Intrigue

◀ **Examples.**

A. Immersions. A smooth map $f : V \to W$ of an n-dimensional manifold V into a q-dimensional manifold W, $n \leq q$, is called an *immersion* if its differential has the maximal rank n at every point. Two immersions are called *regularly homotopic* if one can be deformed to the other through a smooth family of immersions.

A1. For an immersion $f : S^1 \to \mathbb{R}^2$ we denote by $G(f)$ its *tangential degree*, i.e., the degree of the corresponding Gauss map $S^1 \to S^1$. Then *two immersions $f, g : S^1 \to \mathbb{R}^2$ are regularly homotopic if and only if $G(f) = G(g)$*; see [**Wh37**] and Section 7.1.

A2. On the other hand, *any two immersions $S^2 \to \mathbb{R}^3$ are regularly homotopic*; see [**Sm58**] and Section 4.2. In particular, the standard 2-sphere in $\mathbb{R}^3$ can be inverted outside in through a family of immersions.

A3. Consider now pairs of immersions $(f_0, f_1) : D^2 \to \mathbb{R}^2$ which coincide near the boundary circle ∂D^2. What is the classification of such pairs up to regular homotopy in this class? The answer turns out to be quite unexpected:
There are precisely two regular homotopy classes of such pairs. One is represented by the pair (j, j) where j is the inclusion $D^2 \hookrightarrow \mathbb{R}^2$, the second one is represented by the pair (f, g) where the immersions f and g are shown in Figure 0.1; see [**El72**].

A4. Let M_1, M_2 be any two orientable connected 3-manifolds with nonempty boundaries and $\varphi : \partial M_1 \to \partial M_2$ a diffeomorphism. *Then there exist*

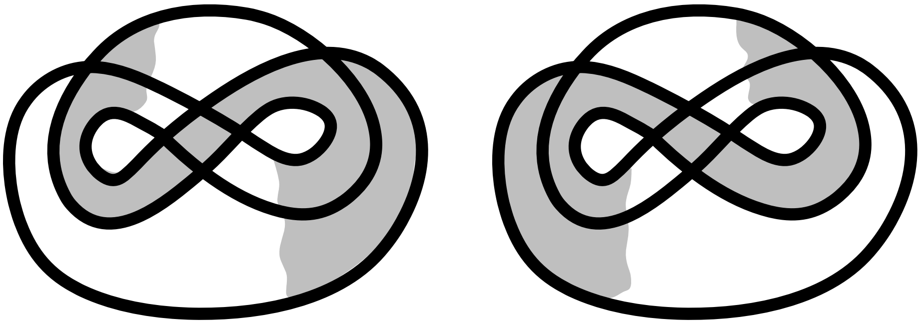

Figure 0.1. The immersions f and g.

immersions $f_j : M_j \to \mathbb{R}^3$, $j = 1, 2$, *such that* $f_1|_{\partial M_1} = f_2 \circ \varphi$; see [**El70**] and Section 14.2.

B. Isometric C^1-immersions. Is there a regular homotopy $f_t : S^2 \to \mathbb{R}^3$ which begins with the inclusion f_0 of the unit sphere and ends with an isometric immersion f_1 into the ball of radius $\frac{1}{2}$? Here the word *isometric* means *preserving the length of all curves.* The answer, of course, is negative if f_1 is required to be C^2-smooth. Indeed, in this case the Gaussian curvature of the metric on S^2 should be ≥ 4 at least somewhere. However, surprisingly, *the answer is "yes" in the case of C^1-immersions* (when the curvature is not defined but the curve length is); see [**Na54**, **Ku55**] and Chapter 29.

C. Mappings with a prescribed Jacobian. *Let Ω be an n-form on a closed oriented stably parallelizable n-dimensional manifold M such that* $\int_M \Omega = 0$, *and let*

$$\eta = dx_1 \wedge \cdots \wedge dx_n$$

be the standard volume form on $\mathbb{R}^n$. Then there exists a map $f : M \to \mathbb{R}^n$ such that $f^*\eta = \Omega$; see [**GE73**].

