Chapter 1

Introduction

In this book, we study the Monge–Ampère equation

$$\det D^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.1}$$

for an unknown function $u : \Omega \to \mathbb{R}$, and its linearization, that is, the linearized Monge–Ampère equation

$$\sum_{i,j=1}^{n} U^{ij} D_{ij} v = g \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.2}$$

where $(U^{ij})_{1\leq i,j\leq n}$ is the cofactor matrix of the Hessian matrix D^2u of a given potential function $u : \Omega \to \mathbb{R}$ where now $v : \Omega \to \mathbb{R}$ is the unknown.

The first equation is fully nonlinear in u while the second equation is linear in v under a "nonlinear background" due to the structure of the coefficient matrix. These equations can be put into the abstract form

$$F(D^2 w(x), x) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.3}$$

with $F(r, x) : \mathbb{S}^n \times \Omega \to \mathbb{R}$, where $\mathbb{S}^n$ is the space of real, symmetric $n \times n$ matrices. We assume $n \geq 2$ in this chapter.

Besides the existence theory for solutions under suitable data, our main focus is on regularity theory in the framework of elliptic partial differential equations (PDEs). For (1.3) to be elliptic, the usual requirement that

$$F(r + A, x) > F(r) \quad \text{for all positive definite matrices } A \in \mathbb{S}^n$$

turns into requiring $\left(\frac{\partial F}{\partial r_{ij}}\right)_{1\leq i,j\leq n}$—the coefficient matrix of the linearization of F—to be positive definite. Specializing this to (1.1) where

$$F(r, x) = \det r - f(x)$$

and (1.2) where

$$F(r,x) = \sum_{i,j=1}^{n} U^{ij} r_{ij} - g(x), \quad \text{with } r = (r_{ij})_{1 \le i,j \le n},$$

we are led to requiring that $U = (U^{ij})_{1 \le i,j \le n}$ be positive definite. In this case, due to $U = (\det D^2 u)(D^2 u)^{-1}$, it is natural to consider $D^2 u$ to be positive definite. This is the reason why we will study (1.1) and (1.2) among the class of convex functions u. In particular, the function f in (1.1) must be nonnegative. This book will study mostly the case of f being bounded between two positive constants.

1.1. The Monge–Ampère Equation

The Monge–Ampère equation in the general form

$$\det D^2 u(x) = G(x, u(x), Du(x))$$

arises in many important problems in analysis, partial differential equations, geometry, mathematical physics, and applications. The study of the Monge–Ampère equation began in two dimensions with the works of Monge [**Mon**] in 1784 and Ampère [**Amp**] in 1820. It seems that Lie was the first to use the terminology "Monge–Ampère equation" in his 1877 paper [**Li**]. In this section, we provide some examples where the Monge–Ampère equation appears, and we give some impressionistic features and difficulties of this equation. This is not meant to be a survey.

1.1.1. Examples.

We list below some examples concerning the Monge–Ampère equation.

Example 1.1 (Prescribed Gauss curvature equation)**.** Let $u \in C^2(\Omega)$ where $\Omega \subset \mathbb{R}^n$. Suppose that the graph of u has Gauss curvature $K(x)$ at the point $(x, u(x))$ where $x \in \Omega$. Then u satisfies the equation

$$\det D^2 u = K(1 + |Du|^2)^{(n+2)/2}.$$

Example 1.2 (Affine sphere of parabolic type)**.** The classification of complete affine spheres of parabolic type is based on the classification of global convex solutions to

$$\det D^2 u = 1 \quad \text{in } \mathbb{R}^n.$$

Example 1.3 (Special Lagrangian equation)**.** Let $\lambda_i(D^2 u)$ $(i = 1, \ldots, n)$ be the eigenvalues of the Hessian matrix $D^2 u$ of a function $u : \mathbb{R}^n \to \mathbb{R}$. The special Lagrangian equation

$$\sum_{i=1}^{n} \arctan \lambda_i(D^2 u) = \text{constant } c$$

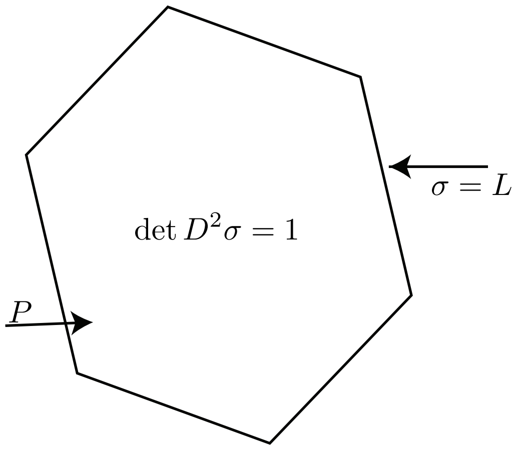

Figure 1.1. Surface tension in the dimer model solves a Monge–Ampère equation on a planar polygon with piecewise affine boundary data.

has close connections with the Monge–Ampère equation. When $n = 2$ and $c = \pi/2$, it is exactly

$$\det D^2 u = 1.$$

When $n = 3$ and $c = \pi$, it becomes $\det D^2 u = \Delta u$.

Example 1.4 (Surface tension in the dimer model)**.** In the dimer model in combinatorics and statistical physics, the surface tension σ is a convex solution to the Monge–Ampère equation

$$\begin{cases} \det D^2\sigma = 1 & \text{in } P, \\ \sigma = L & \text{on } \partial P. \end{cases}$$

Here $P \subset \mathbb{R}^2$ is a convex polygon in the plane, and L is a piecewise affine function on the boundary ∂P. See Figure 1.1.

Example 1.5 (Affine hyperbolic sphere)**.** Let Ω be a bounded, convex domain in $\mathbb{R}^n$. If $u \in C(\overline{\Omega})$ is a convex solution to

$$\det D^2 u = |u|^{-n-2} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

then the Legendre transform of u is a complete affine hyperbolic sphere.

Example 1.6 (The Monge–Ampère eigenvalue problem)**.** Let Ω be a convex, bounded domain in $\mathbb{R}^n$. The Monge–Ampère eigenvalue problem consists of finding constants $\lambda > 0$ and nonzero convex functions $u \in C(\overline{\Omega})$ such that

$$\begin{cases} \det D^2 u = \lambda |u|^n & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Example 1.7 (The Minkowski problem)**.** Assume we are given a bounded, positive function K on the unit sphere $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$, satisfying the integral conditions

$$\int_{S^n} \frac{x_i}{K} \, d\mathcal{H}^n = 0, \quad \text{for all } i = 1, \ldots, n+1,$$

where each x_i is the coordinate function. The Minkowski problem asks to find a closed convex hypersurface $\mathcal{M} \subset \mathbb{R}^{n+1}$ such that its Gauss curvature at $p \in \mathcal{M}$ is equal to $K(\nu_p)$ where ν_p is the outer unit normal of $\mathcal{M}$ at p. This problem reduces to solving the Monge–Ampère equation

$$\det(\nabla^2 u + uI) = K^{-1} \quad \text{on } S^n,$$

where $u(x) = \sup\{x \cdot p : p \in \mathcal{M}\}$ is the support function of $\mathcal{M}$, I is the identity matrix, and ∇ is the covariant derivative in a local orthogonal frame.

Example 1.8 (The Monge–Kantorovich optimal transportation problem)**.** Suppose we are given two bounded, convex domains Ω and Ω' in $\mathbb{R}^n$ and bounded, positive functions f_1 and f_2 in Ω and Ω', respectively, with the normalized compatibility condition

$$\int_{\Omega} f_1(x)\, dx = \int_{\Omega'} f_2(y)\, dy = 1.$$

Thus, $\mu(x) = f_1(x)\, dx$ and $\nu(y) = f_2(y)\, dy$ are probability measures on Ω and Ω', respectively, with densities f_1 and f_2. The *optimal transportation problem* with quadratic cost between (Ω, μ) and (Ω', ν) consists of finding a measurable map $T : \Omega \to \Omega'$ such that $T_\sharp \mu = \nu$, that is, $\nu(A) = \mu(T^{-1}(A))$ for all measurable subsets $A \subset \Omega'$ (equivalently, $\int_{\Omega'} f\, d\nu = \int_{\Omega} f \circ T\, d\mu$ for all $f \in L^1(\Omega')$) such that T minimizes the quadratic transportation cost:

$$\int_{\Omega} |x - T(x)|^2 \, d\mu(x) = \min_{S_\sharp \mu = \nu} \int_{\Omega} |x - S(x)|^2 \, d\mu(x),$$

where the minimum is taken over all measurable maps $S : \Omega \to \Omega'$ such that $S_\sharp \mu = \nu$. A classical theorem of Brenier [**Br**] shows that the optimal map T exists and is given by the gradient of a convex function u; that is, $T = Du$. Moreover, u is a weak solution to the second boundary value problem for the Monge–Ampère equation:

$$\det D^2 u = f_1/(f_2 \circ Du) \quad \text{in } \Omega \quad \text{and} \quad Du(\Omega) = \Omega'.$$

Example 1.9 (Two-dimensional incompressible Navier–Stokes equations)**.** Consider the two-dimensional incompressible Navier–Stokes equations

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot D)\mathbf{u} - \nu \Delta \mathbf{u} + Dp - \mathbf{f} = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \text{div } \mathbf{u} = 0 & \text{in } \mathbb{R}^2 \times (0, \infty), \\ \mathbf{u}(x, 0) = \mathbf{u}_0(x) & \text{in } \mathbb{R}^2. \end{cases}$$

Here, $\mathbf{u}(x,t) = (u^1(x,t), u^2(x,t))$ is the velocity vector field, $\nu \geq 0$ is the viscosity, p is the pressure, $\mathbf{f}$ is the force, and $\mathbf{u}_0$ is a divergence-free initial velocity. The incompressibility condition $\operatorname{div}\mathbf{u} = 0$ implies the existence of a stream function $\Psi : \mathbb{R}^2 \times [0,\infty) \to \mathbb{R}$ such that $\mathbf{u} = (D\Psi)^\perp \equiv \big(-D_2\Psi, D_1\Psi\big)$. Taking the divergence of the Navier–Stokes equations yields

$$\operatorname{div}[(\mathbf{u}\cdot D)\mathbf{u}] + \Delta p - \operatorname{div}\mathbf{f} = 0.$$

Since $\operatorname{div}[(\mathbf{u}\cdot D)\mathbf{u}] = -2\det D^2\Psi$, we obtain from the incompressible Navier–Stokes equations the following Monge–Ampère equation:

$$\det D^2\Psi = (\Delta p - \operatorname{div}\mathbf{f})/2.$$

Example 1.10 (Dual semigeostrophic equations in meteorology)**.** Let ρ^0 be a probability measure on the two-dimensional torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The *dual semigeostrophic equations* of the semigeostrophic equations on $\mathbb{T}^2$ are the following system of nonlinear transport equations for (ρ_t, P_t^*):

$$\begin{cases} \partial_t\rho_t(x) + \operatorname{div}[\rho_t(x)(x - DP_t^*(x))^\perp] &= 0, \qquad (x,t) \in \mathbb{T}^2 \times [0,\infty),\\ \det D^2 P_t^*(x) &= \rho_t(x), \quad (x,t) \in \mathbb{T}^2 \times [0,\infty),\\ P_t^*(x) \text{ convex}, & \qquad\qquad (x,t) \in \mathbb{T}^2 \times [0,\infty),\\ \rho_0(x) &= \rho^0(x), \quad x \in \mathbb{T}^2, \end{cases}$$

with $P_t^*(x) - |x|^2/2$ being $\mathbb{Z}^2$-periodic. Here, we use the notation $\rho_t(x) = \rho(x,t)$, $P_t^*(x) = P^*(x,t)$, and $w^\perp = (-w_2, w_1)$ for $w = (w_1, w_2) \in \mathbb{R}^2$.

1.1.2. Existence of solutions. We will introduce various concepts of weak solutions to (1.1) but will work mostly with Aleksandrov solutions. The other types of solutions are viscosity and variational solutions. Using the concept of normal mapping

$$\partial u(x) = \{p \in \mathbb{R}^n : u(y) \geq u(x) + p\cdot(y - x) \quad \text{for all } y \in \Omega\},$$

we can define for each convex function u the Monge–Ampère measure μ_u via

$$\mu_u(E) = \Big|\bigcup_{x\in E} \partial u(x)\Big| \quad \text{for each Borel subset } E \subset \Omega.$$

Here $|K|$ denotes the Lebesgue measure of $K \subset \mathbb{R}^n$. We say that u is an Aleksandrov solution to (1.1) if, as Borel measures, we have

$$\mu_u = f.$$

Here we identify an integrable function f with its Borel measure $f\,dx$. Therefore, it is natural to consider (1.1) for f being a nonnegative Borel measure.

Using the Perron method and maximum principles including those of Aleksandrov, we can solve the Monge–Ampère equation (1.1) with suitable boundary data φ on $\partial\Omega$ for any nonnegative Borel measure f with finite mass. This general construction does not give much information about the smoothness of solutions when the data is smooth. In other words, can we solve (1.1) in the classical sense? The interior smoothness issue was settled in the works of Pogorelov [**P2**, **P5**] and Cheng and Yau [**CY**]. Ivočkina [**Iv**], Krylov [**K3**], and Caffarelli, Nirenberg, and Spruck [**CNS1**] provided a positive answer with regards to global smoothness to this question. They proved the existence of smooth, uniformly convex solutions to the Monge–Ampère equation based on higher-order a priori global estimates.

1.1.3. Interior regularity theory of Aleksandrov solutions. We will study the regularity of Aleksandrov solutions to (1.1) both in the interior and at the boundary, especially when

$$0 < \lambda \leq f \leq \Lambda, \tag{1.4}$$

where λ and Λ are constants. Particular attentions will be paid to the right-hand side f not having much regularity. Our study somehow mirrors the weak solutions and regularity theory of Poisson's equation

$$\Delta w = g \quad \text{in } \Omega \subset \mathbb{R}^n, \tag{1.5}$$

but as the book progresses, we will see dramatic differences between the two equations in the regime of limited regularity of the data. Note that, however, if u in (1.1) is known to be $C^{2,\alpha}$, then any partial derivative $v := D_k u$ of u is a solution of $U^{ij} D_{ij} v = D_k f$ which is locally uniformly elliptic with C^α coefficients, for which the classical Schauder theory is applicable. Therefore, it suffices to focus our attention on estimates up to second-order derivatives and their continuity.

Convex Aleksandrov solutions to (1.1) are locally Lipschitz but going beyond Lipschitz regularity turns out to be quite a challenging task. Moreover, the heuristic picture that "solutions of Poisson's equation have two derivatives more than the right-hand side" fails spectacularly for the Monge–Ampère equation. Pogorelov [**P5**] constructed an example in all dimensions at least three with f being a positive analytic function and a non-C^2 solution to (1.1). One reason for this failure of regularity is the existence of a line segment in the graph of solutions to (1.1). Ruling out this behavior, which means showing that the solutions are strictly convex, is a key to the interior regularity. A deeper reason for this failure comes from a crucial geometric fact that the Monge–Ampère equation (1.1) is invariant under affine transformations: We can "stretch" u in one direction and at the same time "contract" it in other directions to get another solution. Thus, it is possible that some singular eigenvalues of $D^2 u$ are compensated by zero eigenvalues.

When the right-hand side of (1.1) is C^2, Pogorelov [**P2**] discovered very important second derivative estimates for strictly convex solutions. Then, Calabi's third derivative estimates [**Cal1**] yield interior $C^{2,\alpha}$ estimates when f is C^3, and higher-order regularity estimates follow for more regular f. If f is only C^2, then interior $C^{2,\alpha}$ estimates can also be obtained via the use of the Evans–Krylov Theorem for concave, uniformly elliptic equations [**E1**, **K2**]. All these estimates rely on differentiating the equation, so it is not clear if the above estimates are valid when f is less regular.

In 1990, Caffarelli proved a striking localization theorem for the Monge–Ampère equation [**C2**] which implies strict convexity of solutions under quite general conditions on f and the boundary data, and he established a large part of the interior regularity theory for Aleksandrov solutions.

Consider (1.1) with (1.4). Then Caffarelli [**C2**, **C3**] showed that strictly convex solutions to (1.1) are $C^{1,\alpha}$ while they are $C^{2,\alpha}$ when f is further assumed to be C^α. Moreover, when f is close to a positive constant or continuous, strictly convex solutions to (1.1) belong to $W^{2,p}_{\mathrm{loc}}$ for all finite p. In Caffarelli's deep regularity theory, sections (or cross sections) of solutions to the Monge–Ampère equation plays an important role. They are defined as sublevel sets of convex solutions after subtracting their supporting hyperplanes. As it turns out in further developments of the Monge–Ampère and linearized Monge–Ampère equations, sections have the same fundamental role that Euclidean balls have in the classical theory of second-order, uniformly elliptic equations. In general, sections have degenerate geometry. However, the affine invariance property of the Monge–Ampère equation in combination with John's Lemma allows us to focus on settings where sections are roughly Euclidean balls.

When f is allowed to have arbitrarily large oscillations, Wang's counterexamples [**W3**] show that strictly convex solutions to (1.1) are not expected to be in $W^{2,p}_{\mathrm{loc}}$ for any fixed exponent $p > 1$. It was an open question whether strictly convex solutions to (1.1) with (1.4) belong to $W^{2,1+\varepsilon}_{\mathrm{loc}}$ for some positive constant ε depending only on n, λ, and Λ. This was not resolved until 2013 when De Philippis, Figalli, and Savin [**DPFS**] and Schmidt [**Schm**] independently established this result. These papers built on the groundbreaking work of De Philippis and Figalli [**DPF1**] on $W^{2,1}_{\mathrm{loc}}$ regularity.

1.1.4. Boundary regularity theory. Another example of the dramatic difference between (1.1) and (1.5) is concerned with boundary regularity theory. When Ω is a triangle in the plane and $g \equiv 1$, the solution to (1.5) with zero boundary data is C^∞ around any boundary point P away from the vertices; however, the corresponding convex solution to (1.1) is not even Lipschitz (it is only log-Lipschitz) around P, though it is analytic in the interior! To establish global $W^{2,p}$ estimates for (1.5) where $1 < p < \infty$, it suffices to require $\partial\Omega$ to be C^2 and the boundary value of w on $\partial\Omega$ to be $W^{2,p}$. However, for the Monge–Ampère equation, for $p \geq 3$, there is a counterexample of Wang [**W4**] in the case of either $\partial\Omega$ or the boundary value of u failing to be C^3. In the linear, uniformly elliptic equations, to prove a boundary regularity result, we can typically guess what the optimal regularity assumptions on the boundary data and domain should be. This is not the case for the Monge–Ampère equation. In this regard, we have important results of Trudinger and Wang [**TW5**] and Savin [**S4**, **S5**] on global second derivative estimates. In particular, Savin [**S4**] established a boundary localization theorem for the Monge–Ampère equation under sharp conditions. This theorem paved the way for extending previous interior regularity results to the boundary.

1.2. The Linearized Monge–Ampère Equation

The linearized Monge–Ampère equation arises from several fundamental problems in different subjects including: the affine maximal surface equation in affine geometry, Abreu's equation in the problem of finding Kähler metrics of constant scalar curvature in complex geometry, the semigeostrophic equations in fluid mechanics, and the approximation of minimizers of convex functionals with a convexity constraint in the calculus of variations, to name a few.

The linearized Monge–Ampère equation associated with a C^2 strictly *convex potential* u defined on some convex domain of $\mathbb{R}^n$ is of the form

$$L_u v := \sum_{i,j=1}^{n} U^{ij} D_{ij} v \equiv \operatorname{trace}(U D^2 v) = g. \tag{1.6}$$

Throughout, the cofactor matrix $\operatorname{Cof} D^2 u$ of the Hessian matrix $D^2 u = (D_{ij} u)_{1\leq i,j\leq n}$ is denoted by

$$U = (U^{ij})_{1\leq i,j\leq n} = (\det D^2 u)(D^2 u)^{-1}.$$

The coefficient matrix U of L_u arises from linearizing the Monge–Ampère operator $F(r) = \det r$ where $r = (r_{ij})_{1\leq i,j\leq n} \in \mathbb{S}^n$ because $U^{ij} = \frac{\partial F}{\partial r_{ij}}(D^2 u)$.

Put differently, $L_u v$ is the coefficient of t in the expansion

$$\det D^2(u + tv) = \det D^2 u + t \operatorname{trace}(U D^2 v) + \cdots + t^n \det D^2 v.$$

In this book, we mainly focus on the case that the convex potential u solves the Monge–Ampère equation

$$\det D^2 u = f \text{ for some function } f \text{ satisfying } 0 < \lambda \leq f \leq \Lambda < \infty. \tag{1.7}$$

Given the bounds in (1.7), U is a positive definite matrix, but we cannot expect to obtain structural bounds on its eigenvalues, as indicated by the discussions in Section 1.1.3. Hence, L_u is a linear elliptic partial differential operator which can be both degenerate and singular. *Contrary to what the name might suggest, the linearized Monge–Ampère equation is actually more general than the Monge–Ampère equation. In fact, it includes the Monge–Ampère equation as a special case* because of the identity

$$L_u u = n \det D^2 u,$$

but the difficulty here lies in the fact that one typically does not require any convexity of solutions to the linearized Monge–Ampère equation. On the other hand, in the special case where u is the quadratic polynomial $u(x) = |x|^2/2$, L_u becomes the Laplace operator:

$$L_{|x|^2/2} = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Thus, the linearized Monge–Ampère operator L_u captures two of the most important second-order equations in PDEs from the simplest linear equation to one of the most significant fully nonlinear equations.

As U is divergence-free, that is, $\sum_{i=1}^n D_i U^{ij} = 0$ for all $j = 1, \ldots, n$, the linearized Monge–Ampère equation can be written in both divergence and double divergence form:

$$L_u v = \sum_{i,j=1}^{n} D_i(U^{ij} D_j v) = \sum_{i,j=1}^{n} D_{ij}(U^{ij} v).$$

1.2.1. Examples. We list here some examples where the linearized Monge–Ampère operator appears.

Example 1.11 (Affine maximal surface equation)**.**

$$\sum_{i,j=1}^{n} U^{ij} D_{ij}\big[(\det D^2 u)^{-\frac{n+1}{n+2}}\big] = 0.$$

This equation arises in affine geometry.

Example 1.12 (Abreu's equation)**.**

$$\sum_{i,j=1}^{n} U^{ij} D_{ij}\big[(\det D^2 u)^{-1}\big] = -1.$$

This equation arises in the problem of finding Kähler metrics of constant scalar curvature in complex geometry. A more familiar form of Abreu's equation is

$$\sum_{i,j=1}^{n} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = -1,$$

where $(u^{ij})_{1\leq i,j\leq n} = (D^2 u)^{-1}$ is the inverse matrix of $D^2 u$.

