. (5.9.4) implies that, for each $i \in N$,

$$\varphi_i(v) = v(i) + \varphi_i(v^0(N)e_c^N) = v(i) + \frac{v^0(N)}{n} = \Phi_i^{\text{ES}}(v),$$

which concludes the proof. □

Recently, Alonso-Meijide et al. (2020) extended these two values for games with a priori unions and applied them to an elevator maintenance cost sharing problem in a condominium. Also, Hu and Li (2021) extended the equal surplus division value for games with levels.

5.10. The Nucleolus

In this section we present the nucleolus (Schmeidler 1969), perhaps the second most important allocation rule for TU-games, just behind the Shapley value. We present its definition, an existence result, and a procedure to compute it. Despite following a natural fairness principle, the definition of the nucleolus is already somewhat involved. Hence, the analysis of its properties is harder than the analysis we developed for the Shapley value and we do not cover it in detail here. For an axiomatic characterization of the nucleolus we refer the reader to Snijders (1995).

Let $v \in G^N$ and let $x \in \mathbb{R}^N$ be an allocation. Given a coalition $S \subset N$, the *excess of coalition S with respect to x* is defined by

$$e(S,x) := v(S) - \sum_{i \in S} x_i.$$

This is a measure of the degree of dissatisfaction of coalition S when the allocation x is realized. Note that, for each $x \in I(v)$, $e(N,x) = 0$. Moreover, if $x \in C(v)$, then, for each $S \subset N$, $e(S,x) \le 0$. Now, we define the *vector of ordered excesses* $\theta(x) \in \mathbb{R}^{2^N}$ as the vector whose components are the excesses of the coalitions in 2^N arranged in nonincreasing order.

Given $x, y \in \mathbb{R}^N$, y is larger than x according to the lexicographic order, denoted $\theta(y) \succ_L \theta(x)$, if there is $l \in \mathbb{N}$, $1 \le l \le 2^n$, such that, for each $k \in \mathbb{N}$ with $k < l$, $\theta_k(y) = \theta_k(x)$ and $\theta_l(y) > \theta_l(x)$. We write $\theta(y) \succeq_L \theta(x)$ if either $\theta(y) \succ_L \theta(x)$ or $\theta(y) = \theta(x)$. We use the game in Example 5.5.1 to illustrate the above concepts.

Example 5.10.1. Let $v \in G^N$, where $N = \{1,2,3\}$ and $v(1) = v(2) = 0$, $v(3) = 1$, $v(12) = 2$, $v(13) = v(23) = 1$, and $v(N) = 2$. Let $x = (1,0,1)$ and $y = (3/2,1/2,0)$. The excess of each coalition with respect to the allocations x and y is depicted in Figure 5.10.1. Then, the corresponding vectors of ordered excesses are $\theta(x) = (1,0,0,0,0,0,-1,-1)$ and $\theta(y) = (1,1/2,0,0,0,-1/2,-1/2,-3/2)$ and, hence, $\theta(y) \succ_L \theta(x)$; that is, the most dissatisfied coalitions with respect to x and y are equally dissatisfied, but the second most dissatisfied coalition is more dissatisfied in y than it is in x. The idea of the nucleolus is to lexicographically minimize the degree of dissatisfaction of the different coalitions. With this fairness idea in mind, x would be more desirable than y. ◇

Coalitions	∅	{1}	{2}	{3}	{1,2}	{1,3}	{2,3}	{1,2,3}
$v(S)$	0	0	0	1	2	1	1	2
$e(S,x)$	0	-1	0	0	1	-1	0	0
$e(S,y)$	0	-3/2	-1/2	1	0	-1/2	1/2	0

Figure 5.10.1. Excesses of the coalitions with respect to x and y.

The next result concerning the function θ will prove to be very useful to analyze the nucleolus.

Lemma 5.10.1. *Let $v \in G^N$. Let $x, y \in \mathbb{R}^N$ be such that $x \neq y$ and $\theta(x) = \theta(y)$. Let $\alpha \in (0,1)$. Then, $\theta(x) \succ_L \theta(\alpha x + (1-\alpha)y)$.*

Proof. Note that, for each $S \subset N$, $e(S, \alpha x + (1-\alpha)y) = \alpha(v(S) - \sum_{i \in S} x_i) + (1-\alpha)(v(S) - \sum_{i \in S} y_i) = \alpha e(S,x) + (1-\alpha)e(S,y)$. Assume that

$$\theta(x) = (e(S_1,x), e(S_2,x), \ldots, e(S_{2^n},x)),$$

where $S_1, \ldots, S_{2^n}$ is a permutation of all the coalitions defined as follows. For each $k \in \{1, \ldots, 2^n - 1\}$,

- $e(S_k,x) \geq e(S_{k+1},x)$, and
- if $e(S_k,x) = e(S_{k+1},x)$, then $e(S_k,y) \geq e(S_{k+1},y)$.

Since $x \neq y$, there is $S \subset N$ such that $e(S,x) \neq e(S,y)$. Let k be the smallest index such that $e(S_k,x) \neq e(S_k,y)$. Then, for each $l \in \{1,\ldots,k-1\}$, $e(S_l, \alpha x + (1-\alpha)y) = e(S_l,x)$. Since $\theta(x) = \theta(y)$, it has to be the case that $e(S_k,y) < e(S_k,x)$. Moreover, the choice of the ordering $S_1, \ldots, S_{2^n}$ ensures that, for each $l > k$ such that $e(S_l,x) = e(S_k,x)$, we have $e(S_l,y) \leq e(S_k,y) < e(S_k,x)$. Hence, for each $l > k$, either i) $e(S_l,x) = e(S_k,x)$ and $e(S_l,y) < e(S_k,x)$ or ii) $e(S_l,x) < e(S_k,x)$ and $e(S_l,y) \leq e(S_k,x)$. Then, for each $l \geq k$,

$$e(S_l, \alpha x + (1-\alpha)y) = \alpha e(S_l,x) + (1-\alpha)e(S_l,y) < e(S_k,x).$$

Hence, for each $l \in \{1,\ldots,k-1\}$, $\theta_l(x) = \theta_l(\alpha x + (1-\alpha)y)$ and, for each $l \geq k$, $\theta_k(x) > \theta_l(\alpha x + (1-\alpha)y)$. Therefore, $\theta(x) \succ_L \theta(\alpha x + (1-\alpha)y)$. □

Definition 5.10.1. Let $v \in G^N$ be such that $I(v) \neq \emptyset$. The *nucleolus* of v, that we denote by $\eta(v)$, is the set

$$\eta(v) := \{x \in I(v) : \text{ for each } y \in I(v),\ \theta(y) \succeq_L \theta(x)\}.$$

The nucleolus consists of those imputations that minimize the vector of nonincreasingly ordered excesses according to the lexicographic order within the set of imputations, that is, the nucleolus is not defined for games with an empty set of imputations. At the beginning of this section we said that the nucleolus is an allocation rule and yet, we have defined it as a set. Next, we show that the nucleolus is actually a singleton, i.e., it is never empty and it contains a unique allocation. Therefore, with a slight abuse of language, we identify the unique element of the nucleolus with the nucleolus itself and refer to $\eta(v)$ as the nucleolus of v.

Theorem 5.10.2. *Let $v \in G^N$ be such that $I(v) \neq \emptyset$. Then, the set $\eta(v)$ contains a unique allocation.*

Proof. Let $I^0 := I(v)$. For each $k \in \{1,\ldots,2^n\}$, let I^k be the set

$$I^k := \{x \in I^{k-1} : \text{ for each } y \in I^{k-1},\ \theta_k(x) \leq \theta_k(y)\}.$$

For each $k \in \{1,\ldots,2^n\}$, the function that assigns, to each $x \in I(v)$, the k-th coordinate of the vector $\theta(x)$ is continuous. Hence, since $I(v)$ is a nonempty and compact set, I^1 is also nonempty and compact and so are all the I^k sets. We claim that $\eta(v) = I^{2^n}$. Let $x \in I^{2^n}$ and let $y \in I(v)$. If $y \in I^{2^n}$, then $\theta(x) = \theta(y)$ and, so, $\theta(y) \succeq_L \theta(x)$. Hence, suppose that $y \in I(v) \backslash I^{2^n}$. Let k be the smallest index such that $y \notin I^k$. Then, $\theta_k(y) > \theta_k(x)$. Since $x, y \in I^{k-1}$, for each $l \in \{1,\ldots,k-1\}$, $\theta_l(x) = \theta_l(y)$. Thus, $\theta(y) \succ_L \theta(x)$. Therefore, $\eta(v)$ is a nonempty set.

We now show that $\eta(v)$ is a singleton. Suppose it is not. Let $x, y \in \eta(v)$, with $x \neq y$. Then, $\theta(x) = \theta(y)$. Since $I(v)$ is a convex set, for each $\alpha \in (0,1)$, $\alpha x + (1-\alpha)y \in I(v)$. By Lemma 5.10.1, $\theta(x) \succ_L \theta(\alpha x + (1-\alpha)y)$, which contradicts that $x \in \eta(v)$. □

Remark 5.10.1. It is easy to see that if a TU-game has a nonempty core, then the nucleolus is a core element. Just note that the maximum excess at a core allocation can never be positive and that, in any allocation outside the core, there is a positive excess for at least one coalition. Hence, the vector of ordered excesses cannot be lexicographically minimized outside the core.

There are several procedures to compute the nucleolus. A review can be found in Maschler (1992). Under certain conditions of the characteristic function v, some more specific methods have been given, for instance, in Kuipers et al. (2000), Faigle et al. (2001), and Quant et al. (2005). Below we describe a procedure provided by Maschler et al. (1979) as an application of the Kopelowitz algorithm (Kopelowitz 1967). Let $v \in G^N$ be such that $I_0 := I(v) \neq \emptyset$. Consider the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_1 \\ \text{subject to} & \sum_{i\in S} x_i + \alpha_1 \geq v(S), \quad \emptyset \neq S \subsetneq N \\ & x \in I_0. \end{array}$$

This problem has at least one optimal solution. Let $\bar{\alpha}_1$ be the minimum of this linear programming problem. The set of optimal solutions is of the form $\{\bar{\alpha}_1\} \times I_1$. If this set is a singleton, then I_1 coincides with the nucleolus. Otherwise, let $\mathcal{F}_1$ be the collection of coalitions given by

$$\mathcal{F}_1 := \{S \subset N : \text{ for each } x \in I_1, \sum_{i\in S} x_i + \bar{\alpha}_1 = v(S)\}.$$

For $k > 1$, solve the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_k \\ \text{subject to} & \sum_{i\in S} x_i + \alpha_k \geq v(S), \quad \emptyset \neq S \subsetneq N, S \notin \cup_{l<k}\mathcal{F}_l \\ & x \in I_{k-1}, \end{array}$$

where, for each $l < k$, $\mathcal{F}_l$ is the collection of coalitions given by

$$\mathcal{F}_l := \{S \subset N : \text{ for each } x \in I_l, \sum_{i\in S} x_i + \bar{\alpha}_l = v(S)\}.$$

This algorithm finishes once the optimal solution set is a singleton. At the optimal solution at least one of the inequality restrictions has to be binding, i.e., for each $k \geq 1$ and each $x \in I_k$, there is $S \notin \cup_{l<k}\mathcal{F}_l$ such that $\sum_{i\in S} x_i + \bar{\alpha}_k = v(S)$. Hence, at each step, the set $\mathcal{F}_k$ contains at least one new coalition and so the algorithm finishes in at most $2^n - 1$ steps. Note that $\bar{\alpha}_1$ is the largest excess and $\mathcal{F}_1$ is the collection of coalitions with the largest excess, $\bar{\alpha}_2$ is the second largest excess and $\mathcal{F}_2$ is the collection of coalitions with the second largest excess, and so on.

Example 5.10.2. Take again the TU-game $v \in G^N$ given by $N = \{1,2,3\}$ and $v(1) = v(2) = 0, v(3) = 1, v(12) = 2, v(13) = v(23) = 1$, and $v(N) = 2$

(introduced in Example 5.5.1 and already discussed in this section in Example 5.10.1). First, we solve the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_1 \\ \text{subject to} & x_i + \alpha_1 \geq 0, \quad i \in \{1,2\} \\ & x_3 + \alpha_1 \geq 1 \\ & x_1 + x_2 + \alpha_1 \geq 2 \\ & x_1 + x_3 + \alpha_1 \geq 1 \\ & x_2 + x_3 + \alpha_1 \geq 1 \\ & x_i \geq 0, \quad i \in \{1,2\} \\ & x_3 \geq 1 \\ & x_1 + x_2 + x_3 = 2. \end{array}$$

The optimal solution set is given by $\bar{\alpha}_1 = 1$, $x_1 + x_2 = 1$, and $x_3 = 1$. The collection of coalitions $\mathcal{F}_1$ is given by $\mathcal{F}_1 = \{\{1,2\}\}$. Next, we solve the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_2 \\ \text{subject to} & x_i + \alpha_2 \geq 0, \quad i \in \{1,2\} \\ & x_3 + \alpha_2 \geq 1 \\ & x_1 + x_3 + \alpha_2 \geq 1 \\ & x_2 + x_3 + \alpha_2 \geq 1 \\ & x_i \geq 0, \quad i \in \{1,2\} \\ & x_1 + x_2 = 1 \\ & x_3 = 1. \end{array}$$

The optimal solution set is given by $\bar{\alpha}_2 = 0$, $x_1 + x_2 = 1$, and $x_3 = 1$. The collection of coalitions $\mathcal{F}_2$ is given by $\mathcal{F}_2 = \{\{3\}\}$. Finally, after some simplifications, we get to the linear programming problem

$$\begin{array}{ll} \text{Minimize} & \alpha_3 \\ \text{subject to} & x_1 + \alpha_3 \geq 0 \\ & x_2 + \alpha_3 \geq 0 \\ & x_i \geq 0, \quad i \in \{1,2\} \\ & x_1 + x_2 = 1 \\ & x_3 = 1. \end{array}$$

The optimal solution set is given by $\bar{\alpha}_3 = -1/2$, $x_1 = x_2 = 1/2$, and $x_3 = 1$. The collection of coalitions $\mathcal{F}_3$ is given by $\mathcal{F}_3 = \{\{1\},\{2\},\{1,3\},\{2,3\}\}$. Then, the nucleolus of this game is $\eta(v) = (1/2, 1/2, 1)$. ◇

5.11. The Core-Center

Maschler et al. (1979) showed that, for games with a nonempty core, the nucleolus can be characterized as a "lexicographic center" of the core. This naturally suggests to study the geometric center of the core, i.e., its centroid. This allocation rule, known as the *core-center*, is formally introduced in González-Díaz and Sánchez-Rodríguez (2007).