D. Families of functions with simple singularities. An individual function $f : M \to \mathbb{R}$ on a manifold M can always be C^∞-perturbed to have only nondegenerate (Morse) critical points. In 1-parametric families one unavoidably may also encounter the so-called *birth-death* type critical points, as in the family $f_t(x) = x^3 + tx$, $t \in [-1, 1]$. In families depending on more parameters, the possible singularities become more and more complicated; in fact, as it is proven in singularity theory (see, e.g., [**Tho55**],[**AGZV85**]), they become (smoothly) unclassifiable with a growing number of parameters and dimensions. However, it turns out that if topology permits, then by a C^0 (and sometimes even C^1) perturbation one can get rid of all singularities except Morse and birth-death ones; see [**Ig84**, **Ig87**, **EM97**, **Lu09**, **EM12**] and Chapter 16. For instance, one has the following statement (see Section 16.5):

Let $f_t : \mathbb{R}^n \to \mathbb{R}$ *be a family of functions parameterized by* t *in the sphere* $S^{k-1} = \partial D^k$*. Suppose that for each* $t \in S^{k-1}$ *the function* f_t *has no critical points and coincides with the coordinate function* x_1 *outside a compact set. Then the family* f_t *extends to* $t \in D^k$ *such that for each* $t \in D^k$ *the function* f_t *has only Morse and birth-death critical points and coincides with the coordinate function* x_1 *outside a compact set.* ▶

All the above statements are examples of the *homotopy principle*, or the *h-principle.* Despite the fact that all these problems are asking for the solution of certain differential equations or inequalities, they can be reduced to problems of a purely homotopy theoretic nature which then can be dealt with using the methods of algebraic topology. For instance, the regular homotopy classification of immersions $S^2 \to \mathbb{R}^3$ can be reduced to the computation of the homotopy group $\pi_2(\mathbb{R}P^3)$, which is trivial.

We are teaching in this book how to deal with these problems. In particular, the three general methods which we describe here will be sufficient to handle all the above examples, except **A3** and **C**, though we hope that after studying the book, the reader would be able to solve **A3** and **C** as advanced exercises.

Another, maybe even more important goal of this book is to teach the reader *how to recognize* the problems which may satisfy the h-principle. Of course, in the most interesting cases this is a very difficult question. As we will see below there are plenty of open problems where one can neither establish the h-principle nor find a single instance of *rigidity.* Nevertheless we are confident that the reader should develop a pretty good intuition for the problems which may satisfy the h-principle.

Here are some more examples where the h-principle holds, fails, or is unknown.

◀ **Examples.**

E. Totally real, Lagrangian and ε-Lagrangian embeddings. Let $T^2 = (\mathbb{R}/\mathbb{Z}) \times (\mathbb{R}/\mathbb{Z})$ be the 2-torus with the cyclic coordinates $x_1, x_2 \in \mathbb{R}/\mathbb{Z}$. Given an embedding $f : T^2 \to \mathbb{C}^2$, consider the vectors

$$v_1(x) = \frac{\partial f}{\partial x_1}(x) \quad \text{and} \quad v_2(x) = \frac{\partial f}{\partial x_2}(x), \quad x \in T^2.$$