Example 1.13 (Dual semigeostrophic equations)**.** The time derivative $\partial_t P_t^*$ of the potential function P_t^* in the two-dimensional semigeostrophic equations in Example 1.10 satisfies

$$\sum_{i,j=1}^{2} (\operatorname{Cof} D^2 P_t^*)^{ij} D_{ij}(\partial_t P_t^*) = \operatorname{div} \left[\det D^2 P_t^* (DP_t^*(x) - x)^{\perp} \right],$$

where $w^{\perp}$ denotes the vector $(-w_2, w_1)$ for $w = (w_1, w_2) \in \mathbb{R}^2$.

Example 1.14 (Singular Abreu equation)**.**

$$\sum_{i,j=1}^{n} U^{ij} D_{ij}[(\det D^2 u)^{-1}] = (-\Delta u)\chi_{\Omega_0} + \frac{1}{\delta}(u - \varphi)\chi_{\Omega \setminus \Omega_0} \quad \text{in } \Omega.$$

Here $\delta > 0$ and Ω_0, Ω are smooth, bounded, convex domains in $\mathbb{R}^n$ such that $\Omega_0 \Subset \Omega$. This equation arises from approximating minimizers of convex functionals subjected to a convexity constraint.

1.2.2. Mixed Monge–Ampère measure. We define the mixed Monge–Ampère measure $\tilde{\mu}_{u_1,\ldots,u_n}$ of n convex functions $u_1, \ldots, u_n$ on $\mathbb{R}^n$ by the polarization formula

$$\tilde{\mu}_{u_1,\ldots,u_n} = \frac{1}{n!} \sum_{k=1}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{n-k} \mu_{u_{i_1} + \cdots + u_{i_k}}.$$

Clearly,

$$\mu_u = \tilde{\mu}_{u,\ldots,u}. \tag{1.8}$$

Example 1.15 (The linearized Monge–Ampère operator and mixed Monge–Ampère measures)**.** The linearized Monge–Ampère operator is a special case of mixed Monge–Ampère measures since

$$U^{ij} D_{ij} v = n \tilde{\mu}_{u,\ldots,u,v}. \tag{1.9}$$

An interesting application of the mixed Monge–Ampère measures to *amoeba* can be found in Passare–Rullgård [**PR**]. More general estimates concerning the mixed Hessian can be found in Trudinger–Wang [**TW3**].

1.2.3. The Caffarelli–Gutiérrez theory and beyond. The regularity theory for the linearized Monge–Ampère equation was initiated in the fundamental paper [**CG2**] by Caffarelli and Gutiérrez. It is worth mentioning that one of their motivations was Lagrangian models of atmospheric and oceanic flows in the work of Cullen, Norbury, and Purser [**CNP**], including the dual semigeostrophic equations in Examples 1.10 and 1.13 that we will discuss in detail in Section 15.4. Caffarelli and Gutiérrez developed an interior Harnack inequality theory for nonnegative solutions of the homogeneous equation $L_u v = 0$ in terms of the structure, called the A_∞-condition, of the

Monge–Ampère measure μ_u of the convex potential function u. This A_∞-condition is clearly satisfied when we have the pinching $\lambda \leq \det D^2 u \leq \Lambda$ of the Hessian determinant. Their approach is based on that of Krylov and Safonov [**KS1**, **KS2**] on Hölder estimates for linear, uniformly elliptic equations in nondivergence form, with sections of u replacing Euclidean balls.

When the right-hand side f of the Monge–Ampère equation is only bounded between two positive constants, sections of Aleksandrov solutions to $\det D^2 u = f$ can have degenerate geometry. The key issue in the regularity of the linearized Monge–Ampère equation is to prove that these sections have properties similar to Euclidean balls as in uniformly elliptic equations. These are covered in Chapter 5 for interior sections and in Chapter 9 for boundary sections.

After [**CG2**], many works on the interior regularity of the linearized Monge–Ampère equation have appeared, including: a Liouville theorem [**S2**], interior $W^{2,\delta}, C^{1,\alpha}$, and $W^{2,p}$ estimates [**GTo**, **GN1**, **GN2**], interior Hölder estimates under minimal geometric conditions or to equations with lower-order terms [**Md2**, **Md3**, **L5**], Green's function and Monge–Ampère Sobolev inequality [**TiW**, **Md1**, **Md4**, **L2**, **L6**], a Harnack inequality for the parabolic linearized Monge–Ampère equation [**Hu1**, **Md5**]. Corresponding global Hölder, $C^{1,\alpha}$, $W^{1,p}$, and $W^{2,p}$ estimates for the linearized Monge–Ampère equation were established in [**L1**, **LS1**, **LS3**, **LN2**, **LN3**]. Many results obtained in the linearized Monge–Ampère equation parallel their analogues in the uniformly elliptic equations. *The overall picture is that information on individual eigenvalues of the coefficient matrix of the uniformly elliptic equations is replaced by information on the Hessian determinant of the convex potential in the linearized Monge–Ampère equation.*

The Caffarelli–Gutiérrez interior Harnack inequality plays a crucial role in Trudinger and Wang's resolution of Chern's conjecture in affine geometry concerning affine maximal surfaces in three dimensions. Trudinger and Wang proved in 2000 that locally uniformly convex and smooth solutions u to the affine maximal surface equation in Example 1.11 must be quadratic functions when $n = 2$. In addition to applications to Chern's conjecture and the dual semigeostrophic equations, this book will discuss other applications arising in affine geometry and the monopolist's problem in economics.

1.2.4. Geometric tools and convexity of the potential functions. In general, for estimations of solutions to the linearized Monge–Ampère equation that depend only on structural quantities but not on the smoothness of the potential u, we work with its sections instead of Euclidean balls. Moreover, their deep geometric properties will be of vital importance. In particular, for interior estimates, we rely on the results in Chapter 5, while for boundary estimates, we rely on the results of Chapters 8 and 9.

In this book, we focus on the linearized Monge–Ampère equations associated with a convex potential function. Of course, there are equations involving the linearized Monge–Ampère operator where no convexity on the potential function is assumed. One example, as described in Berger [**Bg**], is concerned with von Kármán equations in elasticity. These are two fourth-order quasi-linear elliptic equations that describe the equilibrium states of thin shallow elastic shells under the action of applied forces.

Example 1.16 (Thin elastic shells)**.** Consider a thin shallow shell S of arbitrary shape in three-dimensional $\mathbb{R}^3$ space whose plane projection is a bounded domain Ω in the x_1x_2-plane with boundary $\partial\Omega$. Suppose the shell is acted on by an external force $F(x)$ ($x \in \mathbb{R}^2$) and by forces along $\partial\Omega$. Then, subject to appropriate boundary conditions, the equilibrium states of S will be determined by solving the following system of fourth-order equations:

$$\begin{cases} \Delta^2 f &= -\det D^2 w - D_1(k_1 D_1 w) - D_2(k_2 D_2 w) \quad \text{in } \Omega, \\ \Delta^2 w &= L_f w + D_1(k_1 D_1 f) + D_2(k_2 D_2 f) + F \quad \text{in } \Omega, \end{cases}$$

where Δ^2 denotes the biharmonic operator on $\mathbb{R}^2$. Here $w(x)$ represents the deflection of the shell from its initial state, $f(x)$ is the Airy stress function, and k_1 and k_2 denote the initial curvatures of the shell in cross sections parallel to the x_1x_3- and x_2x_3-planes, respectively. The right-hand sides of the above system contain the linearized Monge–Ampère operator

$$L_f w = D_{11} f D_{22} w + D_{22} f D_{11} w - 2 D_{12} f D_{12} w.$$

1.3. Plan of the Book

This book consists of a preliminary Chapter 2 and two main parts. In **Part 1** which consists of Chapters 3–11, we cover basic interior and boundary regularity theories of the Monge–Ampère equation. The spectral theory of the Monge–Ampère operator is also introduced. A few applications are presented in Chapters 4 and 7. **Part 2**, consisting of Chapters 12–15, is devoted to the regularity theory of the linearized Monge–Ampère equation and some select applications. The list of applications where the linearized Monge–Ampère equation appears naturally is far from being comprehensive. The applications presented here reflect the author's interests and they cover several problems in geometry, physics, and economics.

To make the book self-contained, we start from scratch the modern theory of the Monge–Ampère equation. Thus, Chapters 3–7 overlap with the books by Figalli [**F2**], Gutiérrez [**G2**], Han [**H**], Le, Mitake, and Tran [**LMT**] and survey papers by De Philippis and Figalli [**DPF2**], Liu and Wang [**LW**], and Trudinger and Wang [**TW6**], though some proofs here are different. The reader may consult the books by Bakelman [**Ba5**] and Pogorelov [**P1**, **P4**,

P5] for earlier treatments of the Monge–Ampère equation, and the book by Schulz [**Schz**] for the Monge–Ampère equation in two dimensions. Many books such as Aubin [**Au2**], Figalli [**F2**], and Villani [**Vi**] cover diverse applications of the Monge–Ampère equation so we will not repeat them here. Our applications are mostly concerned with the linearized Monge–Ampère equation.

The book will address the following **major themes**:

- For the Monge–Ampère equation:
  - (i) *basic concepts and existence results for solutions* (Chapters 3 and 4);
  - (ii) *localization theorems* (Chapters 5 and 8);
  - (iii) *geometry of sublevel sets (or sections)* (Chapters 5 and 9);
  - (iv) *regularity results in the interior and at the boundary* (Chapters 5, 6, and 10);
  - (v) *classification of global solutions* (Chapter 7);
  - (vi) *spectral theory* (Chapter 11).
- For the linearized Monge–Ampère equation:
  - (i) *regularity results in the interior and at the boundary* (Chapters 12 and 13);
  - (ii) *analysis from nondivergence form and divergence form perspectives* (Chapters 14 and 15).

Below is a more detailed description of each chapter.

Chapter 2 collects basic facts and techniques on convex sets, convex functions, analysis, and classical PDE theory that we will use in studying Monge–Ampère equations. Topics in convex sets include: existence of supporting hyperplanes at boundary points, extremal points, John's Lemma, the Hausdorff distance, the Blaschke Selection Theorem, defining function, and approximation of convex bodies by smooth, uniformly convex sets. Topics in convex functions include: supporting hyperplane, the normal mapping, Lipschitz continuity, and Legendre transform. Topics in analysis include: the Hausdorff measure, the area formula, the change of variables formula, the Sobolev Embedding Theorem, and the Implicit Function Theorem. We review the classical PDE theory including: the maximum principle, Perron's method, Schauder estimates, Calderon–Zygmund estimates, Evans–Krylov estimates, pointwise estimates, and perturbation arguments.

Part 1: The Monge–Ampère Equation

Chapter 3 lays the foundations for our investigation of Aleksandrov solutions to the Monge–Ampère equation. We will introduce the important notion of the Monge–Ampère measure of a convex function. With this notion, we can speak of Aleksandrov solutions to a Monge–Ampère equation without the requirement of having two classical derivatives everywhere for the solutions. Then we will prove the Aleksandrov maximum principle, the Aleksandrov–Bakelman–Pucci maximum principle, and compactness results. We will prove the comparison principle and establish various optimal global Hölder estimates. Then we discuss the solvability of the inhomogeneous Dirichlet problem with continuous boundary data via Perron's method.

Chapter 4 discusses the solvability of classical solutions to the Monge–Ampère equation. We will prove the theorem of Ivočkina, Krylov, and Caffarelli, Nirenberg, and Spruck on the existence of smooth, uniformly convex solutions to the Monge–Ampère equation. To prepare for later study of boundary regularity theory, we introduce the notion of the *special subdifferential* of a convex function at a boundary point of a convex domain. We establish the quadratic separation of the function from its tangent hyperplanes at boundary points under suitable hypotheses. Using the existence of classical solutions and the Aleksandrov maximum principle, we give a proof of the classical isoperimetric inequality. We also prove a nonlinear integration by parts for the Monge–Ampère operator. We will discuss Pogorelov's counterexamples to interior regularity which demonstrate that there are no purely interior estimates for the Monge–Ampère equation.

Chapter 5 introduces the notion of sections of convex functions. This notion plays a crucial role in the study of geometry of solutions to the Monge–Ampère equation. We discuss properties of sections including: volume estimates, the size of sections, the engulfing property of sections, the inclusion and exclusion properties of sections, and a localization property of intersecting sections. Two real analysis results for sections will be proved: Vitali's Covering Lemma, and the Crawling of Ink-spots Lemma. The existence of centered sections will be also established. We start the interior regularity theory of the Monge–Ampère equation by proving the important Localization Theorem of Caffarelli. From this together with properties of sections, we obtain Caffarelli's $C^{1,\alpha}$ regularity of strictly convex solutions. We will present Caffarelli's counterexamples to show the failure of $C^{1,\alpha}$ regularity when solutions are not strictly convex.

Chapter 6 is the culmination of the interior regularity theory of the Monge–Ampère equation where we establish critical estimates for second-order derivatives. We will prove Pogorelov's second derivative estimates, Caffarelli's $W^{2,p}$, $C^{2,\alpha}$ estimates, and De Philippis–Figalli–Savin–Schmidt $W^{2,1+\varepsilon}$ estimates. An application is given to interior regularity and uniqueness of degenerate Monge–Ampère equations. We will discuss Wang's counterexample to interior $W^{2,p}$ estimates and Mooney's counterexample to $W^{2,1}$ estimates.

In Chapter 7, we introduce the concept of viscosity solutions to the Monge–Ampère equation and show that Aleksandrov solutions are also viscosity solutions, and vice versa (for a strictly positive, continuous right-hand side). We will prove a classical theorem of Jörgens, Calabi, and Pogorelov on the classification of global solutions to the Monge–Ampère equation with constant right-hand side. We will discuss the relation between Jörgens's Theorem and other Liouville-type theorems in two dimensions such as Bernstein's Theorem on linearity of solutions to the minimal surface equation. We also give a brief introduction to the Legendre–Lewy rotation.

Chapter 8 is devoted to proving Savin's Boundary Localization Theorem. This theorem, which is concerned with the shape of boundary sections, will play a very important role in the boundary regularity theory for the Monge–Ampère equation. The proof uses compactness arguments and viscosity solutions for the Monge–Ampère equation with boundary discontinuities whose groundwork has been built in Chapter 7.

Chapter 9 proves the global $C^{1,\alpha}$ estimates for the Monge–Ampère equation and investigates several important geometric properties of boundary sections and maximal interior sections of convex solutions to the Monge–Ampère equation. We will establish the following properties of boundary sections: dichotomy of sections, volume estimates, engulfing and separating properties, inclusion and exclusion properties, and a chain property. As applications, we prove Besicovitch's Covering Lemma and employ it to prove a covering theorem and a strong-type (p,p) estimate for the maximal function with respect to boundary sections. Moreover, we will introduce a quasi-distance induced by boundary sections and show that the structure of our Monge–Ampère equation gives rise to a space of homogeneous type.

Chapter 10 proves several boundary second derivative estimates for the Monge–Ampère equation using Savin's Boundary Localization Theorem. We will prove the global $W^{2,1+\varepsilon}$ and $W^{2,p}$ estimates, the pointwise $C^{2,\alpha}$ estimates at the boundary, and global $C^{2,\alpha}$ estimates. The latter estimates were previously established by Trudinger and Wang with a different method. We will present Wang's examples to show that the conditions imposed on the boundary in our analysis are in fact optimal.

Chapter 11 is concerned with the Monge–Ampère eigenvalue and eigenfunctions. Similar to the variational characterization of the first Laplace eigenvalue using Rayleigh quotient, we will show that the Monge–Ampère eigenvalue also has a variational characterization using a Monge–Ampère functional. We will study basic properties of this Monge–Ampère functional. The existence of the Monge–Ampère eigenfunctions is established using compactness and solutions to certain degenerate Monge–Ampère equations. In turn, these solutions are found using a parabolic Monge–Ampère flow.

Part 2: The Linearized Monge–Ampère Equation

Chapter 12 proves the Caffarelli–Gutiérrez interior Harnack inequality for the linearized Monge–Ampère equation with the aid of Savin's method of sliding paraboloids. Then we obtain interior Hölder estimates for the inhomogeneous linearized Monge–Ampère equations with L^n right-hand side. Using the Caffarelli–Gutiérrez interior Hölder estimates and Caffarelli's interior $C^{2,\alpha}$ estimates, we sketch the Trudinger–Wang resolution of Chern's conjecture that a locally uniformly convex solution to the affine maximal surface equation in two dimensions must be a quadratic polynomial.

Chapter 13 establishes boundary Hölder, Harnack, and gradient estimates and regularity for solutions to the linearized Monge–Ampère equations under natural assumptions on the domain, Monge–Ampère measures, and boundary data. We briefly describe an application of the boundary Hölder estimates to the Sobolev solvability of global solutions to the second boundary value problems of the prescribed affine mean curvature equation and Abreu's equation. We also briefly describe applications of the boundary Hölder gradient estimates to problems in the calculus of variations concerning minimizers of linear functionals with prescribed determinant.

Chapter 14 is concerned with the Green's function of the linearized Monge–Ampère operator. We prove sharp pointwise estimates and study the integrability of the Green's function and its gradient. In particular, we show that the Green's function of the linearized Monge–Ampère operator has the same integrability range as that of the Green's function of the Laplace operator. The Monge–Ampère Sobolev inequality will be proved. We apply properties of the Green's function to establish local and global Hölder estimates for solutions to the linearized Monge–Ampère equation in terms of the low integrability norm of the inhomogeneity.

Chapter 15 focuses on Hölder estimates for solutions to the inhomogeneous linearized Monge–Ampère equation with right-hand side being the divergence of a bounded vector field. They will be applied to prove the Hölder estimates for the dual semigeostrophic equations in two dimensions when the initial density is bounded away from zero and infinity. We also apply these estimates and those in Chapter 14 to study the solvability of second boundary value problems of singular fourth-order equations of Abreu type arising from the approximation of several convex functionals whose Lagrangians depend on the gradient variable, subject to a convexity constraint.

1.4. Notes

As can be seen in Examples 1.9 and 1.16, solutions of Monge–Ampère equations arising in fluid mechanics and elasticity are not a priori required to be convex. The Monge–Ampère equation with nonconvex solutions is an important direction but so far, it is not well understood. However, we refer the reader to interesting papers [**LMP**, **LP**, **Lwk**] and their references for recent developments in this direction. In this book, the only result where we treat the Monge–Ampère equation with nonconvex solutions is Theorem 3.43 at the cost of requiring the high integrability of the Hessian of the solutions.

For boundary value problems concerning the Monge–Ampère and linearized Monge–Ampère equations, we only consider the Dirichlet boundary condition. There is a vast literature on other boundary conditions. As representatives, we refer the reader to [**LTU**] for the Neumann boundary condition, [**U4**] for oblique boundary condition, [**C8**, **CLW**, **SY2**, **U3**] for the second boundary condition, and [**Hua2**, **Ru**] for the Guillemin boundary condition.

It should be emphasized that we cover in **Part 1** the regularity theory for the Monge–Ampère equations mostly for Monge–Ampère measures with density bounded between two positive constants. The general case of degenerate right-hand side is not covered though this is an important topic in itself. However, only a special case of degeneracy will be treated as it will be used for the treatment of the Monge–Ampère eigenvalue in Chapter 11. Likewise, for the linearized Monge–Ampère equations in **Part 2**, we focus on the equations whose convex potentials have Hessian determinants being bounded between two positive constants. These potentials appear in many applications and they form a general class of settings for the linearized Monge–Ampère equation. To keep the book at a reasonable length, we omit some interesting results dealing with potentials having more regularities such as those having continuous Monge–Ampère measures or potentials having less regularities such as those having Monge–Ampère measures satisfying only a doubling condition. Moreover, we only focus on regularity estimates for smooth solutions with smooth potentials, and we manage to obtain estimates that are independent of the assumed smoothness. These are sufficient for applications where solvability relies on regularity estimates and fixed point arguments.

1.5. Problems

Problem 1.1. Let Ω be a bounded, measurable set in $\mathbb{R}^n$. Show that for any real $n \times n$ matrix A we have

$$|A(\Omega)| = |\det A||\Omega|.$$

Problem 1.2. Let ω_n be the volume of the unit ball in $\mathbb{R}^n$. Show that

$$\int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^{\frac{n+2}{2}}}\, dx = \omega_n.$$

Problem 1.3. Let Ω be a bounded domain in $\mathbb{R}^n$, and let $K, \varphi \in C(\overline{\Omega})$. Suppose that there is a convex solution $u \in C^{0,1}(\overline{\Omega}) \cap C^2(\Omega)$ to

$$\det D^2 u = K(1 + |Du|^2)^{(n+2)/2} \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega.$$

Show that K must satisfy $\int_\Omega K\,dx < \omega_n$.
Hint: Observe that

$$\int_\Omega K\,dx = \int_{Du(\Omega)} \frac{1}{(1 + |y|^2)^{\frac{n+2}{2}}}\,dy.$$

(This problem is based upon Trudinger–Urbas [**TU**].)

Problem 1.4. In Example 1.12, deduce the second form of Abreu's equation from the original one.

Problem 1.5. Verify formulae (1.8) and (1.9).

Problem 1.6. Let u be a C^2 function on $\mathbb{R}^n$. Recall $D_j u = \frac{\partial u}{\partial x_j}$. Prove that, as differential forms,

$$dD_1 u \wedge \cdots \wedge dD_n u = (\det D^2 u) dx_1 \wedge \cdots \wedge dx_n.$$

Problem 1.7. Consider the function $u(x) = |x| - 1$ on $B_1(0) \subset \mathbb{R}^n$ where $n \geq 2$. Show that

(a) $\det D^2 u(x) = 0$ for all $x \neq 0$, $u = 0$ on $\partial B_1(0)$, and
(b) $u \in W^{2,p}(B_1(0))$ for $1 \leq p < \infty$, if and only if $p < n$.

Chapter 3

Aleksandrov Solutions and Maximum Principles

This chapter aims to answer very basic questions in the theory of the Monge–Ampère equation: *What are suitable concepts of solutions to a Monge–Ampère equation? Given a Monge–Ampère equation, does there exist a solution to it?*

We motivate various notions of convex solutions to the Monge–Ampère equation. We will introduce the important notion of Monge–Ampère measure of a convex function. With this notion, we can define Aleksandrov solutions to a Monge–Ampère equation without requiring them to be twice differentiable everywhere. Then we will prove the Aleksandrov maximum principle which is a basic estimate in the Monge–Ampère equation. When coupled with the weak continuity property of Monge–Ampère measure, it gives a compactness result for the Monge–Ampère equation which will turn out to be vital in later chapters. Various maximum principles will also be established including the Aleksandrov–Bakelman–Pucci maximum principle. We will prove the comparison principle and establish various optimal global Hölder estimates. Then we discuss the solvability of the inhomogeneous Dirichlet problem with continuous boundary data via Perron's method. A comparison principle for $W^{2,n}$ nonconvex functions is also provided.