Recall that, by definition, any core allocation is stable in the sense that no coalition has incentives to block it. Further, we saw in Section 5.5 that, within the set of superadditive games (for which the set of imputations is nonempty), the core coincides with the set of undominated imputations (Proposition 5.5.1). Now, assume that, given the desirable properties above, we have narrowed our attention to core allocations. How can we select one such allocation in a fair way? The core-center regards all of them as equally valuable and selects the expectation of a uniform distribution defined over the core of the game, i.e., its center of gravity. Let $U(A)$ denote the uniform distribution defined over the set A.

Definition 5.11.1. Let v be a game with nonempty core. The *core-center* of v, $\mu(v)$, is defined by $\mu(v) := \mathbb{E}\big(U(C(v))\big)$.

Remark 5.11.1. Note that the nucleolus is only defined for games with a nonempty set of imputations and the core-center requires, moreover, that also the core is nonempty.

The nucleolus and the core-center are not the first allocations rules that have been motivated geometrically. The Shapley value is the center of gravity of the vectors of marginal contributions and, for convex games, it coincides with the center of gravity of the vertices of the core, taking their multiplicities into account. Also, González-Díaz et al. (2005) show that the τ-value (Tijs 1981) is the center of gravity of the edges of the core-cover (Tijs et al. 1982), again considering their multiplicities.

Example 5.11.1. Recall the saving game associated with the visiting professor in Example 5.4.4, which was given by $v(1) = v(2) = v(3) = 0$, $v(12) = 1500$, $v(13) = 500$, $v(23) = 500$, and $v(N) = 2000$. We have already seen that its Shapley value is given by $\Phi^S(v) = (5000/6, 5000/6, 2000/6)$. Exercise 5.15 asks the reader to compute the nucleolus and the core-center of this game, which are given, respectively, by $\eta(v) = (875, 875, 250)$ and $\mu(v) = (5200/6, 5200/6, 1600/6)$. The core of game v, along with $\Phi^S(v)$, $\eta(v)$, and $\mu(v)$ are depicted in Figure 5.11.1.

It is no coincidence that each of the three allocation rules assigns the same value to player 1 and player 2, since they are symmetric in game v. Thus, since the three allocation rules satisfy SYM (Exercise 5.14), they must propose the same allocation for these two players. By definition, these three allocation rules also satisfy EFF and, hence, they can just differ on the share of $V(N) = 2000$ that player 3 gets with respect to the equal shares of players 1 and 2. ◇

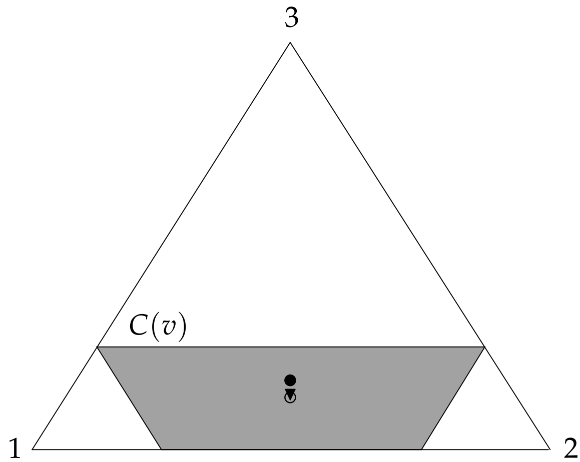


Figure 5.11.1. Core ($C(v)$), Shapley value (•), nucleolus (○), and core-center (▼) for the visiting professor saving game.

It is straightforward to check that, on top of SYM and EFF, the core-center satisfies, among others, the following properties: NPP, CAT, individual rationality, and coalitional rationality. They are inherited from the properties of the core and of the mathematical expectation.

The study of the core-center in González-Díaz and Sánchez-Rodríguez (2007) is mainly concerned with other less trivial properties. In particular, the main focus is on continuity, since it is not easy to prove that the core-center, seen as a function $\mu\colon \mathbb{R}^{2^n-1} \to \mathbb{R}^n$, is continuous.[15] Delving into the intricacies of continuity is beyond the scope of this book. Instead, we present two results in González-Díaz and Sánchez-Rodríguez (2007) associated to some classic monotonicity properties for allocation rules, which we introduce below.[16]

We define four different monotonicity properties. Let φ be an allocation rule. We say that φ is *strongly monotonic* if, for each pair of games $v, w \in G^n$ and each $i \in N$ such that, for each $S \subseteq N\backslash\{i\}$, $w(S \cup \{i\}) - w(S) \geq v(S \cup \{i\}) - v(S)$, then $\varphi_i(w) \geq \varphi_i(v)$. Now, let $v, w \in G^n$ and let $T \subseteq N$ be such that $w(T) > v(T)$ and, for each $S \neq T$, $w(S) = v(S)$: i) φ satisfies *coalitional monotonicity* if, for each $i \in T$, $\varphi_i(w) \geq \varphi_i(v)$, ii) φ satisfies *aggregate monotonicity* if $T = N$ implies that, for each $i \in N$, $\varphi_i(w) \geq \varphi_i(v)$, and iii) φ satisfies *weak coalitional monotonicity* if $\sum_{i\in T} \varphi_i(w) \geq \sum_{i\in T} \varphi_i(v)$.

[15] Michael (1956) is a seminal contribution on the problem of continuous selection from multifunctions. Further, the regularity problems one faces when working with barycentric selections from convex-compact-valued multifunctions (as the core) is discussed in Gautier and Morchadi (1992).

[16] For an axiomatic characterization of the core-center, refer to González-Díaz and Sánchez-Rodríguez (2009).

Young (1985) characterized the Shapley value using strong monotonicity, EFF, and SYM. Young (1985) and Housman and Clark (1998) showed that coalitional monotonicity is incompatible with always selecting core allocations. Below we provide a direct proof that the core-center satisfies neither strong monotonicity nor coalitional monotonicity, which consists of showing that it does not even satisfy a weaker property: aggregate monotonicity.

Proposition 5.11.1. *In the class of games with $n \geq 4$ players and nonempty core, the core-center does not satisfy aggregate monotonicity.*

Proof. The proof is by means of an example when $n = 4$. If $n > 4$ the example can be adapted by adding null players. Let $v \in G^4$ be defined as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$v(S)$	0	0	0	0	0	1	1	1	1	0	1	1	1	2	2

It is easy to see that $C(v) = \{(0,0,1,1)\}$ and, hence, $\mu(v) = (0,0,1,1)$. Let w be such that $w(N) = 3$ and, for each $S \neq N$, $w(S) = v(S)$. Then,

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$w(S)$	0	0	0	0	0	1	1	1	1	0	1	1	1	2	3

and its core is given by the following convex hull $C(w) = \text{conv}\{(1,0,1,1), (0,0,2,1), (0,0,1,2), (0,1,1,1), (1,1,1,0), (1,1,0,1), (1,2,0,0)\}$. In order to prove that the core-center does not satisfy aggregate monotonicity, we show that $\mu_3(v) > \mu_3(w)$. Let $\hat{w}$ be the game defined as follows:

S	1	2	3	4	12	13	14	23	24	34	123	124	134	234	N
$\hat{w}(S)$	0	0	0	0	0	1	1	1	1	0	1	1	2	2	3

so that $C(\hat{w}) = \text{conv}\{(1,0,1,1), (0,0,2,1), (0,0,1,2), (0,1,1,1), (1,1,1,0), (1,1,0,1)\}$. Game $\hat{w}$ only differs from w in the worth of coalition $\{1,3,4\}$. Since $\hat{w}(\{1,3,4\}) > w(\{1,3,4\})$, $C(\hat{w}) \subsetneq C(w)$. Now, $C(\hat{w})$ is symmetric with respect to the point $(0.5, 0.5, 1, 1)$, i.e.,

$$x \in C(\hat{w}) \Leftrightarrow -\big(x - (0.5, 0.5, 1, 1)\big) + (0.5, 0.5, 1, 1) \in C(\hat{w}).$$

Hence, $\mu(\hat{w}) = (0.5, 0.5, 1, 1)$.

Now, $C(w) \backslash C(\hat{w}) \subsetneq \text{conv}\{(1,1,1,0), (0,1,1,1), (1,1,0,1), (1,2,0,0)\}$. Thus, for each $x \in C(w) \backslash C(\hat{w})$, $x_3 \leq 1$. Moreover, the volume of the points in $C(w) \backslash C(\hat{w})$ whose third coordinate is smaller than 1 is positive. By the definition of the core-center, since $\mu_3(\hat{w}) = 1$, we have $\mu_3(w) < 1 = \mu_3(v)$. Hence, the core-center does not satisfy aggregate monotonicity. □

It is known that the nucleolus also violates the first three monotonicity properties defined above. Mirás Calvo et al. (2021) further explore aggregate monotonicity for both the nucleolus and the core-center, providing additional insights and also some subclasses of games where they satisfy this property. Weak coalitional monotonicity was introduced by Zhou (1991), who showed that the nucleolus satisfies it. The result below shows that the core-center also behaves like the nucleolus for this fourth monotonicity property.

Proposition 5.11.2. *The core-center satisfies weak coalitional monotonicity.*

Proof. Let v and w be two games with a nonempty core and let $T \subseteq N$ be such that $w(T) > v(T)$ and, for each $S \neq T$, $w(S) = v(S)$. If $T = N$ the result follows from efficiency. Hence, we can assume $T \subsetneq N$. If $C(w) = C(v)$, then $\mu(w) = \mu(v)$ and $\sum_{i \in T} \mu_i(w) \geq \sum_{i \in T} \mu_i(v)$. Thus, we can assume that $C(w) \subsetneq C(v)$. Let $x \in C(v) \backslash C(w)$ and $y \in C(w)$, then $\sum_{i \in T} y_i \geq w(T) > \sum_{i \in T} x_i$. Hence, since the core-center is the expectation of the uniform distribution over the core, $\sum_{i \in T} \mu_i(w) \geq \sum_{i \in T} \mu_i(v)$. □

The calculation of the core-center is a computationally hard problem since, as was the case for the Shapley value, it depends on the value of all coalitions of the game. However, for some classes of games one can exploit the special structure of the core to obtain effective algorithms to compute the core-center. This approach has been followed, for instance, in González-Díaz et al. (2016) for airport problems (covered in Section 6.2) and in Mirás Calvo et al. (2022) for claim problems (a generalization of bankruptcy problems, which are covered in Section 6.3).

5.12. Convex Games

This section is devoted to the study of the class of *convex games*, introduced in Shapley (1971). These games have several interesting properties; probably the most important one is that the Shapley value is always an element of the core in this class of games.

Definition 5.12.1. A TU-game $v \in G^N$ is convex if, for each $i \in N$ and each pair $S, T \subset N \backslash \{i\}$ with $S \subset T$,

$$v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T). \tag{5.12.1}$$

It can be easily checked that every convex game is superadditive. We devote the rest of this section to establish two important properties of convex games. First, the core of a convex game is always nonempty. Second, the Shapley value of a convex game is always an element of its core. The vectors of marginal contributions introduced in Definition 5.6.4 are a very useful tool in this section.

Example 5.12.1. Consider the visiting professor game introduced in Example 5.4.4, where $N = \{1, 2, 3\}$, $v(1) = v(2) = v(3) = 0$, $v(12) = 1500$, $v(13) = 500$, $v(23) = 500$, and $v(N) = 2000$. This game is convex. Moreover, as we have already seen, its core is nonempty and the Shapley value belongs to the core (see Figure 5.6.1(b)). We represent the vectors of marginal contributions associated with the different orderings of the players in the table below. As we already know from Eq. (5.6.2), the Shapley value can be computed as the average of these vectors. ◇

Permutation π	$m_1^\pi(v)$	$m_2^\pi(v)$	$m_3^\pi(v)$
123	0	1500	500
132	0	1500	500
213	1500	0	500
231	1500	0	500
312	500	1500	0
321	1500	500	0

Figure 5.12.1. Vectors of marginal contributions in the visiting professor game.

We now present the most important result for convex games. It relates convex games, vectors of marginal contributions, and the core.

Theorem 5.12.1. *Let $v \in G^N$. The following statements are equivalent:*

i) *The game v is convex.*

ii) *For each $\pi \in \Pi(N)$, $m^\pi(v) \in C(v)$.*

iii) $C(v) = \operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\}$*, i.e., the core and the Weber set coincide.*

Proof.[17] i) $\Rightarrow$ ii). Let $\pi \in \Pi(N)$. We have that $\sum_{i \in N} m_i^\pi(v) = v(N)$. Let $S \subsetneq N$. Let $i \in N \backslash S$ be such that, for each $j \in N \backslash (S \cup \{i\})$, $\pi(i) < \pi(j)$. Then, $P^\pi(i)$ is a subset of S. Since v is convex, $v(S \cup \{i\}) - v(S) \geq v(P^\pi(i) \cup \{i\}) - v(P^\pi(i)) = m_i^\pi(v) = \sum_{j \in S \cup \{i\}} m_j^\pi(v) - \sum_{j \in S} m_j^\pi(v)$. Hence,

$$\sum_{j \in S \cup \{i\}} m_j^\pi(v) - v(S \cup \{i\}) \leq \sum_{j \in S} m_j^\pi(v) - v(S).$$

Now, we can apply to $S \cup \{i\}$ the same argument we used above for S. By doing this repeatedly until we get coalition N, we get

$$0 = \sum_{j \in N} m_j^\pi(v) - v(N) \leq \sum_{j \in S} m_j^\pi(v) - v(S).$$

[17] The statement i) $\Rightarrow$ ii) was proved by Shapley (1971), ii) $\Rightarrow$ iii) by Weber (1988), and iii) $\Rightarrow$ i) by Ichiishi (1981).

