Preface to the Second Edition

The book is significantly expanded for its second edition, and several old parts are rewritten. The main addition to the original book is Part 3 entitled "Singularities and Wrinkling", which is devoted to the method of wrinkling and its applications. In particular, we discuss there the construction of maps with prescribed fold singularities and prove a generalized form of Igusa's results about families of functions with generalized Morse critical points. The main ingredient in the proof is the multivalued holonomic approximation theorem, which is introduced in the new Chapter 5. We also added a new chapter (Chapter 10) on foliations, and the "foliated" language is now used throughout the book. Several other chapters and sections have been significantly rewritten, including Chapter 3 on holonomic approximation, Chapter 20 on symplectic and contact embeddings, Section 2.3 on the Thom Transversality Theorem, and Section 8.1 on natural fibrations.

In the more than 20 years since the first edition of this book, many more instances of the h-principle were discovered. Especially interesting new manifestations of flexibility phenomenon were found in symplectic and contact geometry. It became clear to us that it is impossible to address new symplectic geometric applications of the h-principle in the format of the current book. Hence, we decided to leave the symplectic part of this book mostly as is and are currently working on a separate book [**CEM25**] devoted to new developments in symplectic and contact flexibility.

Throughout the rest of the book we made numerous small changes and corrections of typos and errors. We are grateful to many readers who over the years notified us of various typos and errors in the book. We thank Daniel Álvarez-Gavela, Álvaro del Pino Gómez, and Zehan Hu for their comments on the manuscript of the current edition.

Preface to the First Edition

A *partial differential relation* $\mathcal{R}$ is any condition imposed on the partial derivatives of an unknown function. A solution of $\mathcal{R}$ is any function which satisfies this relation.

The classical partial differential relations, mostly rooted in physics, are usually described by (systems of) equations. Moreover, the corresponding systems of equations are mostly *determined*: the number of unknown functions is equal to the number of equations. Given appropriate boundary conditions, such a differential relation usually has a unique solution. In some cases this solution can be found using certain *analytical* methods (potential theory, Fourier method, and so on).

In differential geometry and topology one often deals with systems of partial differential equations, as well as partial differential *inequalities*, which have infinitely many solutions whatever boundary conditions are imposed. Moreover, sometimes solutions of these differential relations are C^0-dense in the corresponding space of functions or mappings. The systems of differential equations in question are usually (but not necessarily) *underdetermined.* We discuss in this book *homotopical* methods for solving this kind of differential relations. Any differential relation has an underlying algebraic relation which one gets by substituting derivatives by new independent variables. A solution of the corresponding algebraic relation is called a *formal* solution of the original differential relation $\mathcal{R}$. Its existence is a necessary condition for the solvability of $\mathcal{R}$, and it is a natural starting point for exploring $\mathcal{R}$. Then one can try to deform the formal solution into a genuine solution. We say that the *h-principle* holds for a differential relation $\mathcal{R}$ if any formal solution of $\mathcal{R}$ can be deformed into a genuine solution.

The *notion* of h-principle (under the name *w.h.e.-principle*) first appeared in [**Gr71**] and [**GE71**]. The *term* "h-principle" was introduced and popularized by M. Gromov in his book [**Gr86**]. The h-principle for solutions of partial differential relations exposed the soft/hard (or flexible/rigid) dichotomy for the problems formulated in terms of derivatives: a particular analytical problem is *soft* or *abides by the h-principle* if its solvability is determined by some underlying *algebraic* or *geometric* data. The softness phenomena was first discovered in the 1950s by J. Nash [**Na54**] for isometric C^1-immersions, and by S. Smale [**Sm58, Sm59**] for differential immersions. However, instances of soft problems appeared earlier (e.g., H. Whitney's paper [**Wh37**]). In the 1960s several new geometrically interesting examples of soft problems were discovered by M. Hirsch, V. Poénaru, A. Phillips, S. Feit, and other authors (see [**Hi59**], [**Po66**], [**Ph67**], [**Fe69**]). In his dissertation [**Gr69**], in the paper [**Gr73**], and later in his book [**Gr86**], Gromov transformed Smale's and Nash's ideas into two powerful general methods for solving partial differential relations: *continuous sheaves* (or the *covering homotopy*) method and the *convex integration* method. The third method, called *removal of singularities*, was first introduced and explored in [**GE71**].

There is an opinion that *"the h-principle is the hardest part of Gromov's work to popularize"* (see [**Be00**]). We have written our book in order to improve the situation. We consider here two geometrical methods: *holonomic approximation*, which is a version of the method of *continuous sheaves*, and *convex integration*. We do not pretend to cover here the content of Gromov's book [**Gr86**], but rather we want to prepare and motivate the reader to look for hidden treasures there. On the other hand, the reader interested in applications will find that with a few notable exceptions (e.g., Lohkamp's theory [**Lo95**] of negative Ricci curvature and Donaldson's theory [**Do96**] of approximately holomorphic sections) most instances of the h-principle which are known today can be treated by the methods considered in the present book.

The first three parts[1] of the book are devoted to a quite general theorem about holonomic approximation of sections of jet-bundles and its applications. Given an arbitrary submanifold $V_0 \subset V$ of positive codimension, the Holonomic Approximation Theorem allows us to solve any *open* differential relations $\mathcal{R}$ near a slightly perturbed submanifold $\widetilde{V}_0 = h(V_0)$ where $h : V \to V$ is a C^0-small diffeomorphism. Gromov's h-principle for open $\operatorname{Diff} V$-invariant differential relations on open manifolds, his directed embedding theorem, as well as some other results in the spirit of the h-principle are immediate corollaries of the Holonomic Approximation Theorem.

[1] In the current edition of the book these are the first, second, and fourth parts.

The method for proving the h-principle based on the Holonomic Approximation Theorem works well for *open* manifolds. Applications to closed manifolds require an additional trick, called *microextension*. It was first used by M. Hirsch in [**Hi59**]. The holonomic approximation method also works well for differential relations which are not open, but *microflexible*. The most interesting applications of this type come from symplectic geometry. These applications are discussed in the third part of the book. For convenience of the reader the basic notions of symplectic geometry are also reviewed in that part of the book.

The fourth[2] part of the book is devoted to *convex integration theory*. Gromov's convex integration theory was treated in great detail by D. Spring in [**Sp98**]. In our exposition of convex integration we pursue a different goal. Rather than considering the sophisticated advanced version of convex integration presented in [**Gr86**], we explore only its simple version for first order differential relations, similar to the first exposition of the theory by Gromov in [**Gr73**]. Nevertheless, we prove here practically all the most interesting corollaries of the theory, including the Nash–Kuiper theorem on C^1-isometric embeddings.

Let us list here some available books and survey papers about the h-principle. Besides Gromov's book [**Gr86**], these are Spring's book [**Sp98**], Adachi's book [**Ad93**], Haefliger's paper [**Ha71**], Poénaru's paper [**Po71**], and, most recently, Geiges' notes [**Ge01**].

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[2] In the second edition it is the fifth.