3.1. Motivations and Heuristics

In linear, second-order PDEs in divergence form, we can define a notion of weak solutions having less than two derivatives using integration by parts. As an example, consider Poisson's equation on a domain Ω in $\mathbb{R}^n$:

$$\Delta u = f \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega. \tag{3.1}$$

If u and v are smooth up to the boundary of Ω and $v = 0$ on $\partial\Omega$, then multiplying both sides of (3.1) by v and integrating by parts, we get

$$-\int_\Omega Du \cdot Dv \, dx = \int_\Omega fv \, dx. \tag{3.2}$$

Identity (3.2) allows us to define a notion of weak solutions to (3.1) with u having only one derivative in a weak sense. Precisely, we require that $u \in W_0^{1,2}(\Omega)$, f belongs to the dual of $W_0^{1,2}(\Omega)$, and (3.2) holds for all test functions $v \in W_0^{1,2}(\Omega)$. The theory of linear, second-order PDEs in divergence form is intimately connected to Sobolev spaces.

Now, we would like to define a suitable concept of weak solutions to the Monge–Ampère equation

$$\det D^2 u = f \geq 0 \quad \text{in } \Omega, \tag{3.3}$$

which is similar to that of (3.1), and we will try to use less than two derivatives for possible convex solutions $u : \Omega \to \mathbb{R}$. (We note that u is twice differentiable almost everywhere by Aleksandrov's Theorem (Theorem 2.89). However, using this almost everywhere pointwise information will not give us a good notion of solutions as it lacks uniqueness. For example, in this notion, the equation $\det D^2 u = 0$ on $B_1(0) \subset \mathbb{R}^n$ with zero boundary data has infinitely many solutions, including 0 and $a(|x| - 1)$ for $a > 0$.) Of course, when f is continuous, we can also speak of a *classical solution* to (3.3) where the convex function u is required to be C^2. Although the Monge–Ampère equation can be written in divergence form $D_i(U^{ij} D_j u) = nf$, we cannot mimic the integration by parts as in (3.2) because this would give

$$-\int_\Omega U^{ij} D_j u D_i v \, dx = \int_\Omega nfv \, dx, \tag{3.4}$$

which still requires two derivatives for u to define the cofactor matrix $U = (U^{ij})_{1 \leq i,j \leq n}$ of the Hessian matrix $D^2 u$ with certain integrability, unless we are in one dimension. Thus, we need to proceed differently.

We first explain Aleksandrov's idea. Assume that in (3.3) we have $u \in C^2(\Omega)$ with positive definite Hessian $D^2 u > 0$ in Ω. Then for any nice set $E \subset \Omega$, we obtain from integrating both sides of (3.3) over E

$$\int_E f \, dx = \int_E \det D^2 u(x) \, dx = \int_{Du(E)} dy = |Du(E)|,$$

where the second equality follows from the change of variables $y = Du(x)$. Thus

$$\int_E f\,dx = |Du(E)|. \tag{3.5}$$

Upon inspecting (3.5), we observe that:

(1) $\int_E f\,dx$ can be defined for any Borel subset $E \subset \Omega$ if $\mu = f\,dx$ is a nonnegative Borel measure.

(2) $|Du(E)|$ can be defined if $Du(E)$ is Lebesgue measurable for any Borel subset $E \subset \Omega$, which is the case if $u \in C^1(\Omega)$.

The problem with observation (2) is that if u is a convex function, then u is generally only Lipschitz but not C^1. Thus, we barely fail. One example of a convex but not C^1 function in $\mathbb{R}^n$ is $u(x) = |x|$. In this case, u is not C^1 at 0. Geometrically, there are infinitely many supporting hyperplanes to the graph of u at 0. The set of slopes of these hyperplanes is the closed unit ball $\overline{B_1(0)}$ in $\mathbb{R}^n$.

Aleksandrov's key insight is to replace $Du(E)$ by $\partial u(E) = \bigcup_{x_0\in E} \partial u(x_0)$ where $\partial u(x_0)$ is the set of all slopes of supporting hyperplanes to the graph of u at $(x_0, u(x_0))$. Then define the Monge–Ampère measure μ_u of u to be

$$\mu_u(E) = |\partial u(E)| \quad \text{for each Borel set } E.$$

Now, u is called an *Aleksandrov solution* to (3.3) if $\mu_u = f\,dx$ in the sense of measures. Section 3.2 will carry out this insight in detail. This is based on the *measure-theoretic* method, with deep *geometric* insights.

We can also define a concept of solutions to the Monge–Ampère equation which is based on the *maximum principle*. Let $u \in C^2(\Omega)$ be a convex solution to (3.3) on a domain Ω in $\mathbb{R}^n$. Suppose $\phi \in C^2(\Omega)$ is convex, and let $x_0 \in \Omega$ be such that $u - \phi$ has a local minimum at x_0. Then, $D^2u(x_0) \geq D^2\phi(x_0)$ and hence from $f(x_0) = \det D^2u(x_0)$, we obtain

$$\det D^2\phi(x_0) \leq f(x_0). \tag{3.6}$$

Similarly, if $\varphi \in C^2(\Omega)$ and $x_0 \in \Omega$ are such that $u - \varphi$ has a local maximum at x_0, then

$$\det D^2\varphi(x_0) \geq f(x_0). \tag{3.7}$$

Inspecting the arguments leading to (3.6) and (3.7), we see that all we need concerning u and f is their continuity. Thus, we can define a concept of *viscosity solutions* to (3.3) based on (3.6) and (3.7). This will be carried out in Section 7.1.

Having defined notions of weak solutions to (3.3), we can now proceed to solve it, say on a bounded convex domain Ω in $\mathbb{R}^n$ with Dirichlet boundary condition $u = 0$ on $\partial\Omega$. As in the classical theory for solving (3.1), we can use Perron's method which is based on *the maximum principle and subsolutions*. For (3.3), this amounts to taking

$$u(x) := \sup\{v(x) : v \in C(\overline{\Omega}) \text{ is convex in } \Omega, v = 0 \text{ on } \partial\Omega, \text{ and } \mu_v \geq f\}.$$

We will discuss this in detail in Section 3.7.

For certain Monge–Ampère-type equations where the right-hand sides also depend on the unknown solution u, the maximum principle might not apply. In some cases, the *variational method* can be useful. To illustrate this idea, let us come back to Poisson's equation (3.1) with the identity (3.2) for its solution. The last equation serves as a starting point for solving (3.1) by looking for a minimizer, in $W_0^{1,2}(\Omega)$, of the functional

$$J_1[u] := \int_\Omega \Big(\frac{1}{2}|Du|^2 + fu\Big)\,dx.$$

If we try to mimic this procedure for (3.3), we end up with the analogue of (3.2), that is, (3.4). Thus, we might consider functionals involving the quantity $I_n[u] := \int_\Omega U^{ij} D_i u D_j u\,dx$ for a convex function u vanishing on $\partial\Omega$. As mentioned above, this functional requires two derivatives for u to define the cofactor matrix $(U^{ij})_{1\leq i,j\leq n}$ of D^2u. As a curious fact, in 1937, without mentioning functional spaces and required regularities for u, Courant and Hilbert [**CH1**, p. 278] (see also [**CH2**, p. 326]) suggested looking for minimizers of the following functional in two dimensions:

$$\int_\Omega [(D_1u)^2 D_{22}u - 2D_1uD_2uD_{12}u + (D_2u)^2D_{22}u + 6fu]\,dx = I_n[u] + \int_\Omega 6fu\,dx.$$

In the original German edition [**CH1**], the term $6fu$ appeared as $-4fu$. A similar functional was considered by Gillis [**Gil**] in 1950 where $6fu$ was replaced by $6F(x, Du)$. It seems that we could not make much use of the variational method for the Monge–Ampère equation due to the serious difficulty in defining the quantity $I_n[u]$ for a general convex function u. However, there is a twist to this. Note that if u is smooth and vanishes on $\partial\Omega$, then, by integrating by parts and using that $(U^{ij})_{1\leq i,j\leq n}$ is divergence-free, we find

$$-I_n[u] = \int_\Omega D_j(U^{ij}D_iu)u\,dx = \int_\Omega U^{ij}D_{ij}uu\,dx = \int_\Omega n(\det D^2u)u\,dx.$$

The last term can be defined for a general convex function u where it is replaced by $\int_\Omega nu\,d\mu_u$. With this, we can consider the following analogue of J_1 in solving (3.3):

$$J_n[u] := \int_\Omega \frac{1}{n+1}(-u)\,d\mu_u + \int_\Omega fu\,dx.$$

Minimizing this functional over all convex functions u vanishing on $\partial\Omega$ gives the unique solution to (3.3) with zero Dirichlet boundary condition; see Bakelman [**Ba3**] in two dimensions, Aubin [**Au1**] (in radially symmetric settings), and Bakelman [**Ba4**] in higher dimensions. Since J_n is lower semicontinuous (as can be seen from Proposition 11.2), the existence of a minimizer $u_{\min}$ of J_n is always guaranteed. Showing that $u_{\min}$ is an Aleksandrov solution of (3.3) is, however, a very nontrivial task. The computation of the first variation of J_n at a critical point $u_{\min}$, which is only convex and has no other a priori smoothness properties, relies on deep geometric ideas. These include dual convex hypersurfaces and the theory of Minkowski mixed volumes (see Schneider [**Schn**]). These are beyond the scope of this book.

On the other hand, the variational analysis motivates Tso [**Ts**] to solve the Monge–Ampère-type equation

$$\det D^2 u = |u|^q \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where $q > 0$, using the functional

$$\int_\Omega \frac{1}{n+1}(-u)\, d\mu_u - \int_\Omega \frac{1}{q+1}|u|^{q+1}\, dx.$$

We will study this functional and its minimizers in a more classical setting in Chapter 11. This is possible due to the smoothness property of solutions, if any, to the above equation. In Chapter 14, we will again encounter the quadratic form $\int_\Omega U^{ij} D_i v D_j v\, dx$ when discussing the Monge–Ampère Sobolev inequality.

Since Aleksandrov's geometric method does not seem to have a complex Monge–Ampère equation counterpart where the natural class of solutions consists of plurisubharmonic functions, Bedford and Taylor [**BT**] introduced a powerful *analytic method*, based on the theory of currents (they are differential forms with distribution coefficients) to define the Monge–Ampère measure for a large class of plurisubharmonic functions in $\mathbb{C}^n$, including continuous plurisubharmonic functions. Their construction gives the following formula to define a Monge–Ampère measure $\mathcal{M}_u$ of a convex function u on $\mathbb{R}^n$:

$$\mathcal{M}_u = dD_1 u \wedge \cdots \wedge dD_n u.$$

We will not discuss this analytic method in this book. We just mention that, due to Rauch and Taylor [**RT**], $\mathcal{M}_u$ coincides with μ_u; see also Problem 1.6.

3.2. The Monge–Ampère Measure and Aleksandrov Solutions

A central notion in the theory of the Monge–Ampère equation is the Monge–Ampère measure. The following definition and its content are due to Aleksandrov [**Al1**] (see also Bakelman [**Ba1**] in two dimensions).

Definition 3.1 (The Monge–Ampère measure)**.** Let Ω be an open set in $\mathbb{R}^n$. Let $u : \Omega \to \mathbb{R}$ be a convex function. Given $E \subset \Omega$, we define

$$\mu_u(E) = |\partial u(E)| \quad \text{where } \partial u(E) = \bigcup_{x \in E} \partial u(x).$$

Then $\mu_u : \mathcal{S} \to [0, \infty]$ is a measure, finite on compact subsets of Ω where $\mathcal{S}$ is a Borel σ-algebra defined by

$$\mathcal{S} = \{E \subset \Omega : \partial u(E) \text{ is Lebesgue measurable}\}.$$

We call μ_u the Monge–Ampère measure associated with u.

Justification of μ_u being a measure. For this, our main observation is the following fact, which is an easy consequence of Aleksandrov's Lemma (Lemma 2.45): If A and B are disjoint subsets of Ω, then $|\partial u(A) \cap \partial u(B)| = 0$, so $\partial u(A)$ and $\partial u(B)$ are also *disjoint in the measure-theoretic sense.*

Let $E \subset \Omega$ be compact. Then, by Lemma 2.40(i), $\partial u(E)$ is compact; hence, it is Lebesgue measurable. Thus $E \in \mathcal{S}$ and $\mu_u(E) = |\partial u(E)| < \infty$. Now, by writing Ω as a union of compact sets, we deduce that $\Omega \in \mathcal{S}$.

Next, we show that $\Omega \backslash E \in \mathcal{S}$ if $E \in \mathcal{S}$. Indeed, since $E, \Omega \in \mathcal{S}$, the set $\partial u(\Omega) \backslash \partial u(E)$ is Lebesgue measurable. From $|\partial u(\Omega \backslash E) \cap \partial u(E)| = 0$ and

$$\partial u(\Omega \backslash E) = (\partial u(\Omega) \backslash \partial u(E)) \cup (\partial u(\Omega \backslash E) \cap \partial u(E)),$$

we find that $\partial u(\Omega \backslash E)$ is Lebesgue measurable and hence $\Omega \backslash E \in \mathcal{S}$.

Finally, we show that μ_u is σ-additive. Let $\{E_i\}_{i=1}^\infty$ be a sequence of disjoint sets in $\mathcal{S}$. We need to show that $\left|\partial u\left(\bigcup_{i=1}^\infty E_i\right)\right| = \sum_{i=1}^\infty |\partial u(E_i)|$. Indeed, this easily follows from the identities

$$\partial u\Big(\bigcup_{i=1}^\infty E_i\Big) = \bigcup_{i=1}^\infty \partial u(E_i) = \partial u(E_1) \cup \bigcup_{i=2}^\infty \Big(\partial u(E_i) \setminus \bigcup_{k=1}^{i-1} \partial u(E_k)\Big)$$

and the fact that $\{\partial u(E_i)\}_{i=1}^\infty$ are disjoint in the measure-theoretic sense. □

Example 3.2 (The Monge–Ampère measure of a cone)**.** Let $\Omega = B_R(x_0)$ and $u(x) = r|x - x_0|$ for $x \in \Omega$ where $r > 0$. By Example 2.35, we have

$$\partial u(x) = \begin{cases} \{r(x - x_0)/|x - x_0|\} & \text{if } 0 < |x - x_0| < R, \\ \overline{B_r(0)} & \text{if } x = x_0. \end{cases}$$

Therefore, $\mu_u = |B_r(0)|\delta_{x_0}$ where δ_{x_0} is the Dirac measure at $x_0 \in \mathbb{R}^n$.

Remark 3.3. Functions with cone-like graphs in Example 3.2 play the role of fundamental solutions to the Monge–Ampère equation; see Section 3.7 on the solvability of the Dirichlet problem. They are consistent with the notion of the fundamental solution of Laplace's equation when $n = 1$ as $\det D^2u = u''$ in one dimension. Recall that the function

$$\Phi_{(n)}(x) := \begin{cases} -\dfrac{1}{2\pi}\log|x| & \text{if } n = 2, \\ \dfrac{1}{n(n-2)\omega_n}\dfrac{1}{|x|^{n-2}} & \text{if } n \geq 3 \end{cases}$$

defined for $x \in \mathbb{R}^n \setminus \{0\}$ is the fundamental solution of Laplace's equation as we can write $-\Delta\Phi_{(n)} = \delta_0$ in $\mathbb{R}^n$, where δ_0 is the Dirac measure at 0. When $n = 1$, $\Phi_{(1)}(x) = -|x|/2$ with $[\Phi_{(1)}(x)]'' = -\delta_0$ becomes the fundamental solution of the one-dimensional Laplace equation. If $n = 1$ and $u(x) = |x|/2$, then we obtain from Example 3.2 that $\mu_u = \delta_0$. For an interesting application of the concavity of the fundamental solution of the one-dimensional Laplace equation, see Problem 3.7.

Example 3.4. Let Ω be an open set in $\mathbb{R}^n$. If $u \in C^{1,1}(\Omega)$ is convex, then $\mu_u = (\det D^2u)\,dx$ in Ω.

Proof. The result follows from the classical change of variables formula in calculus if u is C^2 with positive definite Hessian D^2u. In our situation with less smoothness, we will use the change of variables formula in Theorem 2.74. Since $u \in C^{1,1}(\Omega)$ is convex, we have $\partial u(x) = \{Du(x)\}$ for all $x \in \Omega$. Let $E \subset \Omega$ be a Borel set. Then, by Theorem 2.74, we have

$$\mu_u(E) = |\partial u(E)| = |Du(E)| = \int_E \det D^2u\,dx.$$

This shows that $\mu_u = (\det D^2u)\,dx$. □

Example 3.5. Let $u \in C(\overline{\Omega})$ be a convex function on a bounded domain Ω in $\mathbb{R}^n$. Then for any subdomain $\Omega' \Subset \Omega$, Lemma 2.37 gives $\partial u(\Omega') \subset \overline{B_r(0)}$ where $r = [\operatorname{dist}(\overline{\Omega'}, \partial\Omega)]^{-1} \operatorname{osc}_{\Omega} u$, and hence

$$\mu_u(\Omega') \leq \omega_n[\operatorname{dist}(\overline{\Omega'}, \partial\Omega)]^{-n}(\operatorname*{osc}_{\Omega} u)^n.$$

Remark 3.6 (Approximation by compact sets)**.** Let $u : \Omega \to \mathbb{R}$ be a convex function on an open set Ω in $\mathbb{R}^n$. From the definition of μ_u, we find that it has the approximation by compact sets property. That is, for each open subset $\mathcal{O} \Subset \Omega$, we have

$$\mu_u(\mathcal{O}) = \sup_{K\subset\mathcal{O},\ K \text{ is compact}} \mu_u(K).$$

As such, μ_u is a Radon measure, that is, a measure which is Borel regular and finite on compact sets.

A very basic fact of the Monge–Ampère measure is its weak continuity property:

Theorem 3.7 (Weak continuity of Monge–Ampère measure). *Let $\{u_k\}_{k=1}^{\infty}$ be a sequence of convex functions on an open set Ω in $\mathbb{R}^n$ which converges to a convex function $u : \Omega \to \mathbb{R}$ uniformly on compact subsets of Ω. Then μ_{u_k} converges weakly* to μ_u and we write $\mu_{u_k} \overset{*}{\rightharpoonup} \mu_u$; that is,*

$$\lim_{k\to\infty} \int_\Omega f\, d\mu_{u_k} = \int_\Omega f\, d\mu_u \quad \textit{for all } f \in C_c(\Omega).$$

Proof. Due to Remark 3.6 and Theorem 2.68, we only need to verify the following assertions:

(1) If $K \subset \Omega$ is a compact set, then $\limsup_{k\to\infty} \mu_{u_k}(K) \leq \mu_u(K)$.

(2) If $\mathcal{O} \Subset \Omega$ is an open set, then $\mu_u(\mathcal{O}) \leq \liminf_{k\to\infty} \mu_{u_k}(\mathcal{O})$.

From Remark 3.6, instead of proving assertion (2), we only need to prove:

(2′) If K is a compact set and $\mathcal{O}$ is open such that $K \subset \mathcal{O} \Subset \Omega$, then

$$\mu_u(K) \leq \liminf_{k\to\infty} \mu_{u_k}(\mathcal{O}).$$

The proof of assertions (1) and (2′) uses Aleksandrov's Lemma (Lemma 2.45) together with the Dominated Convergence Theorem and the following inclusions:

(3) $\limsup_{k\to\infty} \partial u_k(K) := \bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} \partial u_k(K) \subset \partial u(K)$.

(4) $\partial u(K)\backslash \mathfrak{S} \subset \liminf_{k\to\infty} \partial u_k(\mathcal{O}) := \bigcup_{i=1}^{\infty} \bigcap_{k=i}^{\infty} \partial u_k(\mathcal{O})$, where

$$\mathfrak{S} = \{p \in \mathbb{R}^n : \text{there are } x \neq y \in \Omega \text{ such that } p \in \partial u(x) \cap \partial u(y)\}.$$

For completeness, we indicate, for instance, how assertion (3) and the Dominated Convergence Theorem give assertion (1). By definition, we have

$$\begin{aligned}\limsup_{k\to\infty} \mu_{u_k}(K) = \limsup_{k\to\infty} |\partial u_k(K)| &:= \lim_{i\to\infty} \sup_{k\geq i} |\partial u_k(K)| \\ &\leq \lim_{i\to\infty} \Big|\bigcup_{k=i}^{\infty} \partial u_k(K)\Big|.\end{aligned}$$

By assertion (3), we have

$$\Big|\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} \partial u_k(K)\Big| \leq |\partial u(K)| = \mu_u(K).$$

Now, we will use the Dominated Convergence Theorem to show that

$$\lim_{i\to\infty} \Big|\bigcup_{k=i}^{\infty} \partial u_k(K)\Big| = \Big|\bigcap_{i=1}^{\infty} \bigcup_{k=i}^{\infty} \partial u_k(K)\Big|.$$

Indeed, let $f_i(p) = \chi_{\bigcup_{k=i}^\infty \partial u_k(K)}(p)$ for $p \in \mathbb{R}^n$. Then

$$\lim_{i\to\infty} f_i(p) = \chi_{\bigcap_{i=1}^\infty \bigcup_{k=i}^\infty \partial u_k(K)}(p).$$

So, it suffices to show that each f_i is bounded from above by an integrable function g on $\mathbb{R}^n$. To do this, let $K' \Subset \Omega$ be a compact set containing K such that $r := \operatorname{dist}(K, \partial K') > 0$. By the uniform convergence of u_k to u on K', there is a constant $M > 0$ such that $\|u_k\|_{L^\infty(K')} \leq M$ for all k. Now, if $p \in \partial u_k(x)$ for some $x \in K$, then by Lemma 2.37, we have $|p| \leq [\operatorname{dist}(x, \partial K')]^{-1} \operatorname{osc}_{K'} u_k \leq 2Mr^{-1} := r'$. It follows that $f_i \leq g := \chi_{B_{r'}(0)}$ and this g is the desired integrable function on $\mathbb{R}^n$.

To prove our theorem, it remains to prove assertions (3) and (4).

For assertion (3), let $p \in \limsup_{k\to\infty} \partial u_k(K)$. Then for each i, there are k_i and $z^{k_i} \in K$ such that $p \in \partial u_{k_i}(z^{k_i})$. Since K is compact, extracting a subsequence of $\{z^{k_i}\}_{i=1}^\infty$, still labeled $\{z^{k_i}\}_{i=1}^\infty$, we have $z^{k_i} \to z \in K$. Thus, using $u_{k_i}(x) \geq u_{k_i}(z^{k_i}) + p \cdot (x - z^{k_i})$ for all $x \in \Omega$ and the uniform convergence of u_{k_i} to u on compact subsets of Ω, we obtain

$$u(x) \geq u(z) + p \cdot (x - z) \quad \text{for all } x \in \Omega$$

and therefore, $p \in \partial u(z) \subset \partial u(K)$.