The embedding f is called *real* or *totally real* if these vectors are linearly independent (over $\mathbb{C}$) for each $x \in T^2$. It is called *Lagrangian* if the *real* planes generated by the vectors $v_1(x), v_2(x)$ and $iv_1(x), iv_2(x)$ are orthogonal for each $x \in T^2$. For $0 < \varepsilon \leq \frac{\pi}{2}$, an embedding f is called *ε-Lagrangian* if the angle between these planes is greater than $\frac{\pi}{2} - \varepsilon$ for each $x \in T^2$. Thus Lagrangian embeddings are real, and real embeddings coincide with those that are $(\pi/2)$-Lagrangian. Identifying $\mathbb{C}^2$ with $\mathbb{R}^4$, we can view a 2×2 complex matrix as a pair of vectors in $\mathbb{R}^4$ and thus consider $\mathrm{GL}(2, \mathbb{C})$ as a subspace of the Stiefel manifold $V_{4,2}$ which is formed by pairs of vectors linearly independent over $\mathbb{C}$. With any embedding $f : T^2 \to \mathbb{C}^2$ we associate the map $v_f : T^2 \to V_{4,2}$ defined by the formula

$$v_f(x) = (v_1(x), v_2(x)) \in V_{4,2}.$$

Then f is real if and only if the image $v_f(T^2)$ is contained in $\mathrm{GL}(2, \mathbb{C})$.

E1. Both real and ε-Lagrangian embeddings satisfy the h-principle: *Let $f : T^2 \to \mathbb{C}^2$ be any embedding. Suppose that the map*

$$v_f : T^2 \to V_{4,2}$$

is homotopic to a map

$$w : T^2 \to \mathrm{GL}(2, \mathbb{C}) \subset V_{4,2}.$$

Then for any $\varepsilon > 0$ the embedding f is isotopic to an ε-Lagrangian embedding. Moreover, let $f, g : T^2 \to \mathbb{C}^2$ be two ε-Lagrangian embeddings which are smoothly isotopic and such that the maps

$$v_f, v_g : T^2 \to \mathrm{GL}(2, \mathbb{C})$$

are homotopic inside $\mathrm{GL}(2, \mathbb{C})$*. Then f and g are isotopic via an ε-Lagrangian isotopy; see* [**Gr86**] *and Section* 27.4.

E2. On the other hand, the h-principle is wrong for Lagrangian embeddings. Indeed, any two Lagrangian embeddings $T^2 \to \mathbb{C}^2$ are Lagrangian isotopic [**DRGI16**], whereas the h-principle would predict the existence of knotted Lagrangian tori.

F. Free maps. A map $T^2 \to \mathbb{R}^n$ is called *free* if the five vectors

$$\frac{\partial f}{\partial x_1}(x), \quad \frac{\partial f}{\partial x_2}(x), \quad \frac{\partial^2 f}{\partial x_1 \partial x_2}(x), \quad \frac{\partial^2 f}{\partial x_1^2}(x), \quad \frac{\partial^2 f}{\partial x_2^2}(x) \in \mathbb{R}^n$$

are linearly independent for all $x \in T^2$. Of course, the minimal dimension n for which free embeddings may exist is equal to 5.

It is an open problem whether there exists a free map $T^2 \to \mathbb{R}^5$. In particular, we do not know whether the h-principle holds for free maps to $\mathbb{R}^5$. On the other hand, free maps to $\mathbb{R}^6$ satisfy the h-principle. We invite the reader to guess what this statement really means, or look at [**GE71**].

G. Contact and Engel structures. A *contact structure* on a $(2n+1)$-dimensional manifold M is a completely nonintegrable tangent $2n$-plane field ξ. A completely nonintegrable tangent 2-plane field on a 4-manifold N is called an *Engel structure.* In the first case complete nonintegrability means that the Lie brackets of vector fields tangent to ξ generate TM at each point of M. In the second case it means that two successive Lie brackets of vector fields tangent to ξ generate TN at each point of N.

In both cases, some forms of the h-principle hold even on closed manifolds. For instance, *any tangent $2n$-plane field equipped with an almost complex structure on a $(2n+1)$-manifold is homotopic through such plane fields to a contact structure* (see [**Lu77**, **BEM15**] and Section 19.2). A similar existence h-principle holds for Engel structures on parallelizable 4-manifolds [**Vo09**, **CPPP17**]. ►