Hence, for each $S \subset N$, $\sum_{j\in S} m_j^\pi(v) \geq v(S)$. Therefore, $m^\pi(v) \in C(v)$.

ii) $\Rightarrow$ iii). By ii) we already have $\operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\} \subset C(v)$. We now show that $C(v) \subset \operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\}$. We proceed by induction on the number of players. If $n = 1$, the result is straigthforward. Assume that the result is true for all games with less than n players and let $v \in G^N$. Since $C(v)$ is a convex set, it suffices to prove that all the points in the boundary of $C(v)$ belong to the set $\operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\}$. Let x be a point in the boundary of $C(v)$. Then, there is $S \subsetneq N$, $S \neq \emptyset$, such that $\sum_{i\in S} x_i = v(S)$. Recall that $v_S \in G^S$ is defined, for each $T \subset S$, by $v_S(T) := v(T)$. Let $v_1 \in G^{N\backslash S}$ be defined, for each $T \subset N\backslash S$, by $v_1(T) := v(T \cup S) - v(S)$. Let x_S be the restriction of x to the players in S. Clearly, $x_S \in C(v_S)$. Since $\sum_{i\in S} x_i = v(S)$ and $x \in C(v)$, we have that, for each $T \subset N\backslash S$,

$$\sum_{i\in T} x_i = \sum_{i\in T\cup S} x_i - \sum_{i\in S} x_i \geq v(T \cup S) - v(S).$$

Hence, $x_{N\backslash S} \in C(v_1)$. By the induction hypothesis,

$$x_S = \sum_{\pi\in\Pi(S)} \lambda_\pi m^\pi(v_S) \text{ and } x_{N\backslash S} = \sum_{\pi\in\Pi(N\backslash S)} \gamma_{N\backslash S} m^\pi(v_1),$$

where, for each $\pi \in \Pi(S)$, $\lambda_\pi \geq 0$ and $\sum_{\pi\in\Pi(S)} \lambda_\pi = 1$ and, for each $\pi \in \Pi(N\backslash S)$, $\gamma_\pi \geq 0$ and $\sum_{\pi\in\Pi(N\backslash S)} \gamma_\pi = 1$. Now, for each $\pi_S \in \Pi(S)$ and each $\pi_{N\backslash S} \in \Pi(N\backslash S)$, let $\pi^* \in \Pi(N)$ be defined by $\pi^* := (\pi_S, \pi_{N\backslash S})$. So defined, we have that, for each $i \in S$, $m_i^{\pi^*}(v) = m_i^{\pi_S}(v_S)$ and, for each $i \in N\backslash S$, $m_i^{\pi^*}(v) = m_i^{\pi_{N\backslash S}}(v_1)$. Hence,

$$x = \sum_{\pi_S\in\Pi(S)} \sum_{\pi_{N\backslash S}\in\Pi(N\backslash S)} \lambda_{\pi_S} \gamma_{\pi_{N\backslash S}} m^{\pi^*}(v),$$

which implies that $x \in \operatorname{conv}\{m^\pi(v) : \pi \in \Pi(N)\}$.

iii) $\Rightarrow$ i). Let $i \in N$ and let $S, T \subset N\backslash\{i\}$ be such that $S \subset T$. Let $\pi \in \Pi(N)$ be such that $\pi = (\pi_S, \pi_{T\backslash S}, \pi_{N\backslash T})$ and, for each $j \in N\backslash T$, $j \neq i$, $\pi(i) < \pi(j)$. Then, $P^\pi(i) = T$, $m_i^\pi(v) = v(P^\pi(i) \cup \{i\}) - v(P^\pi(i)) = v(T \cup \{i\}) - v(T)$. Also, $\sum_{j\in S} m_j^\pi = v(S)$. Since $m^\pi \in C(v)$, then

$$v(S \cup \{i\}) \leq \sum_{j\in S\cup\{i\}} m_j^\pi(v) = v(S) + m_i^\pi(v).$$

Hence, $v(S \cup \{i\}) - v(S) \leq m_i^\pi(v) = v(T \cup \{i\}) - v(T)$. Therefore, v is convex. □

Corollary 5.12.2. *Let $v \in G^N$ be a convex game. Then, $\Phi^S(v) \in C(v)$.*

Proof. It immediately follows from the formula of the Shapley value (given in Eq. (5.6.2)) and statement iii) in Theorem 5.12.1 above. □

We now show that the condition above is sufficient but it is not necessary.

Example 5.12.2. Consider the TU-game (N, v), where $N = \{1, 2, 3\}$ and $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 2$, $v(23) = 6$ and $v(N) = 7$. This game is not convex because $v(23) - v(2) = 6 > 5 = v(N) - v(12)$. However, the reader can easily verify that $\Phi^S(v) = (1, 3, 3) \in C(v)$. ◇

As we have seen above, when restricting attention to convex games one can get results that do not hold in general. The main advantage of dealing with convex games is that, mainly because of Theorem 5.12.1, their cores are easier to study. We refer the reader to Shapley (1971) and González-Díaz and Sánchez-Rodríguez (2008) for two papers in which the special structure of convex games is used to derive a series of geometric and game theoretical properties of their cores.

5.13. Games with Communication Graphs: The Myerson Value

In this section we introduce a variation of TU-games, proposed in Myerson (1977), for situations where players can partially cooperate. The possibilities of cooperation are modeled by an undirected graph. Each link represents bilateral communication and cooperation is feasible.

Definition 5.13.1. An *undirected graph* on N is a set of unordered pairs of different elements of N. These pairs $(i : j)$ are called *links*.

Note that, by definition, we only consider graphs without loops and also $(i : j) = (j : i)$. We denote by L^N the complete graph on N, i.e.,

$$L^N = \{(i : j) : i \in N, j \in N, i \neq j\},$$

and by $\mathcal{L}^N$ the set of all undirected graphs. Let $L \in \mathcal{L}^N$ and $i, j \in N$ such that $(i : j) \in L$. The graph that is obtained by removing the link $(i : j)$ is denoted by $L_{-(i:j)} = L \backslash \{(i : j)\}$.

Let $L \in \mathcal{L}^N$, $S \subset N$, and $i, j \in S$. We say that i and j are *connected in S* by L if and only if there is a path in L which goes from i to j through players in S. That is, i and j are connected in S by L if and only if $i = j$ or there is some $k \geq 1$ and a sequence $i_0, i_1, \ldots, i_k$ with $i_0 = i$, $i_k = j$, $(i_{l-1} : i_l) \in L$, and $i_l \in S$, for each $l = 1, \ldots, k$. A coalition S is *connected* in L if and only if, for each $i, j \in S$, i and j are connected in S by L. Clearly, for each $S \subset N$, there is a unique partition of S in maximal subcoalitions whose members are connected in S by L. Formally, if we denote this partition by S/L, we have that $R \in S/L$ if and only if

- $R \subset S$ and R is connected in L.
- There is no $R' \subset S$ such that R' is connected in L and $R \subsetneq R'$.

Each element in S/L is a *connected component of S with respect to L*. The whole set of connected components is denoted by N/L. The next example illustrates these concepts.

Example 5.13.1. Take $N = \{1,2,3,4\}$ and $L = \{(1:2),(2:3)\}$. Player 1 and player 2 are connected players and coalition $\{1,2\}$ is a connected coalition. Then, $\{1,2\}/L = \{\{1,2\}\}$. An example of a nonconnected coalition is $\{1,4\}$, being $\{1,4\}/L = \{\{1\},\{4\}\}$, i.e., coalition $\{1,4\}$ has two connected components with respect to the graph L. The set of connected components is $N/L = \{\{1,2,3\},\{4\}\}$. ◇

Definition 5.13.2. A *TU-game with a communication graph* or *a communication situation* is a pair (v,L), being $v \in G^N$ and $L \in \mathcal{L}^N$.

We denote by CG^N the class of all communication situations with player set N.

Definition 5.13.3. An *allocation rule* for CG^N is just a map $\psi\colon CG^N \to \mathbb{R}^N$.

In order to obtain a reasonable outcome for a communication situation, Myerson (1977) presented some appealing properties that an allocation rule might satisfy in this framework.

Component Efficiency (CEFF): The allocation rule ψ satisfies CEFF if, for each $(v,L) \in CG^N$ and $S \in N/L$, $\sum_{i\in S} \psi_i(v,L) = v(S)$.

Fairness (FN): The allocation rule ψ satisfies FN if, for each $(v,L) \in CG^N$ and $i,j \in N$ with $(i:j) \in L$,

$$\psi_i(v,L) - \psi_i(v,L_{-(i:j)}) = \psi_j(v,L) - \psi_j(v,L_{-(i:j)}).$$

Stability (ST): The allocation rule ψ satisfies ST if, for each $(v,L) \in CG^N$ and $i,j \in N$ with $(i:j) \in L$,

$$\psi_i(v,L) \geq \psi_i(v,L_{-(i:j)}) \quad \text{and} \quad \psi_j(v,L) \geq \psi_j(v,L_{-(i:j)}).$$

Property CEFF indicates that players in a connected component split among themselves the worth they can obtain from cooperation. Properties FN and ST are related to the change on the amount a player receives when his communication capability is diminished. Property FN is a formulation of the equal-gains principle in the sense that any two players gain the same from their bilateral communication. Property ST establishes that any two players always benefit from bilateral communication.

Given a communication situation $(v,L) \in CG^N$, the *graph-restricted game by L*, $v^L \in G^N$, is defined, for each $S \subset N$, by

$$v^L(S) := \sum_{R\in S/L} v(R)$$

This TU-game reflects the idea that only players that can communicate among themselves through the links in L are able to generate worth. Myerson (1977) proposed an allocation for communication situations based on the Shapley value of the graph-restricted game.

Definition 5.13.4. The *Myerson value*, Φ^{M}, is defined, for each $(v, L) \in CG^N$ and each $i \in N$, by

$$\Phi_i^{\mathrm{M}}(v, L) := \Phi_i^{\mathrm{S}}(v^L). \tag{5.13.1}$$

Theorem 5.13.1. *The Myerson value is the unique allocation rule in CG^N that satisfies* CEFF *and* FN.

Proof. Let $(v, L) \in CG^N$. First, we check that Φ^{M} satisfies CEFF. For each $S \in N/L$, let $r^S \in G^N$ be defined, for each $T \subset N$, by

$$r^S(T) := \sum_{R \in (T \cap S)/L} v(R).$$

By definition, for each $T \subset N$, we have $r^S(T) = r^S(T \cap S)$. In particular, $r^S(N) = r^S(S)$. Moreover, for each $i \notin S$ we have that, for each $T \subset N \backslash \{i\}$, $r^S(T \cup \{i\}) = r^S(T)$, *i.e*, i is a null player in r^S. Since the Shapley value satisfies EFF and NPP, we have

$$\sum_{i \in T} \Phi_i^{\mathrm{S}}(r^S) = \begin{cases} r^S(N) = r^S(S) & \text{if } T = S, \\ 0 & \text{if } S \cap T = \emptyset. \end{cases}$$

Note that any two players connected in $T \subset N$ by L are also connected in N. Thus,

$$T/L = \underset{S \in N/L}{\cup} (T \cap S)/L.$$

Therefore, $v^L = \sum_{S \in N/L} r^S$. Property ADD of the Shapley value implies that, for each $S \in N/L$,

$$\sum_{i \in S} \Phi_i^{\mathrm{S}}(v^L) = \sum_{T \in N/L} \sum_{i \in S} \Phi_i^{\mathrm{S}}(r^T) = r^S(N) = r^S(S) = \sum_{R \in S/L} v(R) = v^L(S).$$

Since $S \in N/L$, we have $v^L(S) = v(S)$, which implies that the Myerson value satisfies CEFF.

Now, we check that Φ^{M} satisfies FN. Let $i, j \in N$ be such that $(i:j) \in L$. Let $\hat{v} = v^L - v^{L_{-(i:j)}} \in G^N$. Clearly, if $\hat{v} = 0$, using properties SYM and ADD of the Shapley value, we have

$$\Phi_i^{\mathrm{S}}(v^L) - \Phi_i^{\mathrm{S}}(v^{L_{-(i:j)}}) = \Phi_i^{\mathrm{S}}(\hat{v}) = \Phi_j^{\mathrm{S}}(\hat{v}) = \Phi_j^{\mathrm{S}}(v^L) - \Phi_j^{\mathrm{S}}(v^{L_{-(i:j)}}).$$

Thus, assume that $\hat{v} \neq 0$. Let $S \subset N$ be such that $|\{i, j\} \cap S| < 2$. Then, $S/L = S/L_{-(i:j)}$ and

$$\hat{v}(S) = v^L(S) - v^{L_{-(i:j)}}(S) = \sum_{R \in S/L} v(R) - \sum_{R \in S/L_{-(i:j)}} v(R) = 0.$$

Hence, if $\hat{v}(S) \neq 0$, then $\{i,j\} \subset S$. In such a case, for each $S \subset N\backslash\{i,j\}$, $\hat{v}(S \cup i) = \hat{v}(S \cup j)$, which means that i and j are symmetric players in $\hat{v}$. By properties SYM and ADD of the Shapley value, we have

$$\Phi_i^S(v^L) - \Phi_i^S(v^{L_{-(i:j)}}) = \Phi_i^S(\hat{v}) = \Phi_j^S(\hat{v}) = \Phi_j^S(v^L) - \Phi_j^S(v^{L_{-(i:j)}}),$$

which means that Φ^M satisfies FN.

Next, we show by contradiction that there is at most one allocation rule satisfying CEFF and FN. Assume that there are two different allocation rules ψ^1 and ψ^2, defined on CG^N, both satisfying CEFF and FN. Let $v \in L^N$ and $L \in \mathcal{L}^N$ be such that i) $\psi^1(v,L) \neq \psi^2(v,L)$ and ii) for each $\hat{L} \in \mathcal{L}^N$ with fewer links than L, $\psi^1(v,L) = \psi^2(v,L)$. The number of links in L cannot be zero since, otherwise, both allocation rules would coincide by CEFF. Let $i,j \in N$ be such that $(i:j) \in L$. Then, $\psi^1(v, L_{-(i:j)}) = \psi^2(v, L_{-(i:j)})$ and, by FN,

$$\begin{aligned}\psi_i^1(v,L) - \psi_j^1(v,L) &= \psi_i^1(v,L_{-(i:j)}) - \psi_j^1(v,L_{-(i:j)}) \\ &= \psi_i^2(v,L_{-(i:j)}) - \psi_j^2(v,L_{-(i:j)}) = \psi_i^2(v,L) - \psi_j^2(v,L).\end{aligned}$$

Thus, for each pair of connected players in L, we have that $\psi_i^1(v,L) - \psi_i^2(v,L) = \psi_j^1(v,L) - \psi_j^2(v,L)$. Then, for each connected component $S \in N/L$ and $i \in S$, we have $\psi_i^1(v,L) - \psi_i^2(v,L) = d_S$. Since both ψ^1 and ψ^2 satisfy CEFF,

$$\sum_{i \in S} \psi_i^1(v,L) = v(S) = \sum_{i \in S} \psi_i^2(v,L).$$

Hence,

$$0 = \sum_{i \in S}(\psi_i^1(v,L) - \psi_i^2(v,L)) = |S| d_S,$$

getting $d_S = 0$. Therefore, for each $i \in S \in N/L$, $\psi_i^1(v,L) = \psi_i^2(v,L)$, which contradicts the assumption that $\psi_i^1(v,L) \neq \psi_i^2(v,L)$. □

None of the properties that characterize the Myerson value is superfluous. Exercise 5.16 asks the reader to formally prove it.

Proposition 5.13.2. *In the class of superadditive TU-games, the Myerson value satisfies* ST.