For assertion (4), let $p \in \partial u(x_0) \subset \partial u(K) \setminus \mathfrak{S}$. Then, $u(x) - l(x) (\geq 0)$ is strictly convex at x_0, where $l(x) = u(x_0) + p \cdot (x - x_0)$; see Definition 2.39. By subtracting $l(x)$ from u_k and u, we can assume $p = 0$, $u(x_0) = 0$, and we need to show that $0 \in \partial u_k(z^k)$ for all k large and some $z^k \in \mathcal{O}$. Recalling Remark 2.34, we prove this by choosing a minimum point z^k of the continuous function u_k in the compact set $\bar{\mathcal{O}}$. It remains to show that $z^k \notin \partial\mathcal{O}$ when k is large. Indeed, from the strict convexity of u at $x_0 \in K \subset \mathcal{O}$, we can find some $\gamma > 0$ such that $u(x) \geq \gamma$ on $\partial\mathcal{O}$. Hence, from the uniform convergence of u_k to u on compact sets, we find that $u_k \geq \gamma/2$ on $\partial\mathcal{O}$ if k is large. On the other hand, since $u(x_0) = 0$, we also find that $u_k(x_0) \leq \gamma/4$ when k is large. Therefore, $z^k \notin \partial\mathcal{O}$ when k is large. □

Definition 3.8 (Aleksandrov solutions)**.** Given an open set $\Omega \subset \mathbb{R}^n$ and a Borel measure ν on Ω, a convex function $u : \Omega \to \mathbb{R}$ is called an *Aleksandrov solution* to the Monge–Ampère equation

$$\det D^2 u = \nu$$

if $\mu_u = \nu$ as Borel measures. When $\nu = f\ dx$, we will simply say that u solves

$$\det D^2 u = f,$$

and this is the notation we use in the book.

Similarly, when writing $\det D^2 u \geq \lambda\ (\leq \Lambda)$ in the sense of Aleksandrov, we mean that $\mu_u \geq \lambda\, dx\ (\leq \Lambda\, dx)$.

Notes. For possibly nonconvex $v \in W^{2,n}_{\mathrm{loc}}(\Omega)$ $(C^{1,1}(\Omega))$, $\det D^2 v$ is interpreted in the usual pointwise sense, and it belongs to $L^1_{\mathrm{loc}}(\Omega)$ $(L^\infty_{\mathrm{loc}}(\Omega))$.

The following lemma is useful in studying the Monge–Ampère equation:

Lemma 3.9 (Monge–Ampère measure under affine transformations)**.** *Let $u : \Omega \to \mathbb{R}$ be a convex function on an open set Ω in $\mathbb{R}^n$. Let $A \in \mathbb{M}^{n\times n}$ be invertible, $\mathbf{b}, \mathbf{c} \in \mathbb{R}^n$, $d \in \mathbb{R}$, and $\gamma > 0$. Denote $Tx = Ax + \mathbf{b}$. Let*

$$v(x) = \gamma u(Tx) + \mathbf{c}\cdot x + d \quad \textit{for } x \in T^{-1}\Omega.$$

(i) *If u is $C^2(\Omega)$, then*

$$\det D^2 v(x) = \gamma^n (\det A)^2 \det D^2 u(Tx).$$

(ii) *If, in the sense of Aleksandrov,*

$$\lambda \le \det D^2 u \le \Lambda \quad \textit{in } \Omega,$$

then we also have, in the sense of Aleksandrov,

$$\lambda\gamma^n |\det A|^2 \le \det D^2 v \le \Lambda\gamma^n |\det A|^2 \quad \textit{in } T^{-1}\Omega.$$

Proof. If $u \in C^2(\Omega)$, then

$$Dv(x) = \gamma A^t Du(Tx) + \mathbf{c} \quad \text{and} \quad D^2 v(x) = \gamma A^t (D^2 u(Tx))A.$$

Taking determinants proves assertion (i).

We now prove (ii) using the normal mapping. From Definition 2.30, we infer that

$$\partial v(x) = \gamma A^t \partial u(Tx) + \mathbf{c} \quad \text{for any } x \in T^{-1}(\Omega). \tag{3.8}$$

Now, let $E \subset T^{-1}\Omega$ be a Borel set. Then, by (3.8) and $\det A^t = \det A$,

$$\mu_v(E) = |\partial v(E)| = |\gamma A^t \partial u(T(E)) + \mathbf{c}| = \gamma^n |\det A||\mu_u(T(E))|.$$

Because $\lambda \le \det D^2 u \le \Lambda$ in the sense of Aleksandrov, we have

$$\lambda|\det A||E| = \lambda|T(E)| \le |\mu_u(T(E))| \le \Lambda|T(E)| = \Lambda|\det A||E|.$$

The assertion (ii) then follows from

$$\lambda\gamma^n|\det A|^2|E| \le \mu_v(E) \le \Lambda\gamma^n|\det A|^2|E|.$$

The lemma is proved. □

We now show that the Monge–Ampère measure is superadditive. It is a consequence of the concavity of $(\det D^2 u)^{1/n}$ on the set of C^2 convex functions.

Lemma 3.10 (Multiplicative and superadditive properties of the Monge–Ampère measure)**.** *Let $u, v : \Omega \to \mathbb{R}$ be convex functions on an open set Ω in $\mathbb{R}^n$. Then*

$$\mu_{\lambda u} = \lambda^n \mu_u \quad \textit{for } \lambda > 0 \quad \textit{and} \quad \mu_{u+v} \ge \mu_u + \mu_v.$$

Proof. The multiplicative property is a consequence of Definition 3.1 and the fact that $\partial(\lambda u)(E) = \lambda \partial u(E)$ for all Borel sets $E \subset \Omega$.

Now we prove the superadditive property. We will reduce the proof to the case where u and v are C^2 because in this case, by Lemma 2.59,

$$\det D^2(u+v) \geq ((\det D^2 u)^{1/n} + (\det D^2 v)^{1/n})^n \geq \det D^2 u + \det D^2 v.$$

In general, u and v may not be C^2. We will use an approximation argument together with Theorem 3.7. Extend u and v to convex functions on $\mathbb{R}^n$. Let $\varphi_{1/k}$ be a standard mollifier with support in $\overline{B_{1/k}(0)}$ and $\int_{\mathbb{R}^n} \varphi_{1/k}\, dx = 1$; see Definition 2.49. Consider $u_k := u * \varphi_{1/k}$ and $v_k = v * \varphi_{1/k}$. Then, by Theorem 2.50, $u_k, v_k \in C^\infty(\Omega)$ are convex functions and hence,

$$\mu_{u_k+v_k} \geq \mu_{u_k} + \mu_{v_k}.$$

Since $u_k \to u$ and $v_k \to v$ uniformly on compact subsets of Ω, by Theorem 3.7, we have

$$\mu_{u_k+v_k} \overset{*}{\rightharpoonup} \mu_{u+v}, \quad \mu_{u_k} \overset{*}{\rightharpoonup} \mu_u, \quad \mu_{v_k} \overset{*}{\rightharpoonup} \mu_v.$$

Therefore, we must have $\mu_{u+v} \geq \mu_u + \mu_v$, as asserted. □

3.3. Maximum Principles

In the analysis of Monge–Ampère equations, maximum principles are indispensable tools. In this section, we discuss the most basic ones involving convex functions and also nonconvex functions.

The following basic maximum principle implies that if two convex functions defined on the same domain and having the same boundary values, the one below the other will have larger image of the normal mapping.

Lemma 3.11. *Let Ω be a bounded open set in $\mathbb{R}^n$, and let $u, v \in C(\overline{\Omega})$.*

(i) (Pointwise maximum principle) *If $u \geq v$ on $\partial\Omega$ and $v(x_0) \geq u(x_0)$ where $x_0 \in \Omega$, then $\partial v(x_0) \subset \partial u(\Omega)$.*

(ii) (Maximum principle) *If $u = v$ on $\partial\Omega$ and $u \leq v$ in Ω, then $\partial v(\Omega) \subset \partial u(\Omega)$.*

Proof. Since assertion (ii) is a consequence of assertion (i), we only need to prove (i). Let $p \in \partial v(x_0)$. Then, p is the slope of a supporting hyperplane $l_0(x)$ to the graph of v at $(x_0, v(x_0))$; that is,

$$v(x) \geq l_0(x) := v(x_0) + p \cdot (x - x_0) \quad \text{for all } x \in \overline{\Omega}.$$

We will slide down l_0 by the amount

$$a := \max_{\overline{\Omega}}(l_0 - u) \geq l_0(x_0) - u(x_0) = v(x_0) - u(x_0) \geq 0$$

to obtain a supporting hyperplane $l(x) := l_0(x) - a$ to the graph of u at some point $(z, u(z))$ where $z \in \Omega$, and hence $p \in \partial u(\Omega)$. See Figure 3.1. To

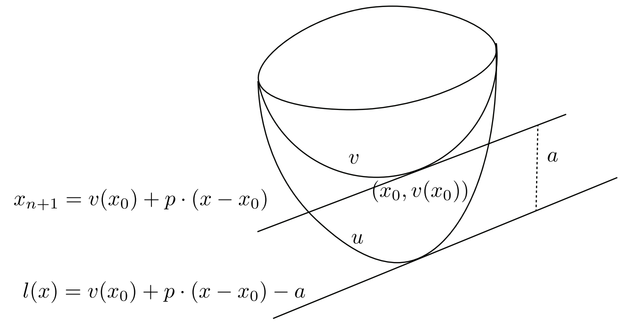

Figure 3.1. Sliding down the hyperplane $v(x_0) + p \cdot (x - x_0)$ to obtain a supporting hyperplane for the graph of u.

see this, we first note that $a \geq l_0(x) - u(x)$ for any $x \in \Omega$, so

$$u(x) \geq l_0(x) - a = l(x) \quad \text{in } \Omega.$$

It suffices to find $z \in \Omega$ such that $u(z) = l(z)$. By our assumptions,

$$\sup_{\partial\Omega}(l_0 - u) \leq \sup_{\partial\Omega}(v - u) \leq 0.$$

If $a = 0$, then $v(x_0) = u(x_0)$, and we can take $z = x_0$. Now, consider $a > 0$. Let $\bar{x} \in \overline{\Omega}$ be such that $l_0(\bar{x}) - u(\bar{x}) = a > 0$. Then, $\bar{x} \notin \partial\Omega$, and $u(\bar{x}) = l_0(\bar{x}) - a = l(\bar{x})$, so we can take $z = \bar{x} \in \Omega$. □

Observe that, for the maximum principles in Lemma 3.11, no convexity on the functions nor the domain is assumed.

The following maximum principle, due to Aleksandrov [**Al4**], is of fundamental importance in the theory of Monge–Ampère equations. It quantifies how much a convex function, with finite total Monge–Ampère measure, will drop its values when stepping inside the domain.

Theorem 3.12 (Aleksandrov's maximum principle)**.** *Let Ω be a bounded and convex domain in $\mathbb{R}^n$. Let $u \in C(\overline{\Omega})$ be a convex function with $u = 0$ on $\partial\Omega$. Consider the diameter-like function $D(x) := \max_{y \in \partial\Omega} |x - y|$. Then for all $x_0 \in \Omega$, we have*

$$|u(x_0)|^n \leq \min \left\{ n\omega_{n-1}^{-1} D^{n-1}(x_0) \operatorname{dist}(x_0, \partial\Omega) |\partial u(\Omega)|, \ \omega_n^{-1} D^n(x_0) |\partial u(\Omega)| \right\}.$$

Proof. Let $v \in C(\overline{\Omega})$ be the convex function whose graph is the cone with vertex $(x_0, u(x_0))$ and the base Ω, with $v = 0$ on $\partial\Omega$. See Figure 3.2. Explicitly, we have $v(tx_0 + (1-t)x) = tu(x_0)$ for all $x \in \partial\Omega$ and $t \in [0, 1]$.

Since u is convex, $0 \geq v \geq u$ in Ω. By the maximum principle in Lemma 3.11, $\partial v(\Omega) \subset \partial u(\Omega)$. Our proof is now based on the following observations:

(1) $\partial v(\Omega) = \partial v(x_0)$, and thus $\partial v(\Omega)$ is convex.

(2) $\partial v(\Omega)$ contains $B_{\frac{|u(x_0)|}{D(x_0)}}(0)$.

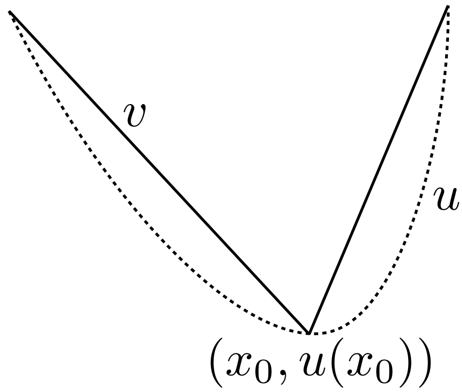

Figure 3.2. Graphs of u and v. The graph of v is the cone with vertex $(x_0, u(x_0))$ and $v = 0$ on $\partial\Omega$.

(3) Take $z \in \partial\Omega$ such that $|z - x_0| = \text{dist}(x_0, \partial\Omega)$. Then

$$p_0 = -u(x_0)\frac{z - x_0}{|z - x_0|^2} \in \partial v(\Omega).$$

Assuming assertions (1)–(3), we see that $\partial v(\Omega)$ contains the convex hull of p_0 and $B_{\frac{|u(x_0)|}{D(x_0)}}(0)$. This convex hull has measure at least

$$\max\left\{\omega_n\left(\frac{|u(x_0)|}{D(x_0)}\right)^n, \frac{\omega_{n-1}}{n}\left(\frac{|u(x_0)|}{D(x_0)}\right)^{n-1}|p_0|\right\}.$$

Since $|\partial v(\Omega)| \leq |\partial u(\Omega)|$ and $|p_0| = |u(x_0)|/\text{dist}(x_0, \partial\Omega)$, the theorem follows.

Let us now verify assertions (1)–(3). To see assertion (1), we note that if $p \in \partial v(\Omega)$, then $p = \partial v(z)$ for some $z \in \Omega$. It suffices to consider the case $z \neq x_0$. Since the graph of v is a cone, $v(x) + p \cdot (x - z)$ is a supporting hyperplane to the graph of v at $(x_0, v(x_0))$; that is, $p \in \partial v(x_0)$.

For assertions (2) and (3), we use the fact from v being a cone that $p \in \partial v(x_0)$ if and only if $v(x) \geq v(x_0) + p \cdot (x - x_0)$ for all $x \in \partial\Omega$. Thus, assertion (2) is clearly valid.

To prove assertion (3), we note that for any $x \in \partial\Omega$, $(x - x_0) \cdot \frac{z - x_0}{|z - x_0|}$ is the vector projection of $x - x_0$ onto the ray $\overrightarrow{x_0 z}$. Since Ω is convex,

$$(x - x_0) \cdot \frac{z - x_0}{|z - x_0|} \leq |z - x_0|,$$

and hence, from the formula for p_0, we find that for all $x \in \partial\Omega$,

$$0 = v(x) = u(x_0) + |p_0||z - x_0| \geq v(x_0) + p_0 \cdot (x - x_0).$$

Therefore $p_0 \in \partial v(x_0)$ as claimed. The theorem is proved. □

Remark 3.13. Theorem 3.12 is sharp. If $\Omega = B_R(x_0)$ and $u(x) = |x - x_0| - R$ (see Example 3.2), then the inequality in Theorem 3.12 becomes equality.

A simple consequence of Aleksandrov's maximum principle is:

Corollary 3.14. *Let Ω be a bounded and convex domain in $\mathbb{R}^n$, and let $u \in C(\overline{\Omega})$ be a convex function. Then, for all $x \in \Omega$, we have*

$$\inf_{\partial\Omega} u - C(n)[\mathrm{dist}(x, \partial\Omega)]^{1/n}[\mathrm{diam}(\Omega)]^{\frac{n-1}{n}}|\partial u(\Omega)|^{1/n} \leq u(x) \leq \sup_{\partial\Omega} u.$$

Proof. Note that the inequality $u(x) \leq \sup_{\partial\Omega} u$ just follows from the convexity of u. For the lower bound on u, let $v(x) = u(x) - \inf_{\partial\Omega} u$. Then $v \geq 0$ on $\partial\Omega$ and $\partial v(\Omega) = \partial u(\Omega)$. If $v(x) \geq 0$ for all $x \in \Omega$, then we are done. If this is not the case, then $E = \{x \in \Omega : v(x) < 0\}$ is a convex domain, with $v = 0$ on ∂E. We apply Theorem 3.12 to conclude that for each $x \in E$,

$$\begin{aligned}(-v(x))^n = |v(x)|^n &\leq C(n)[\mathrm{diam}(E)]^{n-1}\mathrm{dist}(x, \partial E)|\partial v(E)| \\ &\leq C(n)[\mathrm{diam}(\Omega)]^{n-1}\mathrm{dist}(x, \partial\Omega)|\partial u(\Omega)|, \quad C(n) = n\omega_{n-1}^{-1}.\end{aligned}$$

The corollary follows. □

The conclusion of Theorem 3.12 raises the following question: Will a convex function drop its value when stepping inside the domain? Clearly, without a lower bound on the Monge–Ampère measure, the answer is in the negative as can be seen from the constant 0. However, we will provide a positive answer in Lemma 3.24 when the density of the Monge–Ampère measure μ_u has a positive lower bound. By convexity, it suffices to obtain a positive lower bound for $\|u\|_{L^\infty(\Omega)}$ (see Lemma 2.37(i)).

For functions that are not necessarily convex, we will establish in Theorem 3.16 the Aleksandrov–Bakelman–Pucci (ABP) maximum principle which is of fundamental importance in fully nonlinear elliptic equations. Its proof is based on following Aleksandrov-type estimates whose formulation involves the notion of the *upper contact set* Γ^+ of a continuous function u defined on a domain Ω in $\mathbb{R}^n$. It is defined as follows (see Figure 3.3):

$$\Gamma^+ = \{y \in \Omega : u(x) \leq u(y) + p\cdot(x - y) \text{ for all } x \in \Omega, \text{ for some } p = p(y) \in \mathbb{R}^n\}.$$

Lemma 3.15 (Aleksandrov-type estimates)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Then, for $u \in C^2(\Omega) \cap C(\overline{\Omega})$, we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \omega_n^{-1/n}\,\mathrm{diam}(\Omega)\Big(\int_{\Gamma^+} |\det D^2 u|\, dx\Big)^{1/n}, \tag{3.9}$$

where Γ^+ is the upper contact set of u in Ω.

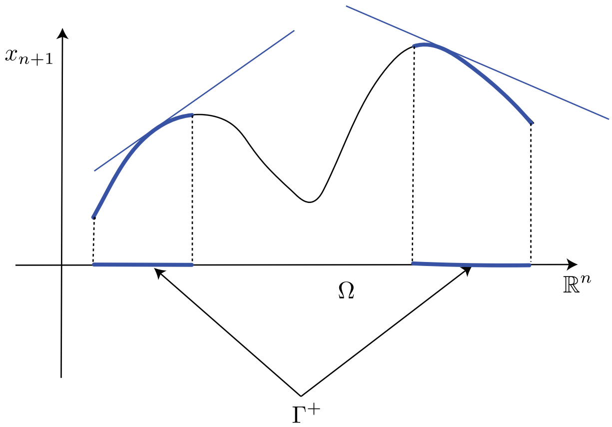

Figure 3.3. The upper contact set Γ^+.

Proof. By considering $u - \sup_{\partial\Omega} u$ instead of u, we can assume $\sup_{\partial\Omega} u = 0$. It suffices to prove (3.9) when the maximum of u on $\overline{\Omega}$ is attained at $x_0 \in \Omega$ with $u(x_0) > 0$. Our main observation is that

$$B_{u(x_0)/d} \subset Du(\Gamma^+), \quad d = \text{diam}(\Omega). \tag{3.10}$$

To see this, let $p \in B_{u(x_0)/d}$. Then, the function $v(x) := u(x) - p \cdot (x - x_0)$ satisfies $v(x_0) = u(x_0)$, while on $\partial\Omega$, $v(x) \leq |p||x - x_0| < u(x_0)$. Thus, v attains its maximum value on $\overline{\Omega}$ at some point $y \in \Omega$. Therefore, $Dv(y) = 0$ which gives $p = Du(y)$. From $v(x) \leq v(y)$ for all $x \in \Omega$, we deduce that $y \in \Gamma^+$. Hence, $p \in Du(T^+)$. The arbitrariness of p establishes (3.10).

Now, taking volumes in (3.10) and then invoking Theorem 2.74, we get

$$\omega_n \Big(\frac{u(x_0)}{d}\Big)^n \leq |Du(\Gamma^+)| \leq \int_{\Gamma^+} |\det D^2 u| \, dx.$$

This implies (3.9) and the proof of the lemma is complete. □

Inclusions of the type (3.10) are the basis for the ABP method in proving geometric inequalities. See Brendle [**Bre**], Cabré [**Cab2**], and Trudinger [**Tr2**]. We refer to Problem 3.6 and Theorem 4.16 for illustrations of this method.

Now, we can establish the Aleksandrov–Bakelman–Pucci maximum principle for elliptic second-order operators with no lower-order terms. *Only the determinant of the coefficient matrix, but not the bounds on its eigenvalues, enters into the ABP estimate.* This is very crucial in many applications.

Theorem 3.16 (Aleksandrov–Bakelman–Pucci maximum principle). *Let Ω be a bounded domain in $\mathbb{R}^n$. Let the coefficient matrix $A(x) = (a^{ij}(x))_{1\leq i,j\leq n}$ be measurable and positive definite in Ω. Let $Lu = a^{ij}D_{ij}u$ and $(Lu)^- = \max\{0, -Lu\}$. Then, the following statements hold:*

(i) *For $u \in C^2(\Omega) \cap C(\overline{\Omega})$, we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{n\omega_n^{1/n}} \left\| \frac{(Lu)^-}{(\det A)^{1/n}} \right\|_{L^n(\Gamma^+)},$$

where Γ^+ is the upper contact set of u in Ω.

(ii) *For $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, we have*

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega)}{n\omega_n^{1/n}} \left\| \frac{(Lu)^-}{(\det A)^{1/n}} \right\|_{L^n(\Omega)}. \tag{3.11}$$

Proof. We prove part (i). Since $u \in C^2(\Omega)$, $D^2u \leq 0$ on the upper contact set Γ^+. Using Lemma 2.57 for $A = (a^{ij})$ and $B = -D^2u$, we have on Γ^+,

$$|\det D^2u| = \det(-D^2u) \leq \frac{1}{\det A}\Big(\frac{-a^{ij}D_{ij}u}{n}\Big)^n \leq \frac{1}{\det A}\Big(\frac{(Lu)^-}{n}\Big)^n.$$

Integrating over Γ^+ and then applying Lemma 3.15, we obtain part (i).

Next, we prove part (ii). If $u \in C^2(\Omega) \cap C(\overline{\Omega})$, then (3.11) follows from part (i). Now, we prove (3.11) for $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$ by approximation arguments. Let $\Lambda(x)$ be the largest eigenvalue of $A(x)$. For $\delta > 0$, let $A_\delta = \delta\Lambda I_n + A$ and $L_\delta u = \operatorname{trace}(A_\delta D^2u)$.

Let $\{u_m\}_{m=1}^\infty \subset C^2(\Omega)$ be a sequence of functions converging to u in $W^{2,n}_{\mathrm{loc}}(\Omega)$; see Theorem 2.50. For any $\varepsilon > 0$, we can assume that u_m converges to u in $W^{2,n}(\Omega_\varepsilon)$ and $u_m \leq \varepsilon + \max_{\partial\Omega} u$ on $\partial\Omega_\varepsilon$ for some domain $\Omega_\varepsilon \Subset \Omega$ where Ω_ε is chosen so that it converges to Ω in the Hausdorff distance as $\varepsilon \to 0$. Let $C_\varepsilon = n^{-1}\omega_n^{-1/n}\operatorname{diam}(\Omega_\varepsilon)$. We can now apply (3.11) to u_m with

$$L_\delta u_m = L_\delta(u_m - u) + L_\delta u$$

in Ω_ε. Observing that $\|A_\delta/[\det A_\delta]^{1/n}\| \leq n(\delta+1)/\delta$ in Ω, we then get

$$\begin{aligned}\max_{\overline{\Omega_\varepsilon}} u_m &\leq \varepsilon + \max_{\partial\Omega} u + C_\varepsilon n\frac{\delta+1}{\delta}\|D^2(u_m - u)\|_{L^n(\Omega_\varepsilon)} \\ &\quad + C_\varepsilon \left\| \frac{(L_\delta u)^-}{(\det A_\delta)^{1/n}} \right\|_{L^n(\Omega_\varepsilon)}.\end{aligned} \tag{3.12}$$

Letting $m \to \infty$ in (3.12) and recalling that $\|D^2(u_m - u)\|_{L^n(\Omega_\varepsilon)}$ converges to 0 and $\{u_m\}$ converges uniformly to u on Ω_ε, we have

$$\max_{\overline{\Omega_\varepsilon}} u \leq \varepsilon + \max_{\partial\Omega} u + C_\varepsilon \left\| \frac{\delta\Lambda\Delta u}{(\det A_\delta)^{1/n}} \right\|_{L^n(\Omega_\varepsilon)} + C_\varepsilon \left\| \frac{(Lu)^-}{(\det A_\delta)^{1/n}} \right\|_{L^n(\Omega_\varepsilon)}.$$

Letting $\delta \to 0$ and using the Dominated Convergence Theorem, we obtain

$$\max_{\overline{\Omega_\varepsilon}} u \leq \varepsilon + \max_{\partial\Omega} u + \frac{\operatorname{diam}(\Omega_\varepsilon)}{n\omega_n^{1/n}} \Big\| \frac{(Lu)^-}{(\det A)^{1/n}} \Big\|_{L^n(\Omega_\varepsilon)}.$$

Letting $\varepsilon \to 0$, we obtain (3.11). The theorem is proved. $\square$

As an application of John's Lemma, we can refine the ABP maximum principle for convex domains where the diameter in Theorem 3.16 is now replaced by the nth root of the volume of the domain.

Lemma 3.17 (ABP estimate for convex domains)**.** *Let Ω be a bounded, convex domain in $\mathbb{R}^n$. Let the $n \times n$ coefficient matrix $A(x)$ be measurable and positive definite in Ω. Then, for all $u \in W^{2,n}_{\mathrm{loc}}(\Omega) \cap C(\overline{\Omega})$, we have*

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C(n)|\Omega|^{1/n} \Big\| \frac{\operatorname{trace}(AD^2u)}{(\det A)^{1/n}} \Big\|_{L^n(\Omega)}.$$

Proof. By Lemmas 2.62 and 2.63, there is an affine transformation $T(x) = Mx + b$, where $M \in \mathbb{S}^n$ is symmetric, positive definite and $b \in \mathbb{R}^n$ such that

$$B_1(0) \subset T(\Omega) \subset B_n(0). \tag{3.13}$$

Let $Lu = \operatorname{trace}(AD^2u)$. For $x \in T(\Omega)$, we define

$$v(x) = u(T^{-1}x), \quad \tilde{A}(x) = MA(T^{-1}x)M^t, \quad \text{and} \quad \tilde{L}v = \operatorname{trace}(\tilde{A}D^2v).$$

Then, $D^2v(x) = (M^{-1})^t D^2u(T^{-1}x)M^{-1}$, and $\tilde{L}v(x) = Lu(T^{-1}x)$. Applying the ABP estimate to v and $\tilde{L}v(x)$ on $T(\Omega)$, we find

$$\begin{aligned} \max_{\overline{T(\Omega)}} v &\leq \max_{\partial T(\Omega)} v + C_1(n)\operatorname{diam}(T(\Omega)) \Big\| \frac{\tilde{L}v}{(\det \tilde{A})^{1/n}} \Big\|_{L^n(T(\Omega))} \\ &= \max_{\partial\Omega} u + \frac{C_1(n)\operatorname{diam}(T(\Omega))}{(\det M)^{1/n}} \Big\| \frac{Lu}{(\det A)^{1/n}} \Big\|_{L^n(\Omega)}. \end{aligned} \tag{3.14}$$

From (3.13), we have $\det M \geq \omega_n|\Omega|^{-1}$ and $\operatorname{diam}(T(\Omega)) \leq 2n$. Using these estimates in (3.14), we obtain the conclusion of the lemma. $\square$

It is important to emphasize that Theorem 3.16 extends the maximum principle in Theorem 2.79 from C^2 functions to $W^{2,n}_{\mathrm{loc}}$ functions.

3.4. Global Hölder Estimates and Compactness

From the Aleksandrov maximum principle, we deduce a global $C^{0,1/n}$ regularity result for convex functions with finite total Monge–Ampère measure.

Lemma 3.18. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let $u \in C(\overline{\Omega})$ be a convex function with $u = 0$ on $\partial\Omega$ and $\mu_u(\Omega) < \infty$. Then $u \in C^{0,1/n}(\overline{\Omega})$ with*

$$\|u\|_{C^{0,1/n}(\overline{\Omega})} \leq C(n, \operatorname{diam}(\Omega), \mu_u(\Omega)).$$

Proof. By Theorem 3.12, we have the boundary Hölder estimate

$$|u(x)| \le C(n, \operatorname{diam}(\Omega), \mu_u(\Omega))\operatorname{dist}^{\frac{1}{n}}(x, \partial\Omega) \quad \text{for all } x \in \Omega.$$

The lemma now follows from Lemma 2.37(iv). □

Note that the exponent $1/n$ in Lemma 3.18 is sharp; see Problem 3.16.

Another consequence of Aleksandrov's maximum principle is the compactness of a family of convex functions on convex domains lying between two fixed balls with zero boundary values and having total Monge–Ampère measures bounded from above.

Theorem 3.19 (Compactness of solutions to the Monge–Ampère equation)**.** *Let λ, Λ, M be positive numbers. Let*

$$\mathcal{C}_{\Lambda,M} = \big\{(\Omega, u, \nu) : \Omega \subset \mathbb{R}^n \text{ is a convex domain}, B_{1/M}(0) \subset \Omega \subset B_M(0), \\ u \in C(\overline{\Omega}) \text{ is convex}, u|_{\partial\Omega} = 0, \mu_u = \nu, \nu(\Omega) \le \Lambda\big\}.$$

Then the set $\mathcal{C}_{\Lambda,M}$ is compact in the following sense: For any sequence $\{(\Omega_i, u_i, \nu_i)\}_{i=1}^{\infty}$ in $\mathcal{C}_{\Lambda,M}$, there exist a subsequence, which is still labeled as $\{(\Omega_i, u_i, \nu_i)\}_{i=1}^{\infty}$, and $(\Omega, u, \nu) \in \mathcal{C}_{\Lambda,M}$ such that

(1) *$\overline{\Omega_i}$ converges to $\overline{\Omega}$ in the Hausdorff distance,*

(2) *u_i converges to u locally uniformly in Ω, and*

(3) *$\nu_i = \mu_{u_i}$ converges weakly* to $\nu = \mu_u$.*

As a consequence, we obtain the compactness of each the following sets:

$$C_{\Lambda,M} = \big\{(\Omega, u) : \Omega \subset \mathbb{R}^n \text{ is a convex domain}, B_{1/M}(0) \subset \Omega \subset B_M(0), \\ u \in C(\overline{\Omega}) \text{ is convex}, u|_{\partial\Omega} = 0, \det D^2 u \le \Lambda\big\}$$

and

$$C_{\lambda,\Lambda,M} = \big\{(\Omega, u) : \Omega \subset \mathbb{R}^n \text{ is a convex domain}, B_{1/M}(0) \subset \Omega \subset B_M(0), \\ u \in C(\overline{\Omega}) \text{ is convex}, u|_{\partial\Omega} = 0, \lambda \le \det D^2 u \le \Lambda\big\}.$$

Proof. Suppose we are given a sequence $\{(\Omega_i, u_i, \nu_i)\}_{i=1}^{\infty} \subset \mathcal{C}_{\Lambda,M}$. By the Blaschke Selection Theorem (Theorem 2.20), we can find a subsequence of $\{\Omega_i\}_{i=1}^{\infty}$, still labeled $\{\Omega_i\}_{i=1}^{\infty}$, such that $\overline{\Omega_i}$ converges to a convex body $\overline{\Omega}$ in the Hausdorff distance. Clearly, $\overline{B_{1/M}(0)} \subset \overline{\Omega} \subset \overline{B_M(0)}$ so $B_{1/M}(0) \subset \Omega \subset B_M(0)$.

By Lemma 3.18, we can find a constant $C(n, \Lambda, M) > 0$ such that

$$\|u_i\|_{C^{0,1/n}(\overline{\Omega_i})} \le C \quad \text{and} \quad |u_i(x)| \le C[\operatorname{dist}(x, \partial\Omega_i)]^{1/n} \quad \text{for all } x \in \Omega_i.$$

It follows from the Arzelà–Ascoli Theorem (Theorem 2.41) that, up to extracting a further subsequence, u_i converges locally uniformly in Ω to a convex function $u \in C^{0,1/n}(\Omega)$ satisfying

$$|u(x)| \leq C[\text{dist}(x, \partial\Omega)]^{1/n} \quad \text{for all } x \in \Omega.$$

Therefore $u \in C(\overline{\Omega})$ and $u = 0$ on $\partial\Omega$.

Since $u_i \to u$ locally uniformly in Ω, Theorem 3.7 gives that the restrictions of $\nu_i = \mu_{u_i}$ on Ω converge weakly* to $\nu = \mu_u$. Since $\overline{\Omega_i}$ converges to $\overline{\Omega}$ in the Hausdorff distance, we deduce that ν_i on Ω converges weakly* to ν. Thus, we have $(\Omega, u, \nu) \in \mathcal{C}_{\Lambda,M}$ with properties (1)–(3). □

A consequence of the above theorem and the uniqueness of solutions to the Dirichlet problem (Corollary 3.23 below) is the following:

Theorem 3.20 (Compactness of solutions to the Monge–Ampère equation: zero boundary data)**.** *Let $\{\Omega_k\}_{k=1}^\infty \subset \mathbb{R}^n$ be a sequence of bounded convex domains that converges to a bounded convex domain Ω in the Hausdorff distance. Let $\{\mu_k\}_{k=1}^\infty$ be a sequence of nonnegative Borel measures with $\sup_k \mu_k(\Omega_k) < \infty$ and which converges weakly* to a Borel measure μ on Ω. For each k, let $u_k \in C(\overline{\Omega_k})$ be the convex Aleksandrov solution of*

$$\det D^2 u_k = \mu_k \quad \text{in } \Omega_k, \qquad u_k = 0 \quad \text{on } \partial\Omega_k.$$

Then u_k converges locally uniformly in Ω to the convex Aleksandrov solution of

$$\det D^2 u = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

3.5. Comparison Principle and Global Lipschitz Estimates

We prove in this section an important converse of Lemma 3.11. It implies that if two convex functions have the same boundary values, the function with larger Monge–Ampère measure is in fact smaller than the other one in the interior. This property roughly says that $(\det D^2)$ is a negative operator on the set of convex functions, which is a nonlinear analogue of the Laplace operator $\Delta = (\text{trace } D^2)$; see also Theorem 2.79.

Theorem 3.21 (Comparison principle for the Monge–Ampère equation)**.** *Let $u, v \in C(\overline{\Omega})$ be convex functions on a bounded open set Ω in $\mathbb{R}^n$ such that*

$$\det D^2 v \geq \det D^2 u \quad \text{in } \Omega,$$

in the sense of Aleksandrov; that is, $\mu_v \geq \mu_u$. Then

$$\min_{\overline{\Omega}}(u - v) = \min_{\partial\Omega}(u - v).$$

In particular, if $u \geq v$ on $\partial\Omega$, then $u \geq v$ in Ω.

Proof. By adding a constant to v, we can assume that $\min_{\partial\Omega}(u-v) = 0$ and hence $u \geq v$ on $\partial\Omega$. We need to show that $u \geq v$ in Ω. Suppose otherwise that $u - v$ attains its minimum at $x_0 \in \Omega$ with $u(x_0) - v(x_0) = -m < 0$. Choose $\delta > 0$ such that $\delta(\text{diam}(\Omega))^2 < m/2$. Consider

$$w(x) := u(x) - v(x) - \delta|x - x_0|^2.$$

If $x \in \partial\Omega$, then $w(x) \geq -\delta(\text{diam}(\Omega))^2 \geq -m/2$, while $w(x_0) = -m < -m/2$. Thus, w attains its minimum value at $y \in \Omega$. Let

$$E = \{x \in \overline{\Omega} : w(x) < -3m/4\}.$$

Then E is open with $y \in E$, and $\partial E = \{x \in \overline{\Omega} : w(x) = -3m/4\} \subset \Omega$. The function u is below $v + \delta|x - x_0|^2 - 3m/4$ in E, but they coincide on ∂E. Consequently, the maximum principle in Lemma 3.11 gives

$$\partial u(E) \supset \partial(v + \delta|x - x_0|^2 - 3m/4)(E) = \partial(v + \delta|x - x_0|^2)(E).$$

It follows from Lemma 3.10 that

$$\begin{aligned}
\mu_u(E) = |\partial u(E)| &\geq |\partial(v + \delta|x - x_0|^2)(E)| \\
&\geq |\partial v(E)| + |\partial(\delta|x - x_0|^2)(E)| \\
&= |\partial v(E)| + (2\delta)^n|E| > |\partial v(E)| = \mu_v(E),
\end{aligned}$$

which is a contradiction. The theorem is thus proved. □

The proof of Theorem 3.21 also implies the following result.

Lemma 3.22. *Let $u, v \in C(\overline{\Omega})$ be convex functions on an open set Ω in $\mathbb{R}^n$ such that $\mu_v \geq \mu_u$. Then for any $x_0 \in \Omega$ and $\delta > 0$, the function $w(x) := u(x) - v(x) - \delta|x - x_0|^2$ cannot attain its minimum value on $\overline{\Omega}$ at an interior point in Ω.*

Proof. Suppose that w attains its minimum value on $\overline{\Omega}$ at $y \in \Omega$. Let $\bar{w}(x) := w(x) + (\delta/2)|x - y|^2$. Then $\bar{w}(y) = w(y) < \min_{\partial\Omega} \bar{w}$. Note that

$$\bar{w}(x) = u(x) - v(x) - (\delta/2)|x|^2 + \text{affine function in } x.$$

The proof of Theorem 3.21 applied to $\bar{w}$ gives a contradiction. □

An immediate consequence of the comparison principle is the uniqueness of solutions to the Dirichlet problem.

Corollary 3.23 (Uniqueness of solutions to the Dirichlet problem)**.** *Let Ω be an open bounded set in $\mathbb{R}^n$. Let $g \in C(\partial\Omega)$ and let ν be a Borel measure in Ω. Then, there is at most one convex Aleksandrov solution $u \in C(\overline{\Omega})$ to the Dirichlet problem*

$$\det D^2 u = \nu \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega.$$

Next, we give a positive answer to the question raised after the proof of Corollary 3.14. The following lemma quantifies how much a convex function with a positive lower bound on the density of its Monge–Ampère measure will drop its values when stepping inside the domain.

Lemma 3.24. *Let Ω be an open set in $\mathbb{R}^n$ that satisfies $B_r(b) \subset \Omega \subset B_R(b)$ for some $b \in \mathbb{R}^n$ and $0 < r \leq R$. Assume that the convex function $u \in C(\overline{\Omega})$ satisfies, in the sense of Aleksandrov,*

$$\lambda \leq \det D^2 u \leq \Lambda \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega,$$

where $0 < \lambda \leq \Lambda$ and $\varphi \in C(\partial\Omega)$. Then

$$(3.15) \quad (\Lambda^{1/n}/2)(|y-b|^2 - R^2) + \min_{\partial\Omega} \varphi \leq u(y) \leq (\lambda^{1/n}/2)(|y-b|^2 - r^2) + \max_{\partial\Omega} \varphi \quad \text{in } \Omega.$$

In particular, if $B_1(b) \subset \Omega \subset B_n(b)$ and $\varphi = 0$ on $\partial\Omega$, then

$$c(\lambda, n) \equiv \lambda^{1/n}/2 \leq |\min_{\Omega} u| \leq C(\Lambda, n) \equiv \Lambda^{1/n} n^2/2.$$

Proof. We prove

$$u(y) \leq v(y) := (\lambda^{1/n}/2)(|y-b|^2 - r^2) + \max_{\partial\Omega} \varphi,$$

the other inequality in (3.15) being similar. Indeed, since $B_r(b) \subset \Omega$,

$$v \geq \max_{\partial\Omega} \varphi \geq u \quad \text{on } \partial\Omega \quad \text{and} \quad \det D^2 v = \lambda \leq \det D^2 u \quad \text{in } \Omega.$$

Thus, by Theorem 3.21, we have $u \leq v$ in Ω as asserted.

Finally, in the special case of $r = 1$, $R = n$, and $\varphi = 0$, (3.15) gives

$$-\Lambda^{1/n} n^2/2 \leq \min_{\Omega} u \leq u(b) \leq -\lambda^{1/n}/2,$$

and the last assertion of the lemma follows. □

We next observe that if a domain is the sublevel set of a convex function u with Monge–Ampère measure bounded from below and above, then the domain is balanced around the minimum point of u; see also Lemma 5.6.

Lemma 3.25 (Balancing)**.** *Let Ω be a bounded, convex domain in $\mathbb{R}^n$. Suppose that the convex function $u \in C(\overline{\Omega})$ satisfies $u = 0$ on $\partial\Omega$ and $0 < \lambda \leq \det D^2 u \leq \Lambda$ in Ω, in the sense of Aleksandrov.*

(i) *Let l be a line segment in Ω with two endpoints $z', z'' \in \partial\Omega$. Let $z \in l$, and let $\alpha \in (0,1)$ be such that $u(z) \leq \alpha \inf_{\Omega} u$. Then, there is $c(n, \alpha, \lambda, \Lambda) > 0$ such that $|z' - z| \geq c(n, \alpha, \lambda, \Lambda)|z'' - z'|$.*

(ii) *Let $x_0 \in \Omega$ be the minimum point of u in $\overline{\Omega}$. Let H_1 and H_2 be two supporting hyperplanes to $\partial\Omega$ that are parallel. Then, there is $c_0(n, \lambda, \Lambda) > 0$ such that $\operatorname{dist}(x_0, H_1)/\operatorname{dist}(x_0, H_2) \geq c_0$.*

Proof. First, we discuss rescalings using John's Lemma. By Lemma 2.63, there is an affine transformation T such that $B_1(0) \subset T^{-1}\Omega \subset B_n(0)$. Let $v(x) = (\det T)^{-2/n}u(Tx)$ for $x \in T^{-1}\Omega$. Then, $\lambda\, dx \leq \mu_v \leq \Lambda\, dx$ in $T^{-1}\Omega$, by Lemma 3.9.

We now prove part (i). Since the ratio $|z' - z|/|z'' - z'|$ is invariant under affine transformations, with the above rescaling, we may assume that $B_1(0) \subset \Omega \subset B_n(0)$. By Lemma 3.24, $\inf_\Omega u \leq -c(n, \lambda)$. Hence, when

$$u(z) \leq \alpha \inf_\Omega u \leq -\alpha c(n, \lambda),$$

we have $\mathrm{dist}(z, \partial\Omega) \geq c_1$ for some $c_1 = c_1(n, \alpha, \lambda, \Lambda) > 0$ by Aleksandrov's maximum principle, Theorem 3.12. It follows that

$$\frac{|z' - z|}{|z'' - z'|} \geq \frac{\mathrm{dist}(z, \partial\Omega)}{\mathrm{diam}(\Omega)} \geq \frac{c_1}{2n}.$$

The proof of part (ii) is similar to the above proof so we skip it. □

Let us now state a strengthening of Aleksandrov's maximum principle for convex functions with Monge–Ampère measure having density bounded from above. A consequence of this result is that the global $C^{0,1/n}$ estimates in Lemma 3.18 are now improved to global $C^{0,2/n}$ estimates, with a slight exception in two dimensions.

Lemma 3.26. *Let Ω be a bounded, convex domain in $\mathbb{R}^n$, and let $u \in C(\overline{\Omega})$ be a convex function satisfying $u = 0$ on $\partial\Omega$ and, in the sense of Aleksandrov, $\det D^2 u \leq 1$ in Ω. Let $\alpha = 2/n$ if $n \geq 3$ and $\alpha \in (0, 1)$ if $n = 2$. Then $u \in C^{0,\alpha}(\overline{\Omega})$ with the estimate*

$$|u(z)| \leq C(n, \alpha, \mathrm{diam}(\Omega))[\mathrm{dist}(z, \partial\Omega)]^\alpha \quad \text{for all } z \in \Omega. \tag{3.16}$$

Proof. We only need to prove the boundary Hölder estimate (3.16) since the global regularity $u \in C^{0,\alpha}(\overline{\Omega})$ then follows from the convexity of u and Lemma 2.37(iv).

Let $z \in \Omega$ be arbitrary. By translation and rotation of coordinates, we can assume the following: The origin 0 of $\mathbb{R}^n$ lies on $\partial\Omega$, the x_n-axis points inward Ω, z lies on the x_n-axis, and $\mathrm{dist}(z, \partial\Omega) = z_n$.