Proof. Take $(v,L) \in CG^N$ with $v \in SG^N$. Let $i,j \in N$ be such that $(i:j) \in L$ and take $S \subset N$. Recall that any two connected components in S/L are disjoint sets. Besides, for each pair $i,j \in N$, every connected component in $S/L_{-(i:j)}$ is a subset of a connected component in S/L and, if $i \notin S$, then $S/L = S/L_{-(i:j)}$. Since $v \in SG^N$, we have

$$v^L(S) = \sum_{T \in S/L} v(T) \geq \sum_{T \in S/L_{-(i:j)}} v(T) = v^{L_{-(i:j)}}(S),$$

with $v^L(S) = v^{L_{-(i:j)}}(S)$ if $i \notin S$. Thus, taking $\hat{v} = v^L - v^{L_{-(i:j)}} \in G^N$, we have: i) for each $S \subset N$, $\hat{v}(S) \geq 0$ and ii) for each $S \subset N \backslash \{i\}$, $\hat{v}(S) = 0$. Then, for each $S \subset N \backslash \{i\}$, $\hat{v}(S \cup \{i\}) \geq \hat{v}(S)$, which implies that $\Phi_i^S(\hat{v}) \geq 0$. Using property ADD of the Shapley value we have

$$0 \leq \Phi_i^S(\hat{v}) = \Phi_i^S(v^L) - \Phi_i^S(v^{L_{-(i:j)}}).$$

Analogously, we also get $0 \leq \Phi_j^S(v^L) - \Phi_j^S(v^{L_{-(i:j)}})$, which concludes the proof. □

Below we revisit Example 5.4.3 by adding a graph that represents bilateral affinities. This example appears in Vázquez-Brage et al. (1996).

Example 5.13.2. The graph in Figure 5.13.1 is a plausible representation of the affinities between any two parties in Example 5.4.3. It corresponds to the links $(PSOE : PAR)$, $(PSOE : IU)$, and $(PP : PAR)$, i.e., $L = \{(1:3),(1:4),(2:3)\}$. The set of connected components is $N/L = \{\{1,2,3,4\}\}$. Both PSOE (player 1) and PAR (player 3) are involved in two links each. The graph-restricted game by L is $v^L(S) = 1$ if there is $T \in \{\{1,3\},\{2,3\}\}$ such that $T \subset S$, otherwise $v^L(S) = 0$. In particular $v^L(\{1,2\}) = 0$. The additional information provided by the communication graph makes players 1 and 2 symmetric in v^L, but not player 3. It is easy to check that the Shapley value of v^L is $\Phi^S(v^L) = (1/6, 1/6, 2/3, 0) = \Phi^M(v, L)$. The Myerson value gives more power to player 3, being this player a member of every coalition S with $v^L(S) = 1$. ◇

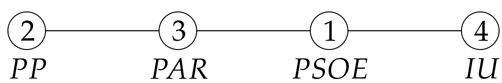


Figure 5.13.1. A communication graph L for Example 5.4.3.

Myerson (1980) generalized the idea of direct communication between two players to direct communication between players in certain groups. Meessen (1988) examined communication situations taking into account the role of the links in the graph. He proposed a new allocation rule, the *position value*, which Borm et al. (1992b) characterized. Different procedures to compute these values can be found in the literature, such as Owen (1986) and van den Nouweland (1993) for the Myerson value, Borm et al. (1992b) for the position value, and Gómez et al. (2004) for both. Note that the model discussed in this section restricts attention to undirected graphs. Versions of the Myerson value and the position value for directed graphs can be seen in González-Arangüena et al. (2008) and Li and Shan (2020) for the former and in Gavilán et al. (2022) for the latter.

5.14. Games with A Priori Unions: The Owen Value

In this section we introduce a variation of TU-games for situations where there is a coalition structure that conditions the cooperation of the players. A coalition structure is a partition of the set of players, which we a call system of *a priori* unions. In a TU-game with a priori unions, cooperation takes place at two levels: there is cooperation between the unions (the classes of the partition) and also cooperation between the players in each union. For these situations, Owen (1977) introduced a variation of the Shapley value that distributes among the players the total worth of the grand coalition by taking into account, for each player, the benefits generated by the union to which he belongs, the benefits that he himself generates being part of his union, and also the benefits that he himself could generate if he were part of other unions.

Definition 5.14.1. A TU-game with a priori unions is a triplet (N, v, P), where (N, v) is a TU-game and P is a partition of N.

When no confusion arises, we denote the game with a priori unions (N, v, P) by (v, P). Let GU^N be the class of TU-games with a priori unions and n players.

Definition 5.14.2. An *allocation rule for n-player TU-games with a priori unions* is just a map $\varphi\colon GU^N \to \mathbb{R}^N$.

Analogous to what was done in the Shapley value, we can identify a list of properties that we consider desirable for an allocation rule in this context. The properties we discuss below are adaptations of Shapley's properties and, likewise, allow to characterize a single allocation rule. Some of them are identical to Shapley's properties, although adapted to the current setting; in such cases we keep their original names. We need two new concepts that we introduce below.

Definition 5.14.3. Let $(v, P) \in GU^N$ with $P = \{P_1, \ldots, P_m\}$. Let $M := \{1, \ldots, m\}$.

i) The *quotient game associated with* (v, P) is the game $(M, v^P) \in G^M$ given, for each $S \subset M$, by $v^P(S) = v(\cup_{k \in S} P_k)$.
ii) Two unions P_k and P_l are *symmetric* if players k and l are symmetric in the quotient game v^P.

Let φ be an allocation rule for n-player TU-games with a priori unions and consider the following properties we might impose on it.

Efficiency (EFF): The allocation rule φ satisfies EFF if, for each game $(v, P) \in GU^N$, $\sum_{i \in N} \varphi_i(v, P) = v(N)$.

Null Player (NPP): The allocation rule φ satisfies NPP if, for each game $(v,P) \in GU^N$ and each null player $i \in N$, $\varphi_i(v,P) = 0$.

Symmetry Within Unions (SWU): The allocation rule φ satisfies SWU if, for each $(v,P) \in GU^N$, each $P_k \in P$ and each pair $i,j \in P_k$ of symmetric players, $\varphi_i(v) = \varphi_j(v)$.

Symmetry Among Unions (SAU): The allocation rule φ satisfies SAU if, for each $(v,P) \in GU^N$ and each pair P_k, P_l of symmetric unions, $\sum_{i\in P_k} \varphi_i(v) = \sum_{i\in P_l} \varphi_i(v)$.

Additivity (ADD): The allocation rule φ satisfies ADD if, for each pair of games $(v,P),(w,P) \in GU^N$, $\varphi(v+w,P) = \varphi(v,P) + \varphi(w,P)$.

Shapley's symmetry property is split into two properties here: SWU calls for equal treatment of equal players belonging to the same union, while SAU calls for equal treatment of equal unions.

We now present the definition of the Owen value as originally introduced in Owen (1977). For a partition P of N and for each $i \in N$, we denote by $P_{k(i)}$ the union of P to which i belongs.

Definition 5.14.4. The *Owen value*, Φ^{O}, is defined, for each $(v,P) \in GU^N$ and each $i \in N$, by

$$\sum_{Q\subset M\backslash\{k(i)\}} \sum_{S\subset P_{k(i)}\backslash\{i\}} A_i(Q,S)\Big(v(\cup_{k\in Q}P_k \cup S \cup \{i\}) - v(\cup_{k\in Q}P_k \cup S)\Big),$$

where $P = \{P_1,\dots,P_m\}$, $M = \{1,\dots,m\}$, and

$$A_i(Q,S) = \frac{|S|!(|P_{k(i)}| - |S| - 1)!|Q|!(m - |Q| - 1)!}{|P_{k(i)}|!m!}.$$

Note that the Owen value can be seen as a generalization of the Shapley value, because they deliver the same allocation when the partition is trivial, i.e., if $P^N := \{N\}$ and $P^n := \{\{1\},\dots,\{n\}\}$, then, for each $v \in G^N$,

$$\Phi^{\mathrm{S}}(v) = \Phi^{\mathrm{O}}(v,P^N) = \Phi^{\mathrm{O}}(v,P^n).$$

Similarly to the interpretation of the Shapley value, the Owen value can be interpreted as follows. The grand coalition is to be formed inside a room, but the players have to enter the room sequentially, one at a time. When a player i enters, he gets his contribution to the coalition of players that are already inside. The order of the players is decided randomly. Yet, now only those orders that do not mix a priori unions are possible, that is, only orders that do not insert elements of other unions between two elements of each union P_k are possible. It is easy to check that the Owen value assigns to each player his expected value under this randomly ordered entry process.

The above discussion suggests an alternative definition of the Owen value, based on the vectors of marginal contributions. A permutation $\pi \in \Pi(N)$ is said to be *compatible with* P if, for each triplet $i, j, l \in N$ with $P_{k(i)} = P_{k(j)} \neq P_{k(l)}$, if $\pi(i) < \pi(l)$ then $\pi(j) < \pi(l)$. Given a partition P of N, we denote by $\Pi_P(N)$ the set of all permutations of the elements in N compatible with P. Note that the permutations compatible with P are precisely those that do not mix unions. Thus, it is easy to check that the formula for the Owen value is equivalent to

$$\Phi_i^{\mathrm{O}}(v, P) = \frac{1}{|\Pi_P(N)|} \sum_{\pi \in \Pi_P(N)} m_i^{\pi}(v). \tag{5.14.1}$$

Similarly to what we did for the Shapley value, we now present an axiomatic characterization of the Owen value.

Theorem 5.14.1. *The Owen value is the unique allocation rule in GU^N that satisfies* EFF, NPP, SWU, SAU, *and* ADD.

Proof. It is not difficult to check that Φ^{O} satisfies EFF, NPP, SWU, SAU, and ADD. Now, let φ be an allocation rule satisfying these properties. Recall that each $v \in G^N$ can be seen as the vector $(v(S))_{S \in 2^N \backslash \{\emptyset\}} \in \mathbb{R}^{2^n - 1}$, and that the unanimity games $U(N) := \{w^S : S \in 2^N \backslash \{\emptyset\}\}$ are a basis of such a vector space. Since φ satisfies EFF, NPP, SWU, and SAU, we have that, for each partition $P = \{P_k\}_{k \in M}$ of N, each $i \in P_{k(i)}$, each $\emptyset \neq S \subset N$, and each $\alpha_S \in \mathbb{R}$,

$$\varphi_i(\alpha_S w^S, P) = \begin{cases} \frac{\alpha_S}{|M_S||P_{k(i)} \cap S|} & i \in S, \\ 0 & \text{otherwise}, \end{cases}$$

where M_S denotes the set $\{k \in M \ : \ P_k \cap S \neq \emptyset\}$. Then, since φ also satisfies ADD, φ is uniquely determined. □

The Owen value has been widely used in practice, especially in cost allocation problems (Costa 2016) and in analysis of voting structures. A seminal paper in the literature of Owen value and political analysis is Carreras and Owen (1988). There are also several papers on alternative axiomatic characterizations of the Owen value and Khmelnitskaya and Yanovskaya (2007), for instance, provided one without the additivity property.

Next we illustrate how the Owen value works by calculating it for the game in Example 5.4.2, considering all possible partitions of the set of players. We omit the detailed computations, which can be easily done using the vectors of marginal contributions and Eq. (5.14.1).

Example 5.14.1. Below we present the Owen value of the glove game v with respect to all nontrivial partitions of the set of players. Recall that the Shapley value of this game is $(2/3, 1/6, 1/6)$.

- $P^1 = \{\{1,2\},\{3\}\}$. Then, $\Phi^O(v, P^1) = (3/4, 1/4, 0)$.
- $P^2 = \{\{1,3\},\{2\}\}$. Then, $\Phi^O(v, P^2) = (3/4, 0, 1/4)$.
- $P^3 = \{\{1\},\{2,3\}\}$. Then, $\Phi^O(v, P^3) = (1/2, 1/4, 1/4)$. ◇

Note that a priori unions, which we study in this section, can be combined with communication graphs, studied in Section 5.13, to model problems in which there are simultaneously restrictions on communication and a structure of coalitions. Such problems have been analyzed, for example, in Vázquez-Brage et al. (1996), where an allocation rule for communication situations with a priori unions based on the Owen value is defined and axiomatically characterized. In a nutshell, a communication situation with a priori unions is a triplet (v, L, P), where $v \in G^N$ is a TU-game, $L \in \mathcal{L}^N$ is a communication graph (as in Section 5.13), and P is a partition of N. The allocation rule proposed and axiomatically characterized in Vázquez-Brage et al. (1996) assigns to each such situation (v, L, P) the Owen value of the corresponding graph-restricted game v^L with the a priori unions given by P, that is, $\Phi^O(v^L, P)$. In the following example we revisit the Parliament of Aragón, which we analyzed in Examples 5.4.3 and 5.13.2, to illustrate the interest of combining a priori unions and communication graphs.

Example 5.14.2. Consider again the game associated with the Parliament of Aragón, where $N = \{1,2,3,4\}$ (1=PSOE, 2=PP, 3=PAR, 4=IU, having 30, 17, 17, and 3 seats, respectively). Bearing in mind that decisions in parliaments are normally taken by majority vote, the game v that describes the sharing of power in this parliament is as follows: $v(S) = 1$ if there is $T \in \{\{1,2\},\{1,3\},\{2,3\}\}$ with $T \subset S$, and $v(S) = 0$ otherwise. The communication graph representing the affinities of political parties is the one given in Figure 5.13.1, i.e., $L = \{(1:3),(1:4),(2:3)\}$. Taking all these elements into account, an alliance of parties (a governing union) will probably be formed in order to be able to govern in Aragón. The two most plausible alliances are the one formed by PSOE and PAR (which will give rise to the system of unions $P^1 = \{\{1,3\},\{2\},\{4\}\}$) and the one formed by PP and PAR (which will give rise to the system of unions $P^2 = \{\{2,3\},\{1\},\{4\}\}$). Figure 5.14.1 shows the Shapley value of this game, the Myerson value of the game with the communication graph given by L, the Owen value of the game with the systems of unions P^1 and P^2, and the value according to the rule in Vázquez-Brage et al. (1996) for the games with the graph L and P^1 and P^2 systems. If we are interested in studying what governing union will be formed, we should use the Owen value for P^1 and P^2 union systems, but we should also take into account the communication graph. The party that can somehow choose the governing union is PAR, given its privileged position in communication L-graph. In view of $\Phi^O(v^L, P^1)$ and $\Phi^O(v^L, P^2)$, it seems that PAR will get the same share of power, higher than

Party	$\Phi^{S}(v)$	$\Phi^{M}(v,L)$	$\Phi^{O}(v,P^1)$	$\Phi^{O}(v,P^2)$	$\Phi^{O}(v^L,P^1)$	$\Phi^{O}(v^L,P^2)$
PSOE	1/3	1/6	1/2	0	1/4	0
PP	1/3	1/6	0	1/2	0	1/4
PAR	1/3	2/3	1/2	1/2	3/4	3/4
IU	0	0	0	0	0	0

Figure 5.14.1. Values in the Parliament of Aragón.

that of its partner, in either option. However, considering that PP and PAR have the same number of seats and that PSOE has remarkably more seats than PAR, it seems that the PAR will have more ability to assert its greater power if P^2 is formed than if P^1 is formed. What eventually happened in Aragón is that the government union of PP and PAR was formed and the presidency of the government was then assumed by PAR, thus reflecting its greater power. $\diamond$

To finish this section, we would like to point out that the model of games with a priori unions can be generalized in a natural way to the model of *games with levels*. In a game with levels, there is a collection of nested partitions of the set of players that conditions their cooperation. Level structures were informally introduced in Owen (1977) and formally introduced and studied in Winter (1989).

5.15. Noncooperative Models in Cooperative Game Theory: Implementation Theory

As we already discussed in the previous sections, a standard way to evaluate different solution concepts for cooperative games is through axiomatic characterizations. *Noncooperative implementation* provides an alternative approach in which the details of some negotiation process among the players participating in the cooperative game are made explicit and then, the tools of noncooperative game theory are used to analyze the new situation. Noncooperative implementation is often referred to as the *Nash program*. In the words of Serrano (2008), "in game theory, 'Nash program' is the name given to a research agenda, initiated in Nash (1953), intended to bridge the gap between the cooperative and noncooperative approaches to the discipline". In this section we provide some examples of key developments in the above agenda. We refer the reader to Serrano (2005) for a comprehensive survey. In all the games we describe in this section we restrict attention to pure strategies.

Remark 5.15.1. Suppose that we want to implement the Shapley value of a given TU-game and consider the following noncooperative game. All the players simultaneously make a proposal. If all the proposals coincide with the Shapley value, then this is what they get; otherwise, they get some bad outcome. The payoff in the unique Nash equilibrium of this game coincides with the Shapley value. Of course, this mechanism would work for any solution and it does not help to get any new insight or motivation. The problem in this mechanism is that the designer needs to know the characteristic function (or at least the corresponding Shapley value) to verify that the players are actually following the rules. In general, one should look at mechanisms that can be monitored even by someone who does not know the particular parameters of the underlying cooperative game.[18] Moreover, one should try to find noncooperative games that are easy to describe and intuitive given the solution concept we are trying to implement.

5.15.1. A first implementation of the Nash bargaining solution. [19] In the seminal paper on the Nash program, Nash (1953) proposed a noncooperative negotiation process to support the Nash bargaining solution, which he had introduced in Nash (1950a). Given, $(F, d) \in B^N$, this negotiation process runs as follows:

- Simultaneously and independently, each player $i \in N$ chooses a demand, $a_i \geq d_i$.
- If the allocation $a = (a_1, \ldots, a_n)$ is feasible, i.e., $a \in F$, then each player receives his demand. Otherwise there is no agreement and each player $i \in N$ gets d_i.

This corresponds with the strategic game $G^{\text{NA}} = (N, A, u)$, with $A_i := \{a_i : a_i \geq d_i\}$ and, for each $i \in N$,

$$u_i(a) := \begin{cases} a_i & a \in F \\ d_i & \text{otherwise.} \end{cases}$$

Proposition 5.15.1. *The set of pure Nash equilibrium payoff vectors of the negotiation game G^{NA} consists of all Pareto efficient allocations of F along with the disagreement point.*

Proof. Let $a \in A$. If a belongs to F but it is not Pareto efficient, then there are $i \in N$ and $\hat{a}_i \in A_i$ such that $(a_{-i}, \hat{a}_i) \in F$ and $\hat{a}_i = u_i(a_{-i}, \hat{a}_i) > u_i(a) = a_i$. Hence, no such strategy profile is a Nash equilibrium of G^{NA}. Suppose

[18] A related issue is that of the representation of the preferences of the players. The predictions of the proposed mechanism should be essentially unaffected by changes in the representation of the preferences of the players. For instance, a change in the units in which we measure the utility of a player (i.e., a rescaling of his utility function), should induce an equivalent change in the predicted payoffs. A simple way of doing this is defining the mechanism directly over the space of outcomes, instead of doing it over the set of payoffs induced by some utility representation.

[19] In this subsection we partially follow the exposition in van Damme (1991, Section 7.5).

that a is such that $|\{i \in N : a_i > \max_{\hat{a}\in F} u_i(\hat{a})\}| \geq 2$. Then, it is straightforward to check that a is a pure Nash equilibrium of G^{NA} with payoff d. Suppose that a is a Pareto efficient allocation. For each, $i \in N$ and each $\hat{a} \in A_i$, $a_i = u_i(a) \geq u_i(a_{-i}, \hat{a}_i)$; indeed, if $\hat{a}_i \neq a_i$, $u_i(a) > u_i(a_{-i}, \hat{a}_i)$. Hence, a is a (strict) pure Nash equilibrium whose payoff vector is a. □

Hence, although the Nash solution is a Nash equilibrium of game G^{NA}, the multiplicity of equilibria poses a serious problem. Actually, all the efficient Nash equilibria are strict equilibria, which is the more demanding refinement of Nash equilibrium that we have discussed in this book. Yet, Nash proposed an elegant way to refine the set of equilibria of the above game. He proposed, in his own words, "to smooth the game G^{NA} to obtain a continuous payoff function and then to study the equilibrium outcomes of the smoothed game as the amount of smoothing approaches 0". For each $\varepsilon > 0$, let B^{ε} be the set of functions $f : A \to (0, 1]$ such that

i) the function f is continuous,
ii) for each $a \in F$, $f(a) = 1$, and
iii) for each $a \notin F$, if $\min_{\hat{a}\in F} \sum_{i\in N}(a_i - \hat{a}_i)^2 > \varepsilon$, i.e., if the distance between a and F is greater than ε, then

$$\max\{f(a), \prod_{i\in N}(a_i - d_i)f(a)\} < \varepsilon.$$

Let $B := \bigcup_{\varepsilon>0} B^{\varepsilon}$. We denote by $G^{\mathrm{NA}}(f)$ the strategic game (N, A, u^f), where, for each $i \in N$,

$$u_i^f(a) := d_i + (a_i - d_i)f(a),$$

Condition iii) above implies that $f(a)$ decreases as a moves away from F. Moreover, as ε goes to 0, the functions in B^{ε} become steeper and, hence, the game $G^{\mathrm{NA}}(f)$ gets closer and closer to the original game G^{NA}. One interpretation of the game $G^{\mathrm{NA}}(f)$ might be that the players think that the feasible set may actually be larger than F; this uncertainty being captured by the function f, which would then represent the probability that a given pair of demands belongs to the real feasible set; just note that $d_i + (a_i - d_i)f(a) = a_i f(a) + d_i(1 - f(a))$.

Definition 5.15.1. Let a^* be a Nash equilibrium of the Nash negotiation game G^{NA}. Consider a pair of sequences $\{\varepsilon_k\}_{k\in\mathbb{N}} \to 0$ and $\{f^{\varepsilon_k}\}_{k\in\mathbb{N}}$, where, for each $k \in \mathbb{N}$, $\varepsilon_k > 0$ and $f^{\varepsilon_k} \in B^{\varepsilon_k}$. Then, a^* is a *B-essential* equilibrium if there is a third sequence $\{a^{\varepsilon_k}\} \subset A$ such that $\{a^{\varepsilon_k}\} \to a^*$ and, for each $k \in \mathbb{N}$, a^{ε_k} is a Nash equilibrium of $G^{\mathrm{NA}}(f^{\varepsilon_k})$.[20]

[20]The name B-essential equilibrium comes from the similarity with the notion of *essential equilibrium* (Wen-Tsun and Jia-He 1962). The difference is that the latter allows for any kind of trembles in the

Theorem 5.15.2. *The Nash solution is a B-essential equilibrium of the negotiation game* G^{NA}.

Proof. Let $z := \mathrm{NA}(F,d)$ and recall that $\prod_{i\in N}(z_i - d_i) > 0$. Suppose that $0 < \varepsilon < \prod_{i\in N}(z_i - d_i)$. Then, for each $f^\varepsilon \in B^\varepsilon$ there is $\bar{a}^\varepsilon \in A$ that maximizes the function $\prod_{i\in N}(a_i - d_i) f^\varepsilon(a)$. For each $a \in A$ such that $\min\{\sum_{i\in N}(a_i - \hat{a}_i)^2 : \hat{a} \in F\} > \varepsilon$, we have $\prod_{i\in N}(a_i - d_i) f^\varepsilon(a) < \varepsilon < \prod_{i\in N}(z_i - d_i) f^\varepsilon(z) = \prod_{i\in N}(z_i - d_i)$. Hence, $\min\{\sum_{i\in N}(\bar{a}^\varepsilon_i - \hat{a}_i)^2 : \hat{a} \in F\} \leq \varepsilon$, i.e., $\bar{a}^\varepsilon$ gets arbitrarily close to F as ε goes to 0. For each $a_i \in A_i$, $\big(\prod_{j\neq i}(\bar{a}^\varepsilon_j - d_j)\big)(a_i - d_i) f^\varepsilon(\bar{a}^\varepsilon_{-i}, a_i) \leq \prod_{j\in N}(\bar{a}^\varepsilon_j - d_j) f^\varepsilon(\bar{a}^\varepsilon)$ and, since $\bar{a}^\varepsilon > d$, we have

$$(a_i - d_i) f^\varepsilon(\bar{a}^\varepsilon_{-i}, a_i) \leq (\bar{a}^\varepsilon_i - d_i) f^\varepsilon(\bar{a}^\varepsilon).$$

Therefore, $\bar{a}^\varepsilon$ is a Nash equilibrium of $G^{\mathrm{NA}}(f^\varepsilon)$. Since, for each $a \notin F$, $0 < f^\varepsilon(a) \leq 1$ and, for each $a \in F$, $f^\varepsilon(a) = 1$, then

$$\prod_{i\in N}(z_i - d_i) = \prod_{i\in N}(z_i - d_i) f^\varepsilon(z) \leq \prod_{i\in N}(\bar{a}^\varepsilon_i - d_i) f^\varepsilon(\bar{a}^\varepsilon) \leq \prod_{i\in N}(\bar{a}^\varepsilon_i - d_i).$$

Recall that, within F, the product $\prod_{i\in N}(a_i - d_i)$ is maximized at $a = z$. Now, as ε goes to 0 the distance between $\bar{a}^\varepsilon$ and F goes to 0 and, hence, $\lim_{\varepsilon\to 0} \bar{a}^\varepsilon = z$. □

Remark 5.15.2. The above proof shows that all the points at which the function $\prod_{i\in N}(a_i - d_i) f(a)$ gets maximized are equilibrium points of the game $G^{\mathrm{NA}}(f)$. Moreover, as ε goes to zero, all these equilibrium points converge to the Nash solution. However, there might be other equilibrium points of the $G^{\mathrm{NA}}(f)$ games, which might converge to a point in F other than the Nash solution. One may wonder if the Nash solution is actually the unique B-essential equilibrium of G^{NA} or not. Nash informally argued that if the smoothing function f goes to zero with some regularity, then the corresponding smoothed games would have a unique equilibrium, which would correspond with the unique maximum of $\prod_{i\in N}(a_i - d_i) f(a)$. Hence, in such a case, the Nash bargaining solution would be the unique B-essential equilibrium of Nash negotiation game. A pair of formalizations of this statement and the corresponding proofs can be found in (Binmore 1987a,b) and (van Damme 1991).

5.15.2. A second implementation of the Nash bargaining solution. Now, we present the implementation based on the *alternating offers game* developed by Rubinstein (1982).[21] A deep discussion on the connections between

utility function of the game, whereas the former only considers trembles defined via the functions in B.

[21] A very similar game had already been introduced years earlier in Ståhl (1972, 1977). The main difference is that Ståhl's model restricts attention to finite horizon games, whereas Rubinstein's approach does not.

the alternating offers game and the Nash bargaining solution can be found in Binmore et al. (1986). This implementation has become much more popular than Nash's implementation, essentially because it is done through an extensive game and, therefore, it can capture some dynamic aspects of negotiation that Nash's static approach cannot. For the sake of exposition, we present a simplified version of the two-player model in Rubinstein (1982). Moreover, extensions of Rubinstein's results to n-players can be found in Chae and Yang (1994) and Krishna and Serrano (1996).

The alternating offers game. There are two players bargaining to share a pie. Let $R := \{(r_1, r_2) : r_1 \geq 0, r_2 \geq 0, \text{ and } r_1 + r_2 \leq 1\}$ denote the set of possible shares of the pie. The game starts in period 0, in which player 1 proposes a share of the pie $(r_1, r_2) \in R$. This proposal can be either accepted or rejected by player 2. Upon acceptance, the proposed share is enforced and the game is over. If the proposal is rejected, period 1 starts, with player 2 making a proposal and player 1 deciding whether to accept or not. As long as no offer has been accepted, player 1 proposes in the even periods and player 2 in the odd ones. Moreover, player i's preferences over the different shares are represented by the utility function $u_i\colon [0,1] \to \mathbb{R}$, which is such that $u_i(0) = 0$. The players are not perfectly patient, in the sense that they discount future utilities according to a discount factor $\delta \in (0,1)$, that is, if share (r_1, r_2) is accepted in period t, the payoffs to the players are given by $(u_1(r_1)\delta^t, u_2(r_2)\delta^t)$. The discount factor δ may not necessarily be attributed to patience. It can also be interpreted as the probability that negotiations continue after a rejection. The u_i functions are assumed to be strictly increasing, concave, and continuous. The first assumption just reflects the fact that the pie is a desirable good. The other two assumptions ensure that the set $F := \{(u_1(r_1), u_2(r_2)) : (r_1, r_2) \in R)\}$ is convex and compact.

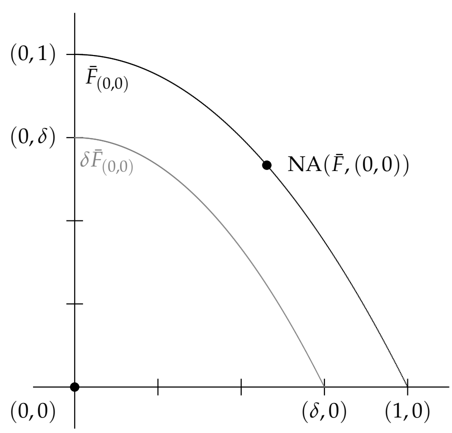


Figure 5.15.1. The bargaining set associated with an alternating offers game.

If we now define $\bar{F}$ as the comprehensive hull of F, we get that $(\bar{F}, (0,0))$ is a well defined bargaining problem. Recall that $\bar{F}_{(0,0)} := \{x \in \bar{F} : x \geq (0,0)\}$. In addition, there is no loss of generality in assuming the normalizations $u_1(1) = u_2(1) = 1$. For the sake of exposition, we do not explicitly describe all the elements of the alternating offers game such as strategy profiles and histories. In Figure 5.15.1 we represent an example of a bargaining set $\bar{F}_{(0,0)}$ with its Nash solution.