To prove (3.16), it suffices to prove that for all $x = (x', x_n) \in \Omega$, we have

$$|u(x)| \leq C(n, \alpha, \mathrm{diam}(\Omega)) x_n^\alpha. \tag{3.17}$$

Let us consider, for $\alpha \in (0, 1)$ and $x \in \Omega$,

$$\phi_\alpha(x) = x_n^\alpha(|x'|^2 - C_\alpha) \quad \text{where } C_\alpha = \frac{1 + 2[\mathrm{diam}(\Omega)]^2}{\alpha(1 - \alpha)}. \tag{3.18}$$

Then, as a special case of Lemma 2.60 with $k = 1$ and $s = n - 1$, we have

$$\det D^2\phi_\alpha(x) = 2^{n-1}x_n^{n\alpha-2}[\alpha(1 - \alpha)C_\alpha - (\alpha^2 + \alpha)|x'|^2] \geq 2^{n-1}x_n^{n\alpha-2} \quad \text{in } \Omega.$$

Moreover, $D^2_{x'}\phi_\alpha = 2x_n^\alpha I_{n-1}$ is positive definite in Ω. Therefore, ϕ_α is convex in Ω with

$$\det D^2\phi_\alpha(x) \geq 2x_n^{n\alpha-2} \quad \text{in } \Omega \quad \text{and} \quad \phi_\alpha \leq 0 \quad \text{on } \partial\Omega. \tag{3.19}$$

Consider now the case $n \geq 3$; the case $n = 2$ is similar.

Then $\det D^2\phi_{2/n} \geq 2 > \det D^2 u$ in Ω, while on $\partial\Omega$, $u = 0 \geq \phi_{2/n}$. By the comparison principle (Theorem 3.21), we have $u \geq \phi_{2/n}$ in Ω. Therefore,

$$|u(x)| = |u(x', x_n)| \leq -\phi_{2/n}(x', x_n) \leq C_{2/n} x_n^{2/n} \quad \text{for all } x \in \Omega,$$

from which (3.17) follows. The lemma is proved. □

Remark 3.27. When $n \geq 3$, the exponent $2/n$ in Lemma 3.26 is optimal; see Proposition 3.42.

Remark 3.28. In the proof of Lemma 3.26, for $u \geq C\phi_\alpha$, we only need the following: u is a viscosity supersolution (see Section 7.1) of $\det D^2 u \leq f$ where $f \in C(\Omega), f \geq 0$ is bounded, and $u \geq 0$ on $\partial\Omega$. This is due to the fact that viscosity solutions satisfy the comparison principle with smooth functions; see Lemma 7.9. Under these relaxed conditions, we have $u(x) \geq -C\text{dist}^\alpha(x, \partial\Omega)$, which improves Corollary 3.14 in the case $\det D^2 u \leq f$.

In two dimensions, we can strengthen the estimate in Lemma 3.26 to a global log-Lipschitz estimate which is sharp (see Example 3.32). This follows from the following general result.

Lemma 3.29 (Global log-Lipschitz estimate)**.** *Let Ω be a bounded convex domain in $\mathbb{R}^n$ $(n \geq 2)$. Let $u \in C(\overline{\Omega})$ be a convex function satisfying $u = 0$ on $\partial\Omega$ and $\det D^2 u \leq M\text{dist}^{n-2}(\cdot, \partial\Omega)$ in Ω, in the sense of Aleksandrov, where $M > 0$. Then*

$$|u(z)| \leq C(M, \text{diam}(\Omega))\,\text{dist}(z, \partial\Omega)(1 + |\log \text{dist}(z, \partial\Omega)|) \quad \textit{for all } z \in \Omega.$$

Proof. As in the proof of Lemma 3.26, we will construct an appropriate log-Lipschitz convex subsolution. Let $z = (z', z_n)$ be an arbitrary point in Ω. By translation and rotation of coordinates, we can assume the following: $0 \in \partial\Omega$, $\Omega \subset \mathbb{R}^n_+ = \{x = (x', x_n) \in \mathbb{R}^n : x_n > 0\}$, the x_n-axis points inward Ω, z lies on the x_n-axis, and $z_n = \text{dist}(z, \partial\Omega)$. Let $d = \text{diam}(\Omega)$ and

$$v(x) = (M + 2d^2)x_n \log(x_n/d) + x_n(|x'|^2 - d^2).$$

Then, $v \leq 0$ on $\partial\Omega$, and

$$D^2 v = \begin{pmatrix} 2x_n & 0 & \cdots & 0 & 2x_1 \\ 0 & 2x_n & \cdots & 0 & 2x_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 2x_n & 2x_{n-1} \\ 2x_1 & 2x_2 & \cdots & 2x_{n-1} & \frac{M+2d^2}{x_n} \end{pmatrix}.$$

By induction in n, we find that v is convex in Ω and

$$\det D^2 v(x) = 2^{n-1} x_n^{n-2}(M + 2d^2 - 2|x'|^2) \geq 2M \text{dist}^{n-2}(x, \partial\Omega).$$

By the comparison principle in Theorem 3.21, we have $u \geq v$ in Ω. Thus

$$|u(z)| \leq |v(z)| = (M + 2d^2) z_n \log(d/z_n) + d^2 z_n \leq C(M, d) z_n (1 + |\log z_n|).$$

Since $z_n = \text{dist}(z, \partial\Omega)$, the lemma is proved. □

We conclude this section by establishing global Lipschitz estimates under suitable conditions on the domain and boundary data.

Theorem 3.30 (Global Lipschitz estimate). *Let Ω be a uniformly convex domain in $\mathbb{R}^n$ with C^2 boundary $\partial\Omega$. Let $\varphi \in C^{1,1}(\overline{\Omega})$. Let $u \in C(\overline{\Omega})$ be the convex solution, in the sense of Aleksandrov, to*

$$\det D^2 u \leq \Lambda \quad \text{in } \Omega, \quad u = \varphi \quad \text{on } \partial\Omega.$$

Then, there is a constant $C = C(n, \Lambda, \Omega, \|\varphi\|_{C^{1,1}(\overline{\Omega})})$ such that

$$\|u\|_{C^{0,1}(\overline{\Omega})} := \sup_{\overline{\Omega}} |u| + \sup_{x \neq y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|} \leq C.$$

Proof. By Lemma 3.24 and the convexity of u, we have

$$C(n, \Lambda, \min_{\partial\Omega} \varphi, \text{diam}(\Omega)) \leq u \leq \max_{\partial\Omega} \varphi \quad \text{in } \overline{\Omega}.$$

For the Lipschitz estimate, thanks to Lemma 2.37(iv), we only need to prove

$$|u(x) - u(x_0)| \leq C|x - x_0| \quad \text{for all } x \in \Omega, x_0 \in \partial\Omega.$$

First, we note that the ray $\overrightarrow{x_0 x}$ intersects $\partial\Omega$ at z and $x = t x_0 + (1-t)z$ for some $t \in (0, 1)$. It follows that $x - x_0 = (1-t)(z - x_0)$. By convexity, $u(x) \leq t u(x_0) + (1-t) u(z)$. Thus, using $u = \varphi$ on $\partial\Omega$, we deduce

$$u(x) - u(x_0) \leq (1-t)(\varphi(z) - \varphi(x_0)) \leq (1-t)\|D\varphi\|_{L^\infty(\Omega)}|z - x_0|.$$

Therefore, $u(x) - u(x_0) \leq \|D\varphi\|_{L^\infty(\Omega)}|x - x_0|$.

Thus, it remains to show $u(x) - u(x_0) \geq -C|x - x_0|$.

Let $K := \|\varphi\|_{C^{1,1}(\overline{\Omega})}$. Let ρ be a $C^2(\overline{\Omega})$ uniformly convex defining function of Ω (see Definition 2.53); that is, $\rho < 0$ in Ω, $\rho = 0$ and $D\rho \neq 0$ on $\partial\Omega$. As in the proof of Lemma 2.54, for $\mu(n, \Lambda, \Omega, K) > 0$ large, the function

$$v = \varphi + \mu(e^\rho - 1)$$

is convex and $\det D^2 v \geq \Lambda \geq \det D^2 u$ in Ω. With this μ and $u = v$ on $\partial\Omega$, we have $u \geq v$ in Ω by the comparison principle in Theorem 3.21.

Using $u = \varphi$ on $\partial\Omega$ and $e^\rho - 1 \geq \rho$, we have, for all $x_0 \in \partial\Omega$ and $x \in \Omega$,

$$u(x) - u(x_0) \geq \varphi(x) - \varphi(x_0) + \mu(\rho(x) - \rho(x_0)) \geq -C|x - x_0|$$

where $C = \|D\varphi\|_{L^\infty(\Omega)} + \mu\|D\rho\|_{L^\infty(\Omega)}$. This concludes the proof. □

3.6. Explicit Solutions

We will see in Section 3.7 general results regarding solvability of the Dirichlet problem for the Monge–Ampère equation. However, due to its highly nonlinear nature, there are very few examples of explicit solutions of the Monge–Ampère equation $\det D^2 u = 1$ with zero boundary condition on bounded convex domains Ω in $\mathbb{R}^n$. We list here two examples.

Example 3.31 (Solution on a ball)**.** Consider the unit ball $B_1(0)$ in $\mathbb{R}^n$. Then, the function $u(x) = (|x|^2 - 1)/2$ is the unique convex solution to

$$\det D^2 u = 1 \quad \text{in } B_1(0), \quad u = 0 \quad \text{on } \partial B_1(0).$$

Example 3.32 (Solution on a triangle)**.** Let T be the open triangle in the plane $\mathbb{R}^2$ with vertices at $(0,0), (1,0)$, and $(0,1)$. Consider the Monge–Ampère equation

$$\det D^2 \sigma_T = 1 \quad \text{in } T, \quad \sigma_T = 0 \quad \text{on } \partial T. \tag{3.20}$$

We can verify that the solution $\sigma_T \in C(\overline{T})$ is given by

$$\sigma_T(x) = -\frac{1}{\pi^2}\big[\mathscr{L}(\pi x_1) + \mathscr{L}(\pi x_2) + \mathscr{L}(\pi(1 - x_1 - x_2))\big],$$

for $x = (x_1, x_2) \in T$, where

$$\mathscr{L}(\theta) = -\int_0^\theta \log|2\sin u|\, du$$

is the Lobachevsky function. This remarkable formula for σ_T comes from the study of the *dimer model* (also known as the lozenge tiling model) in the works of Kenyon, Okounkov, and Sheffield [**Kn**, **KO**, **KOS**]. The function σ_T is called a *surface tension*.

Verification of Example 3.32. By direct calculation, we have

$$D\sigma_T(x) = \frac{1}{\pi}\Big(\log\Big(\frac{\sin(\pi x_1)}{\sin(\pi x_1 + \pi x_2)}\Big), \log\Big(\frac{\sin(\pi x_2)}{\sin(\pi x_1 + \pi x_2)}\Big)\Big) \tag{3.21}$$

and

$$D^2\sigma_T = \begin{pmatrix} \cot(\pi x_1) - \cot(\pi x_1 + \pi x_2) & -\cot(\pi x_1 + \pi x_2) \\ -\cot(\pi x_1 + \pi x_2) & \cot(\pi x_2) - \cot(\pi x_1 + \pi x_2) \end{pmatrix}.$$

Therefore, using the cotangent formula, we find $\det D^2 \sigma_T = 1$.

To verify the boundary condition, we use the fact that

$$\mathscr{L}(\pi) = -\int_0^\pi \log|2\sin u|\, du = 0;$$

see [**Ah**, p. 161]. From this, we see that $\mathscr{L}$ is odd and π-periodic. Thus, for example, on the x_1-axis part of ∂T, we have

$$\sigma_T(x_1, 0) = -\pi^{-2}[\mathscr{L}(\pi x_1) + \mathscr{L}(\pi - \pi x_1)] = 0.$$

Note that $|\sigma_T|$ grows at a log-Lipschitz rate $\operatorname{dist}(\cdot, \partial T)|\log \operatorname{dist}(\cdot, \partial T)|$ from the boundary of T. In view of Lemma 3.29, this is the sharp rate for convex functions with bounded Monge–Ampère measure and which vanish on the boundary of a convex domain in the plane.

3.7. The Dirichlet Problem and Perron's Method

In this section, using the Perron method, we discuss the solvability of the inhomogeneous Dirichlet problem for the Monge–Ampère equation

$$\det D^2 u = \nu \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \tag{3.22}$$

on a bounded convex domain Ω in $\mathbb{R}^n$, with continuous boundary data and right-hand side being a Borel measure with finite mass. This problem was first solved by Aleksandrov and Bakelman.

3.7.1. Homogeneous Dirichlet problem. Before solving the Dirichlet problem with inhomogeneous right-hand side, we consider a simpler problem regarding the solvability of the homogeneous Dirichlet problem for the Monge–Ampère equation with continuous boundary data.

Theorem 3.33 (Homogeneous Dirichlet problem)**.** *Let Ω be a bounded and strictly convex domain in $\mathbb{R}^n$. Then, for any $g \in C(\partial\Omega)$, the problem*

$$\det D^2 u = 0 \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega$$

has a unique convex Aleksandrov solution $u \in C(\overline{\Omega})$.

Remark 3.34. It should be emphasized that Theorem 3.33 asserts the solvability of the Dirichlet problem for all continuous boundary data. For this, the strict convexity of the domain cannot be dropped. For example, let $\Omega \subset \mathbb{R}^2$ be the interior of a triangle with vertices z_1, z_2, z_3. Let $g \in C(\partial\Omega)$ be such that $g(z_1) = g(z_2) = g(z_3) = 0$ and $g((z_1 + z_2)/2) = 1$. Then, there is no convex function $u \in C(\overline{\Omega})$ such that $u = g$ on $\partial\Omega$. Hence, the Dirichlet problem with this boundary data is not solvable.

Proof of Theorem 3.33. The uniqueness of Aleksandrov solutions follows from the comparison principle; see also Corollary 3.23. Now, we show the existence. Heuristically, the sought-after solution has the smallest possible Monge–Ampère measure among all convex functions with fixed boundary value g. The pointwise maximum principle in Lemma 3.11 suggests looking for the maximum of all convex functions with boundary values not exceeding g. This is exactly the Perron method since these convex functions are subsolutions of our equation.

Let us consider

$$\mathcal{S} = \{v \in C(\overline{\Omega}) : v \text{ is convex and } v \le g \text{ on } \partial\Omega\}.$$

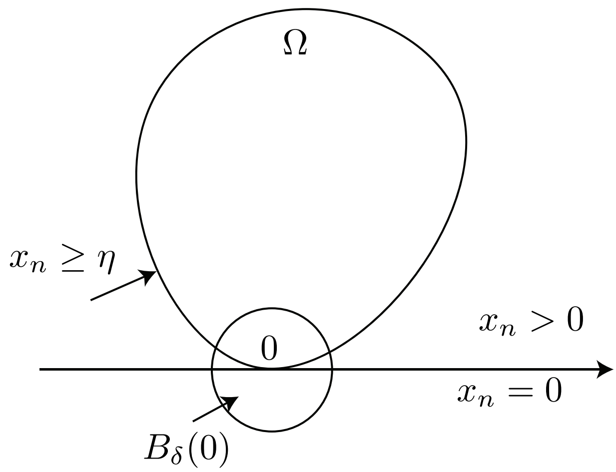

Figure 3.4. $x_n \geq \eta$ for all $x \in \partial\Omega \setminus B_\delta(0)$.

Since g is continuous, $\mathcal{S} \neq \emptyset$ because $v \equiv \min_{\partial\Omega} g \in \mathcal{S}$. Let

$$u(x) = \sup_{v\in\mathcal{S}} v(x), \ \ x \in \overline{\Omega}.$$

Then, by Lemma 2.24, u is convex in Ω, and $u \leq g$ on $\partial\Omega$ so $u : \overline{\Omega} \to \mathbb{R}$ is finite. We show that u is the desired solution in the following steps.

Step 1. For each $x_0 \in \partial\Omega$, we trap the boundary data g between two affine functions that are close to g near x_0. By a translation of coordinates, we can assume that $x_0 = 0 \in \partial\Omega$ and that $\Omega \subset \{x \in \mathbb{R}^n : x_n > 0\}$. From the continuity of g, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$$(3.23) \qquad |g(x) - g(0)| < \varepsilon \quad \text{for all } x \in \partial\Omega \cap B_\delta(0).$$

Recall that Ω is strictly convex. Thus, there exists $\eta > 0$ such that $x_n \geq \eta$ for all $x \in \partial\Omega \setminus B_\delta(0)$. See Figure 3.4.

It can be readily verified that, for $C_1 = 2\|g\|_{L^\infty(\partial\Omega)}/\eta$, we have

$$(3.24) \quad a_1(x) := g(0) - \varepsilon - C_1 x_n \leq g(x) \\ \leq g(0) + \varepsilon + C_1 x_n := A(x) \quad \text{for all } x \in \partial\Omega.$$

For $a_1 \leq g$ to hold, it suffices to assume that g is lower semicontinuous at 0.

Step 2. $u = g$ on $\partial\Omega$. Let $x_0 \in \partial\Omega$. We only need to show $u(x_0) \geq g(x_0)$. Given $\varepsilon > 0$, let us assume that $x_0 = 0 \in \partial\Omega$ and also Ω is as in Step 1. From (3.24), we find $a_1(x) \in \mathcal{S}$. By the definition of u, we have $u(0) \geq a_1(0) = g(0) - \varepsilon$. This holds for all $\varepsilon > 0$ so $u(0) \geq g(0)$. Step 2 is proved.

Step 3. $u \in C(\overline{\Omega})$. Since u is convex in Ω, it is continuous there, so we only need to prove that u is continuous on $\partial\Omega$. Let us assume that $x_0 = 0 \in \partial\Omega$ and also Ω is as in Step 1. Given $\varepsilon > 0$, let a_1 and A be as in Step 1. Then $u \geq a_1$. On the other hand, if $v \in \mathcal{S}$, then $v \leq g \leq A$ on $\partial\Omega$. Since $v - A$ is convex, this implies $v - A \leq 0$ in Ω, so $v \leq A$ on $\overline{\Omega}$. By taking the supremum over $v \in \mathcal{S}$, we find $u \leq A$ on $\overline{\Omega}$. Therefore

$$a_1 \leq u \leq A \quad \text{in } \Omega. \tag{3.25}$$

Since

$$g(0) - \varepsilon \leq \liminf_{x \to 0} a_1(x) \leq \limsup_{x \to 0} A(x) = g(0) + \varepsilon$$

and $\varepsilon > 0$ is arbitrary, we must have $\lim_{x \to 0} u(x) = g(0) = u(0)$.

Step 4. We show

$$\partial u(\Omega) \subset \{p \in \mathbb{R}^n : \text{there are } x \neq y \in \Omega \text{ such that } p \in \partial u(x) \cap \partial u(y)\}.$$

Let $p \in \partial u(\Omega)$. Then $p \in \partial u(x_0)$ for some $x_0 \in \Omega$, and hence

$$u(x) \geq u(x_0) + p \cdot (x - x_0) := a(x) \quad \text{in } \Omega. \tag{3.26}$$

We first show that there is $y \in \partial\Omega$ such that $g(y) = a(y)$. Indeed, from (3.26), together with $u \equiv g$ on $\partial\Omega$ by Step 2, and the continuity of both u and g, we find $g \geq a$ on $\partial\Omega$. If such $y \in \partial\Omega$ does not exist, then by the continuity of g and a, there is $\varepsilon > 0$ such that $g \geq a + \varepsilon$ on $\partial\Omega$. Therefore, $a + \varepsilon \in \mathcal{S}$, which implies $u \geq a + \varepsilon$ on Ω, but this contradicts $u(x_0) = a(x_0)$.

Next, we complete the proof of Step 4 by showing that $a(x)$ is a supporting hyperplane to the graph of u at $(z, u(z))$ for z on a whole open segment I connecting $x_0 \in \Omega$ to $y \in \partial\Omega$ and so $p \in \partial u(z)$ for all $z \in I$. To prove this, we show that $u(z) \leq a(z)$ for all $z \in I$ because we already have $u \geq a$ in Ω. Let $z = tx_0 + (1 - t)y$ where $0 \leq t \leq 1$. By the convexity of u and the fact that a is affine with $a(y) = g(y) = u(y)$, we finally have

$$u(z) \leq tu(x_0) + (1 - t)u(y) = ta(x_0) + (1 - t)a(y) = a(z).$$

From Step 4 and Aleksandrov's Lemma (Lemma 2.45), we have $|\partial u(\Omega)| = 0$, so $\det D^2 u = 0$ in Ω in the sense of Aleksandrov. This completes the proof. □

3.7.2. Inhomogeneous Dirichlet problem. We again use the Perron method to solve (3.22) as in the case $\nu \equiv 0$. Let

$$\mathcal{S}(\nu, g) = \{v \in C(\overline{\Omega}) : v \text{ convex}, \det D^2 v \geq \nu \text{ in } \Omega, v = g \text{ on } \partial\Omega\},$$

and let $u(x) = \sup_{v \in \mathcal{S}(\nu, g)} v(x)$. We wish to show that u is the desired solution.

For this, we first need to show that $\mathcal{S}(\nu, g) \neq \emptyset$. So far, the Monge–Ampère measures of convex functions that we have seen are either Dirac measures, Hessian determinants of $C^{1,1}$ convex functions, or zero. In general, it is not practical to construct explicitly an element in $\mathcal{S}(\nu, g) \neq \emptyset$. Here we will use the superadditivity property of the Monge–Ampère measure to show that $\mathcal{S}(\nu, g) \neq \emptyset$ when ν is a finite combination of Dirac measures with positive coefficients. In fact, we will solve (3.22) in this case first. This is the content of Theorem 3.36. The general case of Borel measure ν follows from Theorem 3.36 and the following compactness result.

Theorem 3.35 (Compactness of solutions to the Monge–Ampère equation: continuous boundary data)**.** *Let Ω be a bounded and strictly convex domain in $\mathbb{R}^n$. Let $\{\mu_j\}_{j=1}^\infty$, μ be Borel measures in Ω such that $\mu_j(\Omega) \leq A < \infty$ and μ_j converges weakly* to μ in Ω. Let $\{g_j\}_{j=1}^\infty \subset C(\partial\Omega)$ be such that g_j converges uniformly to g in $C(\partial\Omega)$. Assume that $u_j \in C(\overline{\Omega})$ is the unique convex Aleksandrov solution to*

$$\det D^2 u_j = \mu_j \quad \textit{in } \Omega, \qquad u_j = g_j \quad \textit{on } \partial\Omega.$$

Then $\{u_j\}_{j=1}^\infty$ contains a subsequence that converges uniformly on compact subsets of Ω to the unique convex Aleksandrov solution $u \in C(\overline{\Omega})$ to

$$\det D^2 u = \mu \quad \textit{in } \Omega, \qquad u = g \quad \textit{on } \partial\Omega.$$

Proof. From Corollary 3.14, we have for all j

$$-\|g_j\|_{L^\infty(\partial\Omega)} - C(n)\mathrm{diam}(\Omega)|\partial u_j(\Omega)|^{1/n} \leq u_j \leq \|g_j\|_{L^\infty(\partial\Omega)} \quad \text{in } \Omega.$$

Due to $\mu_j(\Omega) \leq A < \infty$ and g_j converging uniformly to g in $C(\partial\Omega)$, we deduce that the sequence $\{u_j\}_{j=1}^\infty$ is uniformly bounded. Now, Theorem 2.42 implies that $\{u_j\}_{j=1}^\infty$ contains a subsequence, also denoted by $\{u_j\}_{j=1}^\infty$, such that u_j converges uniformly on compact subsets of Ω to a convex function u in Ω. By the weak continuity property of the Monge–Ampère measure in Theorem 3.7, we have $\det D^2 u = \mu$ in Ω.