Remark 5.15.3. Although it may seem that the class of bargaining sets that can be defined through alternating offers games is quite narrow, this is not the case. The normalizations on the utility functions and the choice of the disagreement point are virtually without loss of generality. On the other hand, despite the symmetry of the strategy sets, the shape of the bargaining set still depends on the utility functions of the players, which are allowed to be very general.[22]

To familiarize the reader with the model and some of the intuitions behind the ensuing implementation, we start by solving a couple of simplifications of the alternating offers game. In both examples we consider that, for each $i \in \{1,2\}$ and each $r_i \in [0,1]$, $u_i(r_i) = r_i$. Moreover, there is a finite-horizon. If no offer is accepted in the first T periods, then the game is over and both players get 0.

Example 5.15.1. (The ultimatum game).[23] Suppose that $T = 0$, that is, if the proposal of player 1 in period 0 is not accepted, then both players get 0. It is easy to see that this game has an infinite number of Nash equilibria. We now show that all but one are based on incredible threats. More precisely, we show that there is a unique subgame perfect equilibrium. We proceed backward. In any subgame perfect equilibrium, when asking whether to accept or not, player 2 will accept any share (r_1, r_2) with $r_2 > 0$. However, no such share can be part of a subgame perfect equilibrium since, given proposal (r_1, r_2) with $r_2 > 0$, player 1 can profitably deviate by offering $(r_1 + \frac{r_2}{2}, \frac{r_2}{2})$. Then, in any subgame perfect equilibrium, player 1 proposes the share $(1,0)$. Player 2 is then indifferent between accepting and rejecting. If he accepts, we have a subgame perfect equilibrium. If he rejects, we would not have an equilibrium, since player 1 would deviate to some proposal (r_1, r_2) with $r_2 > 0$, which, as we have already seen, would be

[22] Note that instead of having the players offering shares of the pie, we could have equivalently defined the alternating offers game with the proposals belonging to a given bargaining set. Actually, this is the approach taken in Myerson (1991). He shows that, for Rubinstein's result to go through, he just needs to impose a mild regularity condition on the bargaining set; this regularity condition plays the role of our assumptions on the utility functions.

[23] This game is often called divide-the-dollar game.

accepted. Therefore, in the unique subgame perfect equilibrium of the ultimatum game, player 1 proposes the share $(1,0)$ and player 2 accepts. ◇

Example 5.15.2. (A simple bargaining game). Suppose that $T = 1$, i.e., the game has two periods, labeled 0 and 1. Again, we proceed backward to show that this game also has a unique subgame perfect equilibrium. If period 1 is reached, the corresponding subgame is an ultimatum game. Thus, in equilibrium, if period 1 is reached, player 2 offers the share $(0,1)$ and player 1 accepts, which leads to payoff 0 for player 1 and δ for player 2. Hence, in period 0, player 1 knows that player 2 will accept any offer (r_1, r_2) with $r_2 > \delta$. However, repeating the arguments above, no such share can be part of a subgame perfect equilibrium since, given proposal (r_1, r_2) with $r_2 > \delta$, player 1 can profitably deviate by offering $(r_1 + \frac{r_2-\delta}{2}, r_2 - \frac{r_2-\delta}{2})$. Thus, it is easy to see that the unique subgame perfect equilibrium of this game has player 1 offering the share $(1-\delta, \delta)$ and player 2 accepting the offer. Again, in order to have a subgame perfect equilibrium, the player who accepts the offer has to be indifferent between accepting and rejecting. Note the role of patience in this example. If the players are very patient, then player 2 gets almost 1, whereas if the players are very impatient, then player 1 gets almost all the pie (actually, in this case the argument only depends on the patience of player 2). ◇

Note that an allocation $x \in \bar{F}$ is Pareto efficient in $\bar{F}$ if and only if $x = (u_1(r_1), u_2(r_2))$, with $(r_1, r_2) \in R$ and $r_1 + r_2 = 1$. The next result highlights the importance of having the parameter δ introducing some "friction" in the bargaining process, in the sense that delay is costly.

Proposition 5.15.3. *Consider the alternating offers game with perfectly patient players (i.e., $\delta = 1$). Then, an allocation $x \in \bar{F}_{(0,0)}$ is a subgame perfect equilibrium payoff if and only if x is Pareto efficient.*

Proof. Let $(r_1, r_2) \in R$ be such that $x = (u_1(r_1), u_2(r_2))$ is Pareto efficient in $\bar{F}$, i.e., $r_1 + r_2 = 1$. Consider the following strategy profile. Both players, when playing in the role of the proposer, propose the share (r_1, r_2). When asked to accept or reject, player i accepts any offer that gives him at least r_i and rejects any other proposal. It is easy to verify that this strategy profile is a subgame perfect equilibrium of the game when the players are perfectly patient. Conversely, suppose that we have a subgame perfect equilibrium delivering an inefficient allocation x. Let $(r_1, r_2) \in R$ be such that $x = (u_1(r_1), u_2(r_2))$ and $r_1 + r_2 < 1$. Consider the proposal $(\bar{r}_1, \bar{r}_2)$ defined, for each $i \in \{1,2\}$, by $\bar{r}_i := r_i + \frac{1-r_1-r_2}{2}$. Then, there has to be a subgame where the proposer at the start of the subgame can profitably deviate by proposing $(\bar{r}_1, \bar{r}_2)$ instead of (r_1, r_2). □

One of the main virtues of Rubinstein's game is that any degree of impatience reduces the set of subgame perfect equilibrium payoff vectors to a single allocation. Moreover, it can be easily checked that even for impatient players, the alternating offers game has a continuum of Nash equilibria. Hence, it is worth noting the strength of subgame perfect equilibrium as a refinement of Nash equilibrium in this particular game. Consider the strategy profile in which player 1 always proposes a share $r^1 \in R$ and player 2 always proposes a share $r^2 \in R$. Player 1 accepts any offer that gives him at least $u_1(r_1^2)$ and player 2 accepts any offer that gives him at least $u_2(r_2^1)$. Note, in particular, that the action of a player at a given period depends neither on the past history of play nor on the number of periods that have elapsed. Moreover, according to this strategy, the first proposal is accepted, so the impatience of the players does not lead to inefficient allocations. Can there be a subgame perfect equilibrium characterized by this simple behavior? The answer is positive and, in addition, the unique subgame perfect equilibrium of the alternating offers game is of this form. Let (s_{r^1}, s_{r^2}) denote the above strategy profile.

Theorem 5.15.4. *The alternating offers game with discount $\delta \in (0,1)$ has a unique subgame perfect equilibrium (s_{r^1}, s_{r^2}), where the shares r^1 and r^2 lead to Pareto efficient allocations satisfying that*

$$u_1(r_1^2) = \delta u_1(r_1^1) \quad \textit{and} \quad u_2(r_2^1) = \delta u_2(r_2^2).$$

In particular, the first proposal is accepted and the equilibrium payoff vector is given by $(u_1(r_1^1), u_2(r_2^1))$.

Proof.[24] Note that all the subgames that begin at an even period (with player 1 offering first) have the same set of subgame perfect equilibria as the whole game. Similarly, all the subgames that begin at an odd period (with player 2 offering first) have the same set of subgame perfect equilibria. Let M_1 and m_1 be the supremum and the infimum of the set of subgame perfect equilibrium payoffs for player 1 in subgames where he proposes first. Similarly, let L_2 and l_2 be the supremum and the infimum of the set of subgame perfect equilibrium payoffs for player 2 in subgames where he proposes first. We now show that $M_1 = m_1$ and $L_2 = l_2$. We also need to define two auxiliary functions h_1 and h_2. Given $z \in [0,1]$, let $h_1(z) := \max\{x_1 : (x_1, z) \in \bar{F}_{(0,0)}\}$ and $h_2(z) := \max\{x_2 : (z, x_2) \in \bar{F}_{(0,0)}\}$, i.e., $h_i(z)$ denotes the maximum utility player i can get conditional on the other player getting z.

[24] This proof is an adaptation of the arguments in the proof of Theorem 8.3 in Myerson (1991).

In any subgame perfect equilibrium, player 2 always accepts any offer giving him more than δL_2, which is the best payoff he can get (in equilibrium) by rejecting the present proposal. Hence, in any equilibrium where player 1 is the first proposer he gets at least $h_1(\delta L_2)$. Moreover, by the definition of L_2, for each $\varepsilon > 0$, there is a subgame perfect equilibrium of the alternating offers game in which player 2 gets at least $L_2 - \varepsilon$ in all the subgames starting at period 1. In such an equilibrium, player 2 can ensure for himself at least payoff $\delta(L_2 - \varepsilon)$ by rejecting the proposal in period 0. Therefore, in this equilibrium player 1 gets, at most, $h_1(\delta(L_2 - \varepsilon))$. Since we have already seen that $h_1(\delta L_2)$ is a lower bound for the payoff of player 1 in any equilibrium when he is the first proposer, by letting ε go to 0, we get that $m_1 = h_1(\delta L_2)$. By a similar argument, we get $l_2 = h_2(\delta M_1)$.

In any subgame perfect equilibrium, player 2 never accepts any offer giving him less than δl_2, since (in equilibrium) he can get δl_2 by rejecting the present proposal. Hence, in any equilibrium where player 1 is the first proposer he gets, at most, $h_1(\delta l_2)$. Moreover, by the definition of l_2, for each $\varepsilon > 0$, there is a subgame perfect equilibrium of the alternating offers game in which player 2 gets at most $l_2 + \varepsilon$ in all the subgames starting at period 1. Therefore, in this equilibrium player 1 gets at least $h_1(\delta(l_2 + \varepsilon))$. Again, by letting ε go to 0, we get that $M_1 = h_1(\delta l_2)$. By a similar argument, we get $L_2 = h_2(\delta m_1)$.

We complete the definitions of the allocations M, m, L, and l. Let $M_2 := h_2(M_1)$, $m_2 := h_2(m_1)$, $L_1 := h_1(L_2)$, and $l_1 := h_1(l_2)$. So defined, these four vectors are Pareto efficient. We claim that, by construction, $M_2 = \delta l_2$, $l_1 = \delta M_1$, $m_2 = \delta L_2$, and $L_1 = \delta m_1$. We do the proof for M_2, with the other cases being analogous. Note that $M_2 = h_2(h_1(\delta l_2))$. By definition, $(h_1(\delta l_2), \delta l_2)$ and $(h_1(\delta l_2), h_2(h_1(\delta l_2)))$ are Pareto efficient allocations in $\bar{F}_{(0,0)}$, but this can only happen if $h_2(h_1(\delta l_2)) = \delta l_2$. Hence, we have

$$M_2 = \delta l_2 \quad \text{and} \quad l_1 = \delta M_1, \tag{5.15.1}$$

$$m_2 = \delta L_2 \quad \text{and} \quad L_1 = \delta m_1. \tag{5.15.2}$$

We now show that there is a unique pair of Pareto efficient allocations x and y in $\bar{F}_{(0,0)}$ such that (see Figure 5.15.2)

$$y_1 = \delta x_1 \quad \text{and} \quad x_2 = \delta y_2.$$

First, note that the above equations can be equivalently rewritten as

$$x_1 - y_1 = (1 - \delta)x_1 \quad \text{and} \quad y_2 - x_2 = \frac{(1 - \delta)x_2}{\delta}.$$

Note that if $x_1 = 0$, then $x_1 - y_1 = 0$. Since both x and y are Pareto efficient, $x_1 - y_1 = 0$ implies that $y_2 - x_2 = 0$. Hence, based on the above

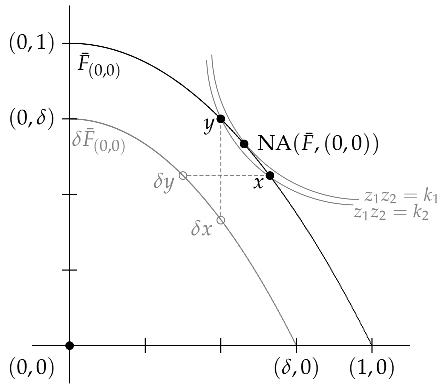


Figure 5.15.2. Subgame perfect equilibrium payoff vectors for a given δ; x is the unique equilibrium payoff vector in the subgames where player 1 is the first proposer and, similarly, y is the unique equilibrium payoff vector in the subgames where player 2 is the first proposer.

equations, monotonically increasing x_1 from 0 to 1 has the following effects. First, $x_1 - y_1$ also increases monotonically from 0. Second, since $x_1 - y_1$ increases, using Pareto efficiency again, $y_2 - x_2$ also increases monotonically from 0.[25] Third, x_2 decreases monotonically from 1 to 0. Therefore, since everything varies continuously, by monotonically increasing x_1 from 0 to 1, there is exactly one value at which the equation $y_2 - x_2 = \frac{(1-\delta)x_2}{\delta}$ is satisfied. Hence, Eqs. (5.15.1) and (5.15.2) immediately imply that $M = m = x$ and $L = l = y$. Therefore, x is the unique equilibrium payoff in each subgame in which player 1 is the first proposer. Similarly, y is the unique equilibrium payoff in each subgame in which player 2 is the first proposer.[26]

Let r^1 and r^2 be the unique shares in R such that $(u_1(r^1_1), u_2(r^1_2)) = x$ and $(u_1(r^2_1), u_2(r^2_2)) = y$, respectively. It is now straightforward to check that (s_{r^1}, s_{r^2}) is a subgame perfect equilibrium of the alternating offers game.

We have already proved the uniqueness of the equilibrium payoff vectors. What is left to show is the uniqueness of the equilibrium strategies. Suppose there is another subgame perfect equilibrium of the game with the property that x is the payoff vector at all subgames where player 1 is the first proposer, and y is the payoff vector at all subgames where player 2 is the first proposer. By the above property, in each subgame perfect equilibrium, in each subgame, player 2 accepts any proposal giving him more

[25]In other words, efficiency implies that, the better player 1 is in x relative to y, the better player 2 is in y relative to x.

[26]Note that we may not get $y_1 = x_2$ and $x_2 = y_1$. Whether these extra equalities are satisfied depends on the shape of the Pareto efficient boundary of $\bar{F}$. For instance, for the bargaining set depicted in Figure 5.15.2, we have that $y_2 > x_1$ and $x_2 > y_1$.

than r_2^1 and rejects any proposal giving him less. Moreover, player 2 also accepts when the proposal gives him r_2^1 since, if he were to reject such an offer at a given subgame, then player 1 would profitably deviate by proposing an efficient allocation giving player 2 something slightly above r_2^1.[27] Similarly, in each subgame perfect equilibrium, player 1 accepts a proposal if and only if it gives him at least r_1^2. Suppose now that there is a subgame in which player 1 is the first proposer and offers less than r_1^2 to player 2; player 2 rejects the proposal and, hence, player 1 gets, at most, δr_1^2. Thus, player 1 can profitably deviate in that subgame by making the proposal r^1, which would be accepted by player 2, and get share r_1^1. Therefore, in equilibrium, player 1 never offers less than r_2^1 to player 2. Also, player 1 never offers more than r_1^2 to player 2, since such an offer would be accepted and player 1 would be better off by proposing r^1 instead. Hence, the only option is that player 1 makes proposal r^1 whenever he is the proposer. Similarly, player 2 makes proposal r^2 whenever he is the proposer. □

The above result highlights a natural property of the equilibrium shares r^1 and r^2. Since $u_1(r_1^2) = \delta u_1(r_1^1)$ and $u_2(r_2^1) = \delta u_2(r_2^2)$, the first mover has an advantage in the alternating offers game. Moreover, this advantage is larger the more impatient the players are. Actually, in the limit, as both players become perfectly patient, the first mover advantage disappears.

In spite of the strength of Theorem 5.15.4, we have not yet found any link between the equilibrium payoff vectors of the alternating offers game and the Nash solution. Actually, Figure 5.15.2 represents a situation in which the equilibrium payoff vectors do not coincide with the Nash solution. Nonetheless, the product of the gains of the players with respect to the disagreement point, the one that the Nash solution maximizes, is implicit in the conditions

$$u_1(r_1^2) = \delta u_1(r_1^1) \quad \text{and} \quad u_2(r_2^1) = \delta u_2(r_2^2).$$

To see why, just note that these conditions imply that

$$\frac{u_1(r_1^2)}{u_1(r_1^1)} = \frac{u_2(r_2^1)}{u_2(r_2^2)} = \delta$$

and, therefore, $u_1(r_1^1)u_2(r_2^1) = u_1(r_1^2)u_2(r_2^2)$ (this feature is illustrated in Figure 5.15.2). Since the disagreement point in the alternating offers game is $(0,0)$, the Nash solution is the share $r^* \in R$ that maximizes the above product of utilities. The result below shows that, as the players become more and more patient, the proposals r^1 and r^2 collapse into r^*. Therefore, the more patient the players are, the closer the unique equilibrium payoff vector is to the Nash solution.

[27] This is exactly the same argument we made for the indifferences in Examples 5.15.1 and 5.15.2.

Corollary 5.15.5. *The equilibrium allocation of the alternating offers game converges to the Nash solution of the bargaining problem* $(\bar{F}, (0,0))$ *as* δ *converges to* 1.

Proof. Given an alternating offers game with discount factor δ, let x^δ be the equilibrium payoff vector when player 1 proposes first, and let y^δ be the equilibrium payoff vector when player 2 proposes first. These payoff vectors satisfy that $x_1^\delta x_2^\delta = y_1^\delta y_2^\delta$. Recall that, by definition, the Nash solution is defined as the unique maximizer of the previous product inside the set $\bar{F}$. Hence, for each discount factor δ, the Nash solution lies in between the corresponding equilibrium allocations x^δ and y^δ (see Figure 5.15.2). Moreover, since $y_1^\delta = \delta x_1^\delta$ and $x_2^\delta = \delta y_2^\delta$, as δ goes to 1, $x_1^\delta - y_1^\delta$ and $x_2^\delta - y_2^\delta$ converge to 0. Therefore, as δ goes to 1, both x^δ and y^δ converge to the Nash solution. □

The relevance of this last result is nicely summarized in the following passage taken from Serrano (2005): "This remarkable result provides a sharp prediction to the bilateral bargaining problem. But unlike Nash's simultaneous-move game, Rubinstein's uniqueness is a result of the use of credible threats in negotiations, together with the assumption of impatience (or breakdown probabilities in the negotiations)".

5.15.3. Implementing the Kalai-Smorodinsky solution. We now present an extensive game introduced by Moulin (1984) to implement the Kalai-Smorodinsky solution. Although the original paper considered general n-player bargaining problems, we restrict attention to the two-player case. Moulin (1984) considered the following noncooperative negotiation process, given by a three-stage extensive game that we call Γ^{KS}.

Stage 1: Simultaneously and independently, each player $i \in \{1,2\}$ submits a bid a_i, with $0 \leq a_i \leq 1$.

Stage 2: The player with the highest bid becomes the proposer. If $a_1 = a_2$, the proposer is randomly chosen. Say i is the proposer and j is not. Player i proposes an allocation $x \in F$. If player j accepts, then the outcome x is realized. If player j rejects x, then with probability $1 - a_i$ the final outcome is the disagreement point and with probability a_i we go to Stage 3.

Stage 3: Player j proposes an allocation. If it is accepted by player i, the proposal is realized, otherwise, the realized outcome is the disagreement point.

In this game both players bid to have the advantage of being the first proposer. Note that, whenever a player is given the chance to make an offer at Stage 3, he is actually a dictator, since the opponent will have to choose between the offer and the disagreement point. Hence, the higher the winning bid is, the less advantageous it is to win the auction, since the probability that the other player becomes a dictator in case of rejection in Stage 2 becomes larger.[28]

Proposition 5.15.6. *Let* $(F, d) \in B^2$ *and let* $x \in F_d$. *Then, there is a Nash equilibrium of* Γ^{KS} *whose payoffs equal* x.

Proof. Let $x \in F_d$. Consider the following strategy profile:

Stage 1: Both players bid 0.

Stage 2: The proposer chooses x. The player who is not the proposer accepts x and rejects any other proposal.

Stage 3: The proposer chooses the disagreement point.

This is a Nash equilibrium of Γ^{KS} and the payoffs are given by x. □

The above result implies that Nash equilibrium does not discriminate among the outcomes in F_d. In addition, in the subgames where player i is a dictator, i.e., when he is given the chance to propose at Stage 3, he can enforce any feasible outcome. Hence, the Nash equilibrium described in Proposition 5.15.6 is not subgame perfect. The next result shows that under a reasonably mild condition, the use of subgame perfect equilibrium is enough to pin down a unique outcome, which, moreover, coincides with the Kalai-Smorodinsky solution.

Recall that, given a bargaining problem $(F, d) \in B^2$, $b(F, d)$ is the vector of utopia payoffs of the different players. Consider the following assumption, which we call A1. Let i and j be the two players in (F, d). For each $x \in F_d$, if $x_i = b_i(F, d)$, then $x_j = d_j$. This assumption is met, for instance, when the players are bargaining on how to share some desirable good. If one player gets everything, then the other player gets nothing, which would give him the same payoff as if no agreement were reached. In the two bargaining problems in Figure 5.3.2 (Section 5.3), the left one satisfies A1 and the right one does not.

Theorem 5.15.7. *Let* $(F, d) \in B^2$ *be a bargaining problem satisfying* A1. Then, the Kalai-Smorodinsky solution of (F, d) is the unique subgame perfect equilibrium payoff of game Γ^{KS}.

Proof. Since the Kalai-Smorodinsky solution satisfies CAT, we can restrict attention, without loss of generality, to bargaining problems where, for each $i \in N$, $d_i = 0$ and $b_i(F, d) = 1$. In a game where the utilities have been normalized in this way, the Kalai-Smorodinsky solution is the unique Pareto efficient allocation that gives both players the same payoff.

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[28] The alternative game in which the probability of player j of becoming a proposer at Stage 3 is given by a_j instead of a_i would lead to the same results (see Exercise 5.17).

For each payoff x_2 that player 2 can get in F_d, let $g_1(x_2)$ be the maximum payoff that player 1 can get in F_d conditional on player 2 getting at least x_2. Define $g_2(x_1)$ in an analogous manner. Suppose that the bids at Stage 1 are such that $a_1 > a_2$. Then, player 2 can ensure himself payoff a_1 by rejecting the offer of player 1 and making offer $(0,1)$ if he is given the chance to (which happens with probability a_1). Moreover, by A1, the only way in which player 2 can offer 1 to himself is by offering 0 to player 1. Hence, the optimal offer of player 1 at Stage 2 is $(g_1(a_1), a_1)$. Therefore, by bidding a_1 at Stage 1, player 1 gets a_2 if $a_1 < a_2$, $g_1(a_1)$ if $a_1 > a_2$, and something in between a_1 and $g_1(a_1)$ if $a_2 = a_1$. Now, let a^* be the unique number such that (a^*, a^*) is efficient, i.e., (a^*, a^*) is the Kalai-Smorodinsky solution. We now claim that, $g_1(a^*) = a^*$. Suppose, on the contrary that there is $\alpha > 0$ such that $g_1(a^*) = a^* + \alpha$. Then, $(a^* + \alpha, a^*) \in F$. Also, $(0,1) \in F$ and, since $a^* + \alpha > 0$, A1 ensures that $a^* < 1$. Let $t := \frac{\alpha}{1+\alpha}$ and consider the allocation $t(0,1) + (1-t)((a^* + \alpha, a^*) = (\frac{a^*+\alpha}{1+\alpha}, \frac{a^*+\alpha}{1+\alpha})$. Since F is convex and $t \in (0,1)$, this allocation belongs to F and, moreover, since $\frac{a^*+\alpha}{1+\alpha} > a^*$, we get a contradiction with the fact that (a^*, a^*) is efficient. Then, since $g_1(a^*) = a^*$, player 1 is guaranteed payoff a^* by bidding a^*. A symmetric argument shows that player 2 is also guaranteed payoff a^* by bidding a^*. Hence, the unique candidate to be a subgame perfect equilibrium payoff of game Γ^{KS} is (a^*, a^*), i.e., the Kalai-Smorodinsky solution. One such subgame perfect equilibrium is the one in which both players submit bid a^* at Stage 0 and the proposal a^* is made and accepted at Stage 2. □

5.15.4. Implementing the Shapley value. Because of the popularity of the Shapley value, a large number of mechanisms implementing it have been discussed in the literature. Here, we restrict attention to the one proposed by Pérez-Castrillo and Wettstein (2001). Other relevant noncooperative approaches to the Shapley value can be found, for instance, in Gul (1989), Evans (1990), Hart and Moore (1990), Winter (1994), Hart and Mas-Colell (1996), Dasgupta and Chiu (1996) and, for more recent contributions, Mutuswami et al. (2004), and Vidal-Puga (2004). It is worth emphasizing the generality of the approach in Hart and Mas-Colell (1996), since they developed their analysis for NTU-games, where their noncooperative game implements a well known extension of the Shapley value. Thus, when restricting attention to TU-games, their approach implements the Shapley value and, moreover, when restricting attention to bargaining problems, it implements the Nash solution. Again, we refer the reader to Serrano (2005) for a detailed survey. Pérez-Castrillo and Wettstein (2001) proposed a bidding mechanism that implements the Shapley value in the class of zero-monotonic games (see Definition 5.4.8). Next, we describe the mechanism in a recursive way as the number of players increases. First, if there is a unique player, the player obtains his value. Suppose that the mechanism has been described when the number of players is at most $n-1$. Now, take a zero-monotonic game $(N,v) \in G^N$ with $N = \{1,\dots,n\}$. Consider the extensive game Γ^{Φ^S} defined as follows.

Stage 1: Simultaneously and independently, each player $i \in N$ submits offers to the players in $N\backslash\{i\}$. These offers are bids to become the proposer at Stage 2. The set of actions of player i at Stage 1 is $\mathbb{R}^{n-1}$. Let b^i_j denote the amount that player i offers to player $j \neq i$. For each $i \in N$, let

$$\alpha^i := \sum_{j\neq i} b^i_j - \sum_{j\neq i} b^j_i.$$

Let p be such that $\alpha^p = \max_{i\in N}\alpha^i$. If there are several maximizers, p is chosen randomly among them. Once p has been chosen, he pays b^p_i to each player $i \neq p$.

Stage 2: Player p makes an offer to any player other than p, that is, his strategy set is again $\mathbb{R}^{n-1}$. Suppose that the offer is a^p.

Stage 3: Players other than p sequentially accept or reject the offer made in Stage 2. The offer is accepted if and only if all the other players accept it. If the offer is rejected, player p leaves the game and players in $N\backslash\{p\}$ play the mechanism again. In this case, player p obtains $v(p) - \sum_{i\neq p} b^p_i$ and each player $i \neq p$ obtains b^p_i plus what he obtains in the ensuing game. If the offer is accepted, the game finishes. Each player $i \neq p$ obtains $b^p_i + a^p_i$ and player p obtains $v(N) - \sum_{i\neq p} b^p_i - \sum_{i\neq p} a^p_i$.

Theorem 5.15.8. *Let (N,v) be a zero-monotonic TU-game. Then, the Shapley value of (N,v) is the unique subgame perfect equilibrium payoff of game Γ^{Φ^S}.*

Proof. First, note that the formula for the Shapley value can be equivalently rewritten, for each $v \in G^N$ and each $i \in N$, as

$$\Phi^S_i(v) = \frac{1}{n}(v(N) - v(N\backslash\{i\})) + \frac{1}{n}\sum_{j\neq i}\Phi^S_i(v_{N\backslash\{j\}}). \tag{5.15.3}$$

We proceed by induction on the number of players. The case $n = 1$ is straightforward. Assume that the result holds for zero-monotonic games with at most $n-1$ players. Let (N,v) be a zero-monotonic game. For the sake of notation, for each $S \subset N$, we denote $\Phi^S(v_S)$ by $\Phi^S(S)$. Since (N,v) is zero-monotonic, all the v_S games are zero-monotonic as well.

Claim 1. The Shapley value is the payoff vector of some subgame perfect equilibrium of Γ^{Φ^S}.

Consider the following strategies.

- At Stage 1, each player $i \in N$ bids b^i, where, for each $j \neq i$ $b^i_j := \Phi^S_j(v) - \Phi^S_j(N\backslash\{i\})$.
- At Stage 2, if the proposer is p, he offers, to each $i \neq p$, $a^p_i := \Phi^S_i(N\backslash\{p\})$.
- At Stage 3, if the proposer is p, every player $i \neq p$ accepts any proposal greater or equal than $\Phi^S_i(N\backslash\{p\})$ and rejects any proposal smaller than $\Phi^S_i(N\backslash\{p\})$.

Under this strategy profile, if p is the proposer, then each player $i \neq p$ gets $\Phi^S_i(v) - \Phi^S_i(N\backslash\{p\}) + \Phi^S_i(N\backslash\{p\}) = \Phi^S_i(v)$. Then, since Φ^S satisfies EFF, the proposer p also gets $\Phi^S_p(v)$. For each $i \in N$,

$$\alpha^i = \sum_{j\neq i}(\Phi^S_j(v) - \Phi^S_j(N\backslash\{i\})) - \sum_{j\neq i}(\Phi^S_i(v) - \Phi^S_i(N\backslash\{j\})).$$

Now, the first sum above equals $\sum_{j\neq i}\Phi^S_j(v) - \sum_{j\neq i}\Phi^S_j(N\backslash\{i\}) = v(N) - \Phi^S_i(v) - v(N\backslash\{i\})$. By Eq. (5.15.3), the second sum equals $\sum_{j\neq i}\Phi^S_i(v) - \sum_{j\neq i}\Phi^S_i(N\backslash\{j\}) = (n-1)\Phi^S_i(v) - n\Phi^S_i(v) + v(N) - v(N\backslash\{i\})$. Therefore, $\alpha^i = 0$.