It remains to show that $u \in C(\overline{\Omega})$ and $u = g$ on $\partial\Omega$. Consider an arbitrary point $x_0 \in \partial\Omega$. We can assume $x_0 = 0$ and $\Omega \subset \{x \in \mathbb{R}^n : x_n > 0\}$. We will show that u is continuous at 0 and $u(0) = g(0)$.

By Theorem 3.33, there is a unique Aleksandrov solution $U_j \in C(\overline{\Omega})$ to

$$\det D^2 U_j = 0 \quad \text{in } \Omega, \qquad U_j = g_j \quad \text{on } \partial\Omega.$$

Since $\det D^2 U_j \leq \det D^2 u_j$ in Ω and $U_j = u_j$ on $\partial\Omega$, the comparison principle in Theorem 3.21 gives

$$u_j \leq U_j \quad \text{in } \Omega. \tag{3.27}$$

Now, we try to obtain a good lower bound for u_j from below that matches U_j locally. Due to the continuity of g, given $\varepsilon > 0$, there exists $\delta > 0$ such that

$|g(x) - g(0)| < \varepsilon/3$ for all $x \in \partial\Omega \cap B_\delta(0)$. From the uniform convergence of g_j to g, there is $j(\varepsilon) > 0$ such that

$$|g_j(x) - g_j(0)| < \varepsilon \quad \text{for all } x \in \partial\Omega \cap B_\delta(0) \text{ and all } j \geq j(\varepsilon).$$

From now on, we only consider $j \geq j(\varepsilon)$. Since Ω is strictly convex, there exists $\eta > 0$ such that $x_n \geq \eta$ for all $x \in \partial\Omega \setminus B_\delta(0)$. As in (3.25), we have

$$g_j(0) - \varepsilon - C_1 x_n \leq U_j(x) \leq g_j(0) + \varepsilon + C_1 x_n, \tag{3.28}$$

where $C_1 = 2\max_j\{\|g_j\|_{L^\infty(\Omega)}\}/\eta < \infty$. Now, consider

$$v_j(x) = u_j(x) - [g_j(0) - \varepsilon - C_1 x_n].$$

Then $v_j \geq 0$ on $\partial\Omega$ and $\det D^2 v_j = \det D^2 u_j = \mu_j$. By Corollary 3.14,

$$v_j(x) \geq -C(n)[\operatorname{dist}(x, \partial\Omega)]^{1/n}(\operatorname{diam}(\Omega))^{\frac{n-1}{n}}|\partial v_j(\Omega)|^{1/n} \quad \text{in } \Omega.$$

Since $|\partial v_j(\Omega)| = |\partial u_j(\Omega)| = \mu_j(\Omega) \leq A$ and $\operatorname{dist}(x, \partial\Omega) \leq x_n$, we obtain

$$u_j(x) \geq g_j(0) - \varepsilon - C_1 x_n - C(n)x_n^{1/n}(\operatorname{diam}(\Omega))^{\frac{n-1}{n}}A^{1/n} \quad \text{in } \Omega. \tag{3.29}$$

From (3.27)–(3.29), upon letting $j \to \infty$, we find that u is continuous at 0 and $u(0) = g(0)$. This completes the proof. □

As mentioned earlier, we first solve the inhomogeneous problem (3.22) when ν is a finite combination of Dirac measures with positive coefficients.

Theorem 3.36 (The Dirichlet problem with Dirac measures). *Let Ω be a bounded and strictly convex domain in $\mathbb{R}^n$, and let $g \in C(\partial\Omega)$. Let $\nu = \sum_{i=1}^N a_i\delta_{x_i}$ where $a_i > 0$ and δ_{x_i} is the Dirac measure at $x_i \in \Omega$. Then, there is a unique convex Aleksandrov solution $u \in C(\overline{\Omega})$ to the problem*

$$\det D^2 u = \nu \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega.$$

Proof. Let

$$\mathcal{S}(\nu, g) = \{v \in C(\overline{\Omega}) : v \text{ convex},\ \det D^2 v \geq \nu \text{ in } \Omega,\ v = g \text{ on } \partial\Omega\},$$

and let

$$u(x) = \sup_{v \in \mathcal{S}(\nu, g)} v(x). \tag{3.30}$$

We show that u is the desired solution in the following steps.

Step 1. There is $v_0 \in \mathcal{S}(\nu, g)$, and u defined by (3.30) is bounded. Note that the convex function $|z - x_i|$ on $\mathbb{R}^n$ has Monge–Ampère measure $\mu_{|z-x_i|} = \omega_n\delta_{x_i}$. Let $u_0(z) = \omega_n^{-1/n}\sum_{i=1}^N a_i^{1/n}|z - x_i|$. Then, the superadditivity property in Lemma 3.10 gives $\mu_{u_0} \geq \nu$.

By Theorem 3.33, there is a unique Aleksandrov solution $U_1 \in C(\overline{\Omega})$ to

$$\det D^2 U_1 = 0 \quad \text{in } \Omega, \qquad U_1 = g - u_0 \quad \text{on } \partial\Omega.$$

Let $v_0 = u_0 + U_1$. Then $v_0 \in C(\overline{\Omega})$, $v_0 = g$ on $\partial\Omega$, and by Lemma 3.10

$$\mu_{v_0} = \mu_{u_0+U_1} \geq \mu_{u_0} + \mu_{U_1} \geq \nu.$$

Therefore, $v_0 \in \mathcal{S}(\nu, g)$.

For u defined by (3.30), we have $u \geq v_0$ so it is bounded from below. Now, we use Theorem 3.33 to obtain a unique convex solution $W \in C(\overline{\Omega})$ to

$$\mu_W \ = 0 \quad \text{in } \Omega, \qquad W = g \quad \text{on } \partial\Omega.$$

For any $v \in \mathcal{S}(\nu, g)$, we have $v \leq W$ by the comparison principle in Theorem 3.21. In particular, v is uniformly bounded from above and so is u.

Step 2. If $v_1, v_2 \in \mathcal{S}(\nu, g)$, then $v := \max\{v_1, v_2\} \in \mathcal{S}(\nu, g)$. Indeed, given a Borel set $E \subset \Omega$, we write $E = E_0 \cup E_1 \cup E_2$, $E_i = E \cap \Omega_i$, where Ω_i $(i = 0, 1, 2)$ are the following subsets of Ω:

$$\Omega_0 = \{v_1 = v_2\}, \quad \Omega_1 = \{v_1 > v_2\}, \quad \Omega_2 = \{v_1 < v_2\}.$$

We show that $\mu_v(E_i) \geq \nu(E_i)$ for each $i = 0, 1, 2$. The cases $i = 1, 2$ are similar, so we only consider $i = 1$. It suffices to show that $\partial v_1(E_1) \subset \partial v(E_1)$. Indeed, if $p \in \partial v_1(x)$ where $x \in E_1$, then $p \in \partial v(x)$. This is because $v(x) = v_1(x)$ and for all $z \in \Omega$, we have

$$v(z) \geq v_1(z) \geq v_1(x) + p \cdot (z - x) = v(x) + p \cdot (z - x).$$

For the case $i = 0$, the same argument as above shows that $\partial v_1(E_0) \subset \partial v(E_0)$ and $\partial v_2(E_0) \subset \partial v(E_0)$, and we are done.

Step 3. Approximation property of u defined by (3.30). We show that

- for each $y \in \Omega$, there exists a sequence $\{v_m\}_{m=1}^\infty \subset \mathcal{S}(\nu, g)$ that converges uniformly on compact subsets of Ω to a function $w \in \mathcal{S}(\nu, g)$ so that $w(y) = u(y)$;
- $u \in C(\overline{\Omega})$.

Let $y \in \Omega$. Then, by the definition of u, there is a sequence $\{\bar{v}_m\}_{m=1}^\infty \subset \mathcal{S}(\nu, g)$ such that $\bar{v}_m(y) \to u(y)$ as $m \to \infty$. Let

$$v_m = \max\{v_0, \max_{1\leq k\leq m} \bar{v}_k\}.$$

By Step 2 (applied finitely many times), we have $v_m \in \mathcal{S}(\nu, g)$. Moreover, $v_m \leq u$ in Ω while $\bar{v}_m(y) \leq v_m(y) \leq u(y)$. Hence, $v_m(y) \to u(y)$ as $m \to \infty$.

Since $v_0 \leq v_m \leq v_{m+1} \leq W$, Theorem 2.42 implies that $\{v_m\}_{m=1}^\infty$ contains a subsequence that converges uniformly on compact subsets of Ω to a convex function $w \in C(\Omega)$. Clearly, $w(y) = u(y)$. By Theorem 3.7, $\mu_w \geq \nu$.

Let w be g on $\partial\Omega$. We show that $w \in C(\overline{\Omega})$. Indeed, assume that $x_0 = 0 \in \partial\Omega$ and $\Omega \subset \{x \in \mathbb{R}^n : x_n > 0\}$ as in the proof of Theorem 3.33.

Given $\varepsilon > 0$, then, as in (3.24), there is $C_1(\varepsilon, g, \Omega) > 0$ such that

$$g(0) - \varepsilon - C_1 x_n \le g(x) \le g(0) + \varepsilon + C_1 x_n \quad \text{for } x \in \partial\Omega.$$

Using the comparison principle, we find that $v \le g(0) + \varepsilon + C_1 x_n$ in Ω for any $v \in \mathcal{S}(\nu, g)$. In particular, this is true for v_m and hence its limit w. Thus, $w \le g(0) + \varepsilon + C_1 x_n$ in Ω. Since $w \ge v_0$, we have

$$v_0(x) - \varepsilon - C_1 x_n \le w(x) \le g(0) + \varepsilon + C_1 x_n \quad \text{in } \Omega. \tag{3.31}$$

It follows from $v_0(0) = g(0)$ that

$$-\varepsilon \le \liminf_{x\to 0}[w(x) - g(0)] \le \limsup_{x\to 0}[w(x) - g(0)] \le \varepsilon.$$

Since ε is arbitrary, w is continuous at 0. Thus, $w \in \mathcal{S}(\nu, g)$.

To show that $u \in C(\overline{\Omega})$, it suffices to show that u is continuous on $\partial\Omega$. With the above notation, we apply (3.31) at y where $w(y) = u(y)$, and then we vary y over Ω to conclude that

$$v_0(x) - \varepsilon - C_1 x_n \le u(x) \le g(0) + \varepsilon + C_1 x_n \quad \text{in } \Omega,$$

so u is continuous at 0.

Step 4. $u \in \mathcal{S}(\nu, g)$. From Step 3, it remains to prove that $\mu_u(\{x_i\}) \ge a_i$ for each $i = 1, \ldots, N$. We prove this for $i = 1$. By Step 3, there is a sequence $\{v_m\}_{m=1}^\infty \subset \mathcal{S}(\nu, g)$ that converges uniformly on compact subsets of Ω to a convex function w with $\mu_w \ge \nu$ so that $w(x_1) = u(x_1)$ and $w \le u$ in Ω. Thus $\mu_w(\{x_1\}) \ge a_1$. If $p \in \partial w(x_1)$, then $p \in \partial u(x_1)$ because for all $x \in \Omega$,

$$u(x) \ge w(x) \ge w(x_1) + p \cdot (x - x_1) = u(x_1) + p \cdot (x - x_1).$$

Therefore $\partial u(x_1) \supset \partial w(x_1)$ and hence

$$\mu_u(\{x_1\}) = |\partial u(x_1)| \ge |\partial w(x_1)| = \mu_w(\{x_1\}) \ge a_1.$$

Step 5. μ_u is concentrated on the set $X = \{x_1, \ldots, x_N\}$. For this, we use a lifting argument. Let $z \in \Omega \setminus X$. We can choose $r > 0$ such that $B_{2r}(z) \subset \Omega \setminus X$. Let $B = B_r(z)$ and let $v \in C(\overline{B})$ be the convex solution to

$$\mu_v = 0 \;\text{ in } B, \quad v = u \;\text{ on } \partial B.$$

Define the lifting w of u and v by

$$w(x) = v(x) \quad \text{if } x \in B \quad \text{and} \quad w(x) = u(x) \quad \text{if } x \in \Omega \setminus B.$$

Then $w \in C(\overline{\Omega})$ with $w = g$ on $\partial\Omega$. We claim that $w \in \mathcal{S}(\nu, g)$. Since $\mu_u \ge 0 = \mu_v$ in B and $u = v$ on ∂B, we have $v \ge u$ in B. Thus w is convex.

We now verify that $\mu_w(E) \ge \nu(E)$ for each Borel set $E \subset \Omega$. Let $E_1 = E \cap B$ and $E_2 = E \cap (\Omega \setminus B)$. As in Step 2, we have $\mu_w(E_1) \ge \mu_v(E_1)$ and $\mu_w(E_2) \ge \mu_u(E_2)$. Hence,

$$\mu_w(E) \ge \mu_v(E_1) + \mu_u(E_2) = \mu_u(E_2) \ge \nu(E_2) \ge \nu(E \cap X) = \nu(E).$$

This shows that $w \in \mathcal{S}(\nu, g)$. From the definition of u, we have $w \leq u$ in Ω. By the above argument, we have $w = v \geq u$ in B which implies that $u = v$ in B. It follows that $\mu_u(B) = 0$ for any ball $B = B_r(z)$ with $B_{2r}(z) \subset \Omega \setminus X$. Hence, if E is a Borel set with $E \cap X = \emptyset$, then $\mu_u(E) = 0$ by the regularity of μ_u. Therefore, μ_u is concentrated on the set X; that is, $\mu_u = \sum_{i=1}^n \lambda_i a_i \delta_{x_i}$, with $\lambda_i \geq 1$ for all $i = 1, \ldots, N$.

Step 6. $\mu_u = \nu$ in Ω. For this, we show that $\lambda_i = 1$ for all i. Suppose otherwise that $\lambda_i > 1$ for some i. For simplicity, we can assume that $a_i = 1$ and in some ball, say $B_r(0)$, we have $\mu_u = \lambda\delta_0$ with $\lambda > 1$ while $\nu = \delta_0$. We will locally insert a cone with Monge–Ampère measure δ_0 that is above u, and this will contradict the maximality of u.

Since $\partial u(0)$ is convex with measure $\lambda > 1$, there is a ball $B_{2\varepsilon}(p_0) \subset \partial u(0)$. Then $u(x) \geq u(0) + p \cdot x$ for all $p \in B_{2\varepsilon}(p_0)$ and all $x \in \Omega$. By subtracting $p_0 \cdot x$ from u and g, we can assume that for all $x \in \Omega$,

$$u(x) \geq u(0) + \varepsilon|x|. \tag{3.32}$$

Indeed, let $w(x) := u(x) - p_0 \cdot x$. Then $w(x) \geq w(0) + (p - p_0) \cdot x$ for all $p \in B_{2\varepsilon}(p_0)$ and all $x \in \Omega$. For any $x \in \Omega \setminus \{0\}$, we take $p - p_0 = \varepsilon x/|x|$ and obtain $w(x) \geq w(0) + \varepsilon|x|$.

Now, assume (3.32) holds. By subtracting a constant, say $u(0) + \varepsilon r/2$ for small $r > 0$, from u and g we can assume that $u(0) < 0$ but $|u(0)|$ is small while $u(x) \geq 0$ for $|x| \geq r$. Then, the convex set $D = \{x \in \Omega : u(x) < 0\}$ contains a neighborhood of 0. On D, we have $\mu_{\lambda^{-1/n}u} = \delta_0$. We now define the lifting v of u and $\lambda^{-1/n}u$ by

$$v(x) = u(x) \quad \text{if } x \in \Omega \setminus D \qquad \text{and} \qquad v(x) = \lambda^{-1/n}u(x) \quad \text{if } x \in D.$$

As in Step 5, we have $v \in \mathcal{S}(\nu, g)$ but $v(0) = \lambda^{-1/n}u(0) > u(0)$, contradicting the definition of u. Therefore $\lambda = 1$, and this completes the proof. $\square$

We are now ready to solve the inhomogeneous Dirichlet problem for the Monge–Ampère equation with continuous boundary data and right-hand side being a Borel measure with finite mass.

Theorem 3.37 (The Dirichlet problem on strictly convex domains)**.** *Let Ω be a bounded and strictly convex domain in $\mathbb{R}^n$. Let μ be a Borel measure in Ω with $\mu(\Omega) < \infty$. Then for any $g \in C(\partial\Omega)$, there is a unique convex Aleksandrov solution $u \in C(\overline{\Omega})$ to the Dirichlet problem*

$$\begin{cases} \det D^2 u = \mu & \text{in } \Omega, \\ \qquad\quad u = g & \text{on } \partial\Omega. \end{cases}$$

Proof. Note that there exists a sequence of Borel measures $\{\mu_j\}_{j=1}^\infty$ converging weakly* to μ such that each μ_j is a finite combination of Dirac measures with positive coefficients and $\mu_j(\Omega) \leq A < \infty$ for some constant A. Indeed, using dyadic cubes, we can find for each $j \in \mathbb{N}$ a partition of Ω into Borel sets $\Omega = \bigcup_{i=1}^{N_j} B_{j,i}$ where $\mathrm{diam}(B_{j,i}) \leq 1/j$. Then set $\mu_j := \sum_{i=1}^{N_j} \mu(B_{j,i})\delta_{x_{j,i}}$ where $x_{j,i} \in B_{j,i}$.

For each j, by Theorem 3.36, there exists a unique Aleksandrov solution $u_j \in C(\overline{\Omega})$ to

$$\det D^2 u_j = \mu_j \text{ in } \Omega, \qquad u_j = g \text{ on } \partial\Omega.$$

The conclusion of the theorem now follows from Theorem 3.35, by letting $j \to \infty$ along a suitable subsequence. □

The strict convexity of Ω was used in Theorems 3.35 and 3.36 to assert the continuity up to the boundary of u and to assert that $u = g$ on $\partial\Omega$ when g is an arbitrary continuous function. In the special case of $g \equiv 0$, these properties of u follow from the Aleksandrov maximum principle in Theorem 3.12. Thus, we have the following basic existence and uniqueness result for the Dirichlet problem with zero boundary data on a general bounded convex domain.

Theorem 3.38 (The Dirichlet problem with zero boundary data)**.** *Let Ω be a bounded convex domain in $\mathbb{R}^n$, and let μ be a nonnegative Borel measure in Ω with $\mu(\Omega) < \infty$. Then there exists a unique convex function $u \in C(\overline{\Omega})$ that is an Aleksandrov solution of*

$$\det D^2 u = \mu \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

More generally, the zero boundary data can be replaced by the trace of a convex function on $\overline{\Omega}$. This is also a necessary condition for the solvability of the Dirichlet problem since the boundary data is the trace of the Aleksandrov solution, if any. We have the following theorem.

Theorem 3.39 (The Dirichlet problem with convex boundary data)**.** *Let Ω be a bounded open convex domain in $\mathbb{R}^n$, and let μ be a nonnegative Borel measure in Ω with $\mu(\Omega) < \infty$. Let $\varphi \in C(\overline{\Omega})$ be a convex function. Then there exists a unique convex Aleksandrov solution $u \in C(\overline{\Omega})$ to*

$$\begin{cases} \det D^2 u = \mu & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega. \end{cases}$$

In the proof of Theorem 3.39, we will use the following consequence of the maximum principle for harmonic functions.

Lemma 3.40. *Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let $u \in C(\overline{\Omega})$ be convex, and let $h \in C(\overline{\Omega})$ be harmonic. If $u \leq h$ on $\partial\Omega$, then $u \leq h$ in Ω.*

Proof. For any $x_0 \in \Omega$, take $p \in \partial u(x_0)$. Then $l(x) := u(x_0) + p \cdot (x - x_0)$ satisfies $l \leq u$ in Ω, and also in $\overline{\Omega}$ by continuity. From $u \leq h$ on $\partial\Omega$, it follows that $h - l \geq 0$ on $\partial\Omega$. Note that $h - l$ is harmonic in Ω. Thus, by the maximum principle for harmonic functions, $h - l \geq 0$ in Ω. In particular $h(x_0) \geq l(x_0) = u(x_0)$. Since x_0 is arbitrary, we infer that $h \geq u$ in Ω. □

Proof of Theorem 3.39. Let $h \in C(\overline{\Omega})$ be the harmonic function with $h = \varphi$ on $\partial\Omega$; see Theorem 2.80.

Step 1. We first prove the theorem for $\mu \equiv 0$. Define u as in the proof of Theorem 3.33; that is, $u(x) = \sup_{v \in \mathcal{S}} v(x)$ for $x \in \overline{\Omega}$, where

$$\mathcal{S} = \{v \in C(\overline{\Omega}) : v \text{ is a convex function and } v \leq \varphi \text{ on } \partial\Omega\}.$$

Instead of Steps 1–3 in the proof of Theorem 3.33, we only need to show that $\varphi \leq u \leq h$ in Ω and the rest is similar to Step 4 there. If $v \in \mathcal{S}$, then v is convex in Ω, with $v \leq h$ on $\partial\Omega$. Thus, by Lemma 3.40, $v \leq h$ in Ω. Hence $u \leq h$. To show that $u \geq \varphi$, let us fix $x_0 \in \Omega$ and take $p \in \partial\varphi(x_0)$. Then $\bar{v}(x) := \varphi(x_0) + p \cdot (x - x_0) \leq \varphi(x)$ for all $x \in \Omega$. This implies that $\bar{v} \in \mathcal{S}$ and $u(x_0) \geq \bar{v}(x_0) = \varphi(x_0)$. Therefore $u \geq \varphi$ as desired.

Step 2. Now, we prove the theorem for general μ. There exists a sequence of measures $\{\mu_j\}_{j=1}^{\infty}$ converging weakly* to μ such that each μ_j is a finite combination of Dirac measures with positive coefficients and $\mu_j(\Omega) \leq A < \infty$ for some constant A. By Theorem 2.51, there exists an increasing sequence of uniformly convex domains $\Omega_j \subset \Omega$ such that all points of mass concentration of μ_j lie in Ω_j and Ω_j converges to Ω in the Hausdorff distance.

For each j, we will apply Theorem 3.37 on Ω_j. There exists a unique convex function $u_j \in C(\overline{\Omega_j})$ that is an Aleksandrov solution of

$$\det D^2 u_j = \mu_j \quad \text{in } \Omega_j, \qquad u_j = \varphi \quad \text{on } \partial\Omega_j.$$

Let $w_j \in C(\overline{\Omega_j})$ be the Aleksandrov solution of

$$\det D^2 w_j = 0 \quad \text{in } \Omega_j, \qquad w_j = \varphi \quad \text{on } \partial\Omega_j.$$

Let $v_j \in C(\overline{\Omega_j})$ be the Aleksandrov solution of

$$\det D^2 v_j = \mu_j \quad \text{in } \Omega_j, \qquad v_j = 0 \quad \text{on } \partial\Omega_j.$$

Then, in view of Lemma 3.10, $\mu_{v_j + w_j} \geq \mu_{v_j} + \mu_{w_j} = \mu_{u_j}$. By the comparison principle in Theorem 3.21, $v_j + w_j \leq u_j$ in Ω_j.

Since $h + \max_{\partial\Omega_j}(\varphi - h)$ is harmonic in Ω_j and not less than the convex function u_j on $\partial\Omega_j$ (which is φ), Lemma 3.40 yields

$$u_j \leq h + \max_{\partial\Omega_j}(\varphi - h) \quad \text{in } \Omega_j.$$

By the comparison principle, we have $w_j \geq \varphi$ in Ω_j. Therefore, all together,

$$\varphi + v_j \leq u_j \leq h + \max_{\partial\Omega_j}(\varphi - h) \quad \text{in } \Omega_j.$$

Now, using the Aleksandrov maximum principle for v_j, we find

$$\varphi - C(n,\Omega)[\operatorname{dist}(\cdot, \partial\Omega_j)]^{1/n} A^{1/n} \leq u_j \leq h + \max_{\partial\Omega_j}(\varphi - h) \text{ in } \Omega_j.$$

As in the first paragraph of the proof of Theorem 3.35, letting $j \to \infty$ along a subsequence, we obtain $u \in C(\overline{\Omega})$, solving our Dirichlet problem. □

3.8. Comparison Principle with Nonconvex Functions

We first record a simple comparison principle for C^2 functions that are possibly nonconvex.

Lemma 3.41. *Let Ω be a bounded and open set in $\mathbb{R}^n$. Let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ be functions such that v is convex and* $\det D^2 v > \det D^2 u$ *in Ω. Then*

$$\min_{\overline{\Omega}}(u - v) = \min_{\partial\Omega}(u - v).$$

Proof. Assume that $u - v$ attains its minimum value on $\overline{\Omega}$ at $x_0 \in \Omega$. Then $D^2 u \geq D^2 v(x_0) \geq 0$, so $\det D^2 u(x_0) \geq \det D^2 v(x_0)$, a contradiction. □

Now, we use Lemma 3.41 to show that the global Hölder estimates in Lemma 3.26 are optimal in dimensions at least 3.

Proposition 3.42. *Let $n \geq 3$, $\alpha = 2/n$, and*

$$\Omega := \{(x', x_n) : |x'| < 2^{-n}, 0 < x_n < (2^{-2n} - |x'|^2)^{\frac{1}{1-\alpha}}\} \subset \mathbb{R}^n.$$

Let $u \in C(\overline{\Omega})$ be the Aleksandrov solution to

$$\det D^2 u = 1 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

Then for $x = (0, x_n) \in \Omega$ sufficiently close to $\partial\Omega$, we have

$$|u(x)| \geq c(n) \operatorname{dist}^{2/n}(x, \partial\Omega).$$

Proof. Let $v(x) = x_n + x_n^\alpha(|x'|^2 - 2^{-2n})$. Then, $v \in C^\infty(\Omega)$, and $v = 0$ on $\partial\Omega$. Computing as in the proof of Lemma 3.26, we have

$$\det D^2 v = 2^{n-1} x_n^{n\alpha - 2}\Big[\alpha(1-\alpha)2^{-2n} - (\alpha^2 + \alpha)|x'|^2\Big] \leq 2^{n-1} 2^{-2n} \leq 1/4.$$

By using Theorems 5.13 and 6.5 in later chapters, we have $u \in C^\infty(\Omega)$. Thus, Lemma 3.41 tells us that $v \geq u$ in Ω. Hence, for $x = (0, x_n)$, we have

$$|u(x)| \geq |v(x)| = x_n^\alpha(2^{-2n} - x_n^{1-\alpha}) \geq x_n^\alpha 2^{-2n-1} = 2^{-2n-1} \operatorname{dist}^{2/n}(x, \partial\Omega)$$

if $x_n > 0$ is sufficiently small. □

The solvability of the Dirichlet problem for the Monge–Ampère equation on strictly convex domains with continuous boundary data allows us to extend the comparison principle in Theorem 3.21 to $W^{2,n}$ nonconvex functions on possibly nonconvex domains.

Theorem 3.43 (Comparison principle for $W^{2,n}$ nonconvex functions)**.** *Let Ω be a bounded and open set in $\mathbb{R}^n$. Let $u \in W^{2,n}(\Omega)$. Let $v \in C(\overline{\Omega})$ be a convex function satisfying $\mu_v(\Omega) < \infty$ and*

$$\mu_v \geq \max\{\det D^2 u, 0\}\, dx \quad \text{in } \Omega.$$

Then

$$\min_{\overline{\Omega}}(u - v) = \min_{\partial\Omega}(u - v).$$

Proof. To simplify, let us denote $f^+ = \max\{0, f\}$.

Step 1. We first prove the theorem for $u \in C^2(\Omega)$. Assume by contradiction that $u - v$ attains its minimum value on $\overline{\Omega}$ at $x_0 \in \Omega$ with

$$u(x_0) - v(x_0) < \min_{\partial\Omega}(u - v).$$

Since Ω is bounded, we can find $\delta > 0$ small such that the function

$$w(x) := v(x) + \delta|x - x_0|^2$$

satisfies the following: $u(x) - w(x)$ attains its minimum value on $\overline{\Omega}$ at $y \in \Omega$ with

$$u(y) - w(y) < \min_{\partial\Omega}(u - w).$$

Note that

$$\mu_w \geq \mu_v \geq (\det D^2 u)^+ \; dx \geq \det D^2 u \; dx.$$

Observe that $D^2 u(y)$ cannot be positive definite; otherwise we obtain a contradiction when applying Lemma 3.22 to u and v in $B_r(y)$ where $D^2 u$ is positive definite and $\mu_u = \det D^2 u \; dx$. Hence $D^2 u(y)$ has an eigenvalue $\lambda \leq 0$ with a corresponding eigenvector $z \in \partial B_1(0)$. By the Taylor expansion,

$$u(y + tz) - u(y) = At + (\lambda/2)t^2 + o(t^2) \quad \text{as } t \to 0, \tag{3.33}$$

where $A := Du(y) \cdot z$. We also have from the convexity of v

$$\begin{aligned} w(y + tz) - w(y) &= v(y + tz) - v(y) + \delta[|y + tz - x_0|^2 - |y - x_0|^2] \\ &\geq p \cdot (tz) + \delta[|y + tz - x_0|^2 - |y - x_0|^2] \quad (p \in \partial v(y)) \\ &\geq \tilde{A}t + \delta t^2 \quad (\tilde{A} = p \cdot z + 2\delta(y - x_0) \cdot z). \end{aligned}$$

From the minimality of $u - w$ at y and (3.33), we find

$$\begin{aligned} u(y) - w(y) &\leq u(y + tz) - w(y + tz) \\ &\leq u(y) - w(y) + (A - \tilde{A})t + (\lambda/2 - \delta)t^2 + o(t^2), \text{ as } t \to 0. \end{aligned}$$

Therefore $(A-\tilde{A})t+(\lambda/2-\delta)t^2 \geq 0$ for all $|t|$ small, so $A = \tilde{A}$ and $\lambda/2-\delta \geq 0$. This contradicts $\lambda \leq 0 < \delta$.

Step 2. Finally, we prove the theorem for $u \in W^{2,n}(\Omega)$. We use an approximation argument and the solvability of the Dirichlet problem for the Monge–Ampère equation on a ball with continuous boundary data.

Let y and w be as above. Let $r > 0$ be such that $B := \overline{B_r(y)} \subset \Omega$. Then

$$\int_B (\det D^2 u)^+ \, dx \leq \mu_v(B) \leq \mu_v(\Omega) < \infty.$$

Let $\{u_k\}_{k=1}^\infty \subset C_c^\infty(\mathbb{R}^n)$ be a sequence of smooth functions that satisfies $\lim_{k\to\infty} \|u_k - u\|_{W^{2,n}(B)} = 0$; see Theorem 2.50. From the Sobolev Embedding Theorem (Theorem 2.76), we also have $\lim_{k\to\infty} \|u_k - u\|_{C(\overline{B})} = 0$. For each k, by Theorem 3.37, there exists a unique convex solution $\tilde{u}_k \in C(\overline{B})$ to

$$\mu_{\tilde{u}_k} = (\det D^2 u_k)^+ \; dx \quad \text{in } B, \quad \tilde{u}_k = u_k \quad \text{on } \partial B.$$

Since

$$\lim_{k\to\infty} \|(\det D^2 u_k)^+ - (\det D^2 u)^+\|_{L^1(B)} = 0 \quad \text{and} \quad \lim_{k\to\infty} \|u_k - u\|_{C(\partial B)} = 0,$$

we can let $k \to \infty$ and then use the compactness result in Theorem 3.35. We find that, up to extracting a subsequence, $\tilde{u}_k \to \tilde{u}$, uniformly on compact subsets of B, where $\tilde{u} \in C(\overline{B})$ is the convex solution to

$$\mu_{\tilde{u}} = (\det D^2 u)^+ \; dx \quad \text{in } B, \quad \tilde{u} = u \quad \text{on } \partial B.$$

Because $u_k \in C^2(\Omega)$, Step 1 gives $\tilde{u}_k \leq u_k$ in B which implies that $\tilde{u} \leq u$ on $\overline{B}$. Hence $\tilde{u}(y) \leq u(y)$ and

$$\begin{aligned} \tilde{u}(y) - w(y) \leq u(y) - w(y) &\leq \min_{\overline{\Omega}}(u - w) \\ &\leq \min_{\partial B}(u - w) = \min_{\partial B}(\tilde{u} - w). \end{aligned} \tag{3.34}$$

On B, $\mu_{\tilde{u}} = (\det D^2 u)^+ \, dx \leq \mu_v \leq \mu_w$, and both $\tilde{u}$ and w are convex. Thus, the comparison principle in Theorem 3.21 implies that

$$\min_{\overline{B}}(\tilde{u} - w) = \min_{\partial B}(\tilde{u} - w).$$

This combined with (3.34) shows that $\tilde{u} - w$ has its minimum value over $\overline{B}$ at an interior point y in B. However, this contradicts Lemma 3.22. □

We can use the comparison principle in Theorem 3.43 to obtain the uniqueness of $W^{2,n}$ solutions to the homogenous Monge–Ampère equation with zero boundary values; see Problem 3.17. In view of Problem 1.7, the exponent n in $W^{2,n}$ is optimal.

3.9. Problems

In Problems 3.3–3.5, we use the concept of polar body of a convex body defined in Problem 2.10; moreover, δ_z is the Dirac measure at $z \in \mathbb{R}^n$.

Problem 3.1. Let $p^1, \ldots, p^{n+1}$ be the vertices of a convex polygon in $\mathbb{R}^2$, and let $u(x) = \sup_{1\leq i\leq n+1}\{x \cdot p^i\}$ where $x \in \mathbb{R}^2$. Find $\partial u(\mathbb{R}^2)$ and μ_u.

Problem 3.2. Let $f : [0, 1) \to (0, \infty)$ be an integrable function. Find the Aleksandrov solution to the Monge–Ampère equation

$$\det D^2 u(x) = f(|x|) \quad \text{in } B_1(0) \subset \mathbb{R}^n, \quad u = 0 \quad \text{on } \partial B_1(0).$$

Problem 3.3 (Polar body and fundamental solution of the Monge–Ampère equation)**.** Let Ω be a bounded, convex domain in $\mathbb{R}^n$. For $z \in \Omega$, let $u^z \in C(\overline{\Omega})$ be the convex Aleksandrov solution to the Monge–Ampère equation

$$\det D^2 u^z = \delta_z \quad \text{in } \Omega, \qquad u^z = 0 \quad \text{on } \partial\Omega.$$

Let $\overline{\Omega}^z = \{x \in \mathbb{R}^n : x \cdot (y - z) \leq 1 \quad \text{for all } y \in \overline{\Omega}\}$ be the polar body of $\overline{\Omega}$ with respect to $z \in \Omega$. Show that

$$\partial u^z(z) = |u^z(z)|\overline{\Omega}^z \quad \text{and} \quad |u^z(z)|^n = |\overline{\Omega}^z|^{-1}.$$

Problem 3.4 (Santaló point)**.** Let K be a convex body in $\mathbb{R}^n$. Prove that there is a unique interior point $s(K)$ of K such that $|K^{s(K)}| \leq |K^z|$ for all interior points z of K. We call $s(K)$ the Santaló point of K.
Hint: Use Problem 3.3, Aleksandrov's maximum principle, and compactness of the Monge–Ampère equation.

Problem 3.5. Let Ω be a bounded, convex domain in $\mathbb{R}^n$ that is centrally symmetric, so that $-x \in \Omega$ whenever $x \in \Omega$. Let $u \in C(\overline{\Omega})$ be the Aleksandrov solution to the Monge–Ampère equation

$$\det D^2 u = \delta_0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

(a) Show that $\|u\|^n_{L^\infty(\Omega)} = |u(0)|^n \leq n^{n/2}|\Omega|$.

(b) The *symmetric Mahler conjecture* states that if K is a centrally symmetric convex body in $\mathbb{R}^n$, then

$$|K||K^0| \geq \frac{4^n}{n!}.$$

This conjecture was proved by Mahler [**Mah**] for $n = 2$ and by Iriyeh and Shibata [**ISh**] for $n = 3$. Prove that the symmetric Mahler conjecture is equivalent to the following estimate for the fundamental solution of the Monge–Ampère equation:

$$|u(0)|^n \leq \frac{n!}{4^n}|\Omega|.$$

Problem 3.6. Let Ω be a bounded domain in $\mathbb{R}^n$ with smooth boundary. This problem aims to prove the classical isoperimetric inequality

$$\frac{\mathcal{H}^{n-1}(\partial\Omega)}{|\Omega|^{(n-1)/n}} \geq \frac{\mathcal{H}^{n-1}(\partial B_1(0))}{|B_1(0)|^{(n-1)/n}} = n\omega_n^{1/n}$$

via the ABP method, following Cabré [**Cab2**]. Proceed as follows.

(a) Let ν be the outer unit normal to $\partial\Omega$. Show that there is a solution $u \in C^2(\overline{\Omega})$ to the Neumann boundary value problem

$$\Delta u = \mathcal{H}^{n-1}(\partial\Omega)/|\Omega| \quad \text{in } \Omega, \qquad Du \cdot \nu = 1 \quad \text{on } \partial\Omega.$$

(b) Let Γ^- be the lower contact set of u:

$$\Gamma^- = \{y \in \Omega : u(x) \geq u(y) + p \cdot (x - y) \text{ for all } x \in \Omega \text{ for some } p(y) \in \mathbb{R}^n\}.$$

Show that

$$B_1(0) \subset Du(\Gamma^-) \cap \{x \in \Omega : |Du(x)| < 1\}.$$

(c) Take volumes in the inclusion $B_1(0) \subset Du(\Gamma^-)$, invoke Theorem 2.74 and use $0 \leq \det D^2 u \leq (\Delta u/n)^n$ on Γ^- to conclude.

Problem 3.7. Let $A(x)$ be a smooth function on $(-1, 1)$. Consider the following fourth-order boundary value problem for a convex function u:

$$\left(\frac{1}{u''}\right)'' = -A \quad \text{on } (-1,1), \qquad u''(x) \to \frac{2}{1-x^2} + w(x) \quad \text{as } x \to \pm 1,$$

for some (also unknown) smooth function w on $[-1, 1]$. This is a *one-dimensional version of the Abreu equation with Guillemin boundary conditions.* Geometrically, this equation corresponds to a rotationally invariant metric on the two-dimensional sphere whose scalar curvature is a given function $A(h)$ of the Hamiltonian h for the circle action.

(a) Suppose that a smooth convex solution u exists on $(-1, 1)$. Show that A must satisfy the moment conditions

$$\int_{-1}^{1} A(x)\,dx = 2, \qquad \int_{-1}^{1} xA(x)\,dx = 0.$$

(b) Assume now that $A(x)$ satisfies the above moment conditions. Then, there exists a convex solution u if and only if

$$\mathcal{L}_A(f) := f(1) + f(-1) - \int_{-1}^{1} fA\,dx > 0$$

for all nonaffine convex functions f on $[-1, 1]$. In this case, show that the solution is an absolute minimum of the functional

$$\mathcal{F}_A(f) := f(1) + f(-1) - \int_{-1}^{1} \Big(fA + \log(f'')\Big)\,dx.$$

Hint for the existence of solutions in part (b): Let $v = 1/u''$. Then $v'' = -A$ with $v(x) \to 0$ and $v'(x) \to \mp 1$ as $x \to \pm 1$. Thus v can be expressed in terms of A and the Green's function $G(x,y)$ of the one-dimensional Laplace equation via $v(x) = \int_{-1}^{1} G(x,y)A(y)\,dy$. Here, for a fixed $x \in (-1,1)$, the function $G(x,\cdot)$ is concave; it is a linear function of y on the intervals $(-1,x)$ and $(x,1)$, vanishing at ± 1. Suppose that $\mathcal{L}_A(f) > 0$ for all nonaffine convex functions f on $[-1,1]$. Then $v(x) = \mathcal{L}_A(-G(x,\cdot))$ is positive in $(-1,1)$. Integrate twice to get a solution u from $u'' = v^{-1}$.

Problem 3.8. Let Ω be a bounded domain in $\mathbb{R}^n$.

(a) Let $L(u;\Omega) = \{(x, Du(x)) : x \in \Omega\}$ be the gradient graph of a function $u : \Omega \to \mathbb{R}$. Prove that if $u \in C^2$, then

$$\int_\Omega |\det D^2 u|\,dx = \lim_{\lambda\to\infty} \frac{\mathcal{H}^n(L(\lambda u;\Omega))}{\lambda^n}.$$

Hint: Show that $\mathcal{H}^n(L(u;\Omega)) = \int_\Omega \sqrt{\det(I_n + (D^2u)^2)}\,dx$.

(b) Let $u \in C^2(\mathbb{R}^n)$ be a function with support in Ω. Prove the following ABP-type estimate:

$$\sup_\Omega |u| \leq C(n)|\Omega|^{1/n}\Big(\int_\Omega |\det D^2 u|\,dx\Big)^{1/n}.$$

Compare with Lemmas 3.15 and 3.17.

Problem 3.9. Let $u, v \in C(\overline{\Omega})$ be convex functions in a bounded, convex domain Ω in $\mathbb{R}^n$ with $u = v = 0$ on $\partial\Omega$. Assume that for some $0 < \varepsilon < 1$,

$$1 - \varepsilon \leq \det D^2 u \leq 1 + \varepsilon \quad \text{in } \Omega \quad \text{and} \quad \det D^2 v = 1 \quad \text{in } \Omega.$$

Prove that there is a positive constant $C(n, |\Omega|)$ such that

$$|u(x) - v(x)| \leq C\varepsilon \quad \text{for all } x \in \Omega.$$

Problem 3.10. Verify that $u(x) = -(1 - |x|^2)^{\frac{1}{2}}$ is the unique convex solution to the following singular Monge–Ampère equation on the unit ball:

$$\det D^2 u = |u|^{-(n+2)} \quad \text{in } B_1(0), \qquad u = 0 \quad \text{on } \partial B_1(0).$$

Problem 3.11 (Nonconvex supersolution)**.** Assume $p > 0$. Let

$$\Omega := \{(x', x_n) : |x'| < 1, 0 < x_n < (1 - |x'|^2)^{\frac{n}{n+p-2}}\} \subset \mathbb{R}^n.$$

Show that the function

$$u(x) = \Big[x_n - x_n^{\frac{2}{n+p}}(1 - |x'|^2)^{\frac{n}{n+p}}\Big]/2$$

is smooth in Ω and satisfies

$$\det D^2 u \leq |u|^{-p} \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega.$$

Problem 3.12 (Supersolutions for a singular Monge–Ampère equation)**.** Assume $p \geq 1$. Let $\Omega = \{(x', x_n) : |x'| < 1, 0 < x_n < 1 - |x'|^2\} \subset \mathbb{R}^n$, $n \geq 2$. Show that there is a constant $C = C(n, p)$ such that the function

$$u = Cx_n - Cx_n^{\frac{2}{n+p}}(1 - |x'|^2)^{\frac{n+p-2}{n+p}}$$

is smooth, convex in Ω, and satisfies

$$\det D^2 u \leq |u|^{-p} \quad \text{in } \Omega \qquad \text{and} \qquad u = 0 \quad \text{on } \partial\Omega.$$

Problem 3.13. Let Ω be a bounded, convex domain in $\mathbb{R}^n$. Let $p > 0$. Assume that $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ solves the singular Monge–Ampère equation

$$\det D^2 u \ = |u|^{-p} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

(a) Show that $u \in C^{\frac{2}{n+p}}(\overline{\Omega})$ with the estimate

$$|u(x)| \leq C(n, p, \Omega) \text{dist}^{\frac{2}{n+p}}(x, \partial\Omega) \quad \text{for all } x \in \Omega.$$

(b) Show that the global Hölder regularity in part (a) is optimal in the following sense: There exist bounded convex domains Ω in $\mathbb{R}^n$ such that $u \notin C^\beta(\overline{\Omega})$ for any $\beta > \frac{2}{n+p}$.
Hint: Use Lemma 3.41 and Problem 3.11.

Problem 3.14. Let Ω be a bounded, convex domain in $\mathbb{R}^n$. Let $q > 0$ and $\lambda(\Omega) > 0$. Assume that $u \in C(\overline{\Omega}) \cap C^\infty(\Omega)$ is a solution to the degenerate Monge–Ampère equation

$$\det D^2 u \ = \lambda(\Omega)|u|^q \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

(a) Assume $q \geq n - 2$. Show that $u \in C^\beta(\overline{\Omega})$ for all $\beta \in (0, 1)$.

(b) Assume $0 < q < n - 2$. Show that $u \in C^\beta(\overline{\Omega})$ for all $\beta \in (0, \frac{2}{n-q})$.

Problem 3.15. Let Δ be the open triangle in $\mathbb{R}^2$ with vertices at $(0, 0)$, $(1, 0)$, and $(1/2, \sqrt{3}/2)$. Write down an explicit formula for the solution to

$$\det D^2 u \ = 1 \quad \text{in } \Delta, \qquad u = 0 \quad \text{on } \partial\Delta.$$

Problem 3.16. Let $n \geq 2$ and $\Omega = \{(x', x_n) : |x'| < 1, 0 < x_n < 1 - |x'|^2\}$. For $0 < p < n$, consider $u(x) = x_