We now show that the above strategy profile is a subgame perfect equilibrium. The induction hypothesis ensures that no player $i \neq p$ can gain by deviating at Stage 3, whatever the offer made by p is. At Stage 2, if p offers $\hat{a}^p \in \mathbb{R}^{n-1}$, with $\hat{a}^p_i < \Phi^S_i(N\backslash\{p\})$ for some $i \neq p$, then i rejects the offer. In this case, p gets $v(p) - \sum_{i\neq p}(\Phi^S_i(v) - \Phi^S_i(N\backslash\{p\}))$, which, by Eq. (5.15.3), equals $v(p) - v(N) + \Phi^S_p(v) + v(N\backslash\{p\})$ and, by zero-monotonicity, this is at most $\Phi^S_p(v)$. Hence, this deviation is not profitable for p. If p's deviation is such that there is $i \neq p$ with $\hat{a}^p_i > \Phi^S_i(N\backslash\{p\})$, and there is no $i \neq p$ who gets offered less than $\Phi^S_i(N\backslash\{p\})$, then p strictly decreases his payoff. Finally, it is straightforward to check that deviations at Stage 1 are not profitable.

Claim 2. Any subgame perfect equilibrium of Γ^{Φ^S} yields the Shapley value.

In the rest of the proof, whenever an arbitrary subgame perfect equilibrium of Γ^{Φ^S} is considered, p denotes the proposer at Stage 2, b^p denotes his bid at Stage 1, and a^p denotes his offer at Stage 2. We proceed by proving several new claims.

Claim 2.1. In any subgame perfect equilibrium, all the players in $N\backslash\{p\}$ accept the offer if, for each $i \neq p$, $a^p_i > \Phi^S_i(N\backslash\{p\})$. Moreover, if there is $i \neq p$ with $a^p_i < \Phi^S_i(N\backslash\{p\})$, then the proposal is rejected.

Let $i \neq p$. By the induction hypothesis, if there is a rejection at Stage 3, then player i gets $b_i^p + \Phi_i^S(N\backslash\{p\})$. Let l be the last player who decides to accept or reject the offer. If l is given the chance to accept or reject, i.e., if there has been no prior rejection, at a best reply l accepts any offer above $\Phi_l^S(N\backslash\{p\})$ and rejects any offer below $\Phi_l^S(N\backslash\{p\})$. Let k be the second to last player. Since player k is also best replying, if $a_k^p > \Phi_k^S(N\backslash\{p\})$, $a_l^p > \Phi_l^S(N\backslash\{p\})$, and the game reaches k, then player k accepts the offer. If $a_k^p < \Phi_k^S(N\backslash\{p\})$ and $a_l^p > \Phi_l^S(N\backslash\{p\})$, then player k rejects the offer. If $a_l^p < \Phi_l^S(N\backslash\{p\})$, then player k is indifferent to rejecting or accepting since he knows that player l will reject anyway. Hence, we can just proceed backward and repeat the same argument to get the proof of Claim 2.1.

Claim 2.2. If $v(N) > v(N\backslash\{p\}) + v(p)$, then, starting at Stage 2, in any subgame perfect equilibrium we have:

- At Stage 2, player p offers, to each $i \neq p$, $a_i^p = \Phi_i^S(N\backslash\{p\})$.
- At Stage 3, each $i \neq p$ accepts any offer $\hat{a}^p \in \mathbb{R}^{n-1}$ with $\hat{a}_i^p \geq \Phi_i^S(N\backslash\{p\})$. Any offer that gives less than $\Phi_i^S(N\backslash\{p\})$ to some $i \neq p$ is rejected.

If $v(N) = v(N\backslash\{p\}) + v(p)$, there are other types of subgame perfect equilibria starting at Stage 2 in which:

- At Stage 2, player p offers $a^p \in \mathbb{R}^{n-1}$, with $a_i^p \leq \Phi_i^S(N\backslash\{p\})$ for some $i \neq p$.
- At Stage 3, each $i \neq p$ rejects any offer $\hat{a}^p \in \mathbb{R}^{n-1}$ with $\hat{a}_i^p \leq \Phi_i^S(N\backslash\{p\})$.

All the above equilibria lead to the same payoff vector. The proposer p gets $v(N) - v(N\backslash\{p\}) - \sum_{i\neq p} b_i^p$ and each $i \neq p$ gets $\Phi_i^S(N\backslash\{p\}) + b_i^p$.

It is now easy to see that all the strategy profiles we have described above are subgame perfect equilibria of the subgame starting at Stage 2. Suppose that we are at a subgame perfect equilibrium. First, suppose that $v(N) > v(N\backslash\{p\}) + v(p)$. If a player $i \neq p$ rejects the offer, then p gets $v(p) - \sum_{i\neq p} b_i^p$. In such a case, if p deviates and proposes, to each $i \neq p$, $\hat{a}_i^p = \Phi_i^S(N\backslash\{p\}) + \varepsilon/(n-1)$, with $0 < \varepsilon < v(N) - v(N\backslash\{p\}) - v(p)$, he increases his payoff because, by Claim 2.1, $\hat{a}^p$ is accepted at Stage 3. Therefore, at Stage 3, the proposal gets accepted in any subgame perfect equilibrium. This implies that player p offers, to each $i \neq p$, $a_i^p \geq \Phi_i^S(N\backslash\{p\})$. However, if there is $i \neq p$ such that $a_i^p > \Phi_i^S(N\backslash\{p\})$, then p increases his payoff by slightly decreasing the offer to i, but still offering him something above $\Phi_i^S(N\backslash\{p\})$. Hence, in any subgame perfect equilibrium, for each $i \neq p$ $a_i^p = \Phi_i^S(N\backslash\{p\})$. Finally, we have already shown that, at Stage 3, the offer gets accepted and, hence, every offer $a^p \in \mathbb{R}^{n-1}$ such that, for each $i \neq p$, $a_i^p \geq \Phi_i^S(N\backslash\{p\})$ gets accepted.

If $v(N) = v(N\backslash\{p\}) + v(p)$, then any proposal $a^p \in \mathbb{R}^{n-1}$ such that $\sum_{i \neq p} a_i^p < v(N\backslash\{p\})$ is rejected in any subgame perfect equilibrium (by Claim 2.1). Otherwise, by the induction hypothesis, each player i with $a_i^p < \Phi_i^S(N\backslash\{p\})$ increases his payoff by rejecting the offer. Thus, at Stage 3 an offer would be accepted if $\sum_{i \neq p} a_i^p \geq v(N\backslash\{p\})$. Paralleling the argument above, acceptance at Stage 3 implies that, for each $i \neq p$, $a_i^p = \Phi_i^S(N\backslash\{p\})$. Also, since p gets $v(p)$ in case of a rejection, any offer that leads to a rejection is also part of a subgame perfect equilibrium.

Claim 2.3. In any subgame perfect equilibrium, for each $i, j \in N$, $\alpha^i = \alpha^j$. Moreover, since $\sum_{i \in N} \alpha^i = 0$, we have that, for each $i \in N$, $\alpha_i = 0$.

For each $i \in N$, let $b^i \in \mathbb{R}^{n-1}$ be the bid of player i at Stage 1 in a subgame perfect equilibrium. Let $S := \{i \in N : \alpha^i = \max_{j \in N} \alpha^j\}$. If $S = N$, since $\sum_{j \in N} \alpha^j = 0$, we have that, for each $i \in N$, $\alpha^i = 0$. Suppose that $S \subsetneq N$. Let $k \notin S$ and let $i \in S$. Let $\varepsilon > 0$ and consider the bid of player i given by $\hat{b}^i \in \mathbb{R}^{n-1}$, where, for each $l \in S\backslash\{i\}$, $\hat{b}_l^i = b_l^i + \varepsilon$, $\hat{b}_k^i = b_k^i - |S|\varepsilon$, and, otherwise, $\hat{b}_l^i = b_l^i$. Hence, for each $l \in S$, $\hat{\alpha}^l = \alpha^l - \varepsilon$, $\hat{\alpha}^k = \alpha^k + |S|\varepsilon$, and, otherwise, $\hat{\alpha}^l = \alpha^l$. Therefore, if ε is small enough, the set S does not change. However, player i increases his payoff, since

$$\sum_{l \neq i} \hat{b}_l^i = \sum_{l \neq i} b_l^i - \varepsilon.$$

Claim 2.4. In any subgame perfect equilibrium, independently of who is chosen as the proposer, each player gets the same payoff.

By Claim 2.3, we know that, for each $i, j \in N$, $\alpha^i = \alpha^j$. Note that Claim 2.2 implies that if the proposer pays b^p at Stage 1, then the rest of the payoffs only depend on who the proposer is, i.e., they are independent of the specific bids made at Stage 1. Hence, if a player i had strict incentives to be the proposer, he could increase his payoff by slightly increasing his bid. Similarly, if i strictly preferred player $j \neq i$ to be the proposer, he could increase his payoff by slightly decreasing b_j^i. Therefore, in any equilibrium all the players have to be indifferent to the identity of the proposer.

Claim 2.5. In any subgame perfect equilibrium, the realized payoff coincides with the Shapley value.

Consider a subgame perfect equilibrium and let $i \in N$. By Claim 2.1, if i is the proposer then he gets $x_i^i = v(N) - v(N\backslash\{i\}) - \sum_{k \neq i} b_k^i$; if the proposer is $j \in N\backslash\{i\}$, then player i gets $x_i^j = b_i^j + \Phi_i^S(N\backslash\{j\})$. Then, using that $\alpha_i = 0$ first and then Eq. (5.15.3), we have

$$\begin{aligned} \sum_{j\in N} x_i^j &= v(N) - v(N\backslash\{i\}) - \sum_{j\neq i} b_j^i + \sum_{j\neq i} b_i^j + \sum_{j\neq i} \Phi_i^S(N\backslash\{j\}) \\ &= v(N) - v(N\backslash\{i\} + \sum_{j\neq i} \Phi_i^S(N\backslash\{j\}) \\ &= n\Phi_i^S(v). \end{aligned}$$

Moreover, by Claim 2.4, for each $j \in N$, $x_i^i = x_i^j$. Hence, for each $j \in N$, $x_i^j = \Phi_i^S(v)$. □

Remark 5.15.4. The last step of the proof above implies that the game Γ^{Φ^S} is not just an implementation of the Shapley value as the expected payoff in any subgame perfect equilibrium. In any subgame perfect equilibrium, the realized payoffs always coincide with the Shapley value.

Exercises of Chapter 5

5.1. Let φ be the allocation rule that, for each bargaining problem (F,d), selects the allocation $\varphi(F,d) := d + \bar{t}(1,\ldots,1)$, where $\bar{t} := \max\{t \in \mathbb{R} : d + t(1,\ldots,1) \in F_d\}$. Show that this allocation rule does not satisfy CAT. In particular, show that a player can benefit by changing the scale in the utility function used to represent his preferences.

5.2. Show that none of the axioms used in Theorem 5.3.5 to characterize the Kalai-Somorodinsky solution is superfluous.

5.3. Consider the bargaining problem (F,d) with $d = (1,0)$ and

$$F_d = \{(x,y) \in \mathbb{R}^2 : x^2 + y \leq 4, 0 \leq y \leq 3, 1 \leq x\}.$$

Obtain $\text{NA}(F,d)$ and $\text{KS}(F,d)$.

5.4. Let (F,d) be the bargaining problem with $d = (0,0)$ and

$$F = \{(x,y) \in \mathbb{R}^2 : 4x + 2y \leq 2\}.$$

Derive $\text{NA}(F,d)$ and $\text{KS}(F,d)$ using the properties that characterize these solutions.

5.5. Prove Lemma 5.4.1.

5.6. Prove Lemma 5.5.4.

5.7. Show that the two formulas of the Shapley value given in Eqs. (5.6.1) and (5.6.2) are equivalent.

5.8. Suppose that $v \in G^N$ can be decomposed as $v = \sum_{S\in 2^N\backslash\{\emptyset\}} \alpha_S w^S$, where $\alpha_S \in \mathbb{R}$ and, for each $S \in 2^N\backslash\{\emptyset\}$, w^S is the unanimity game of coalition S. Now, for the coalitions in $2^N\backslash\{\emptyset\}$, we recursively define the following real numbers, called Harsanyi dividends (Harsanyi 1959)

$$d_S := \begin{cases} v(S) & |S| = 1 \\ \frac{v(S) - \sum_{T \subsetneq S} |T| d_T}{|S|} & |S| > 1. \end{cases}$$

Prove that, for each $S \in 2^N \backslash \{\emptyset\}$, $\alpha_S = |S| d_S$.

5.9. Prove Theorem 5.8.1.

5.10. Show that none of the axioms used in Theorem 5.8.1 to characterize the Banzhaf value is superfluous.

5.11. Use the multilinear extension to obtain the Banzhaf value for the TU-games in Examples 5.4.1, 5.4.2, and 5.4.4.

5.12. Let (N, v) be the TU-game where $v(1) = 1$, $v(2) = v(3) = 2$, $v(12) = v(13) = 4$, $v(23) = 8$, and $v(N) = 10$.

(a) Compute the equal division value and the equal surplus division value for this game.

(b) Take $P = \{\{1\}, \{2, 3\}\}$ and compute the Owen value $\Phi^O(v, P)$.

(c) Take now the communication graph $L = \{(1 : 3), (2 : 3)\}$ and compute $\Phi^O(v^L, P)$.

5.13. Let (N, v) be the TU-game where $v(1) = v(2) = v(3) = 0$, $v(12) = v(13) = 4$, $v(23) = 2$, and $v(N) = 8$.

(a) Is (N, v) a convex game?

(b) Obtain the Shapley value and the Banzhaf value.

(c) Compute the core and the nucleolus.

5.14. Prove that the nucleolus and the core-center satisfy SYM.

5.15. Compute the nucleolus and the core-center of the saving game associated with the visiting professor (Example 5.4.4).

5.16. Show that none of the axioms used in Theorem 5.13.1 to characterize the Myerson value is superfluous.

5.17. Consider the following modification of the mechanism proposed by Moulin to implement the Kalai-Smorodinsky solution (Section 5.15.3). If player j rejects the offer of player i at Stage 2, then the probability that he becomes a proposer is a_j instead of a_i; with the remaining probability, the disagreement point is the realized outcome. Show that Proposition 5.15.6 and Theorem 5.15.7 also hold for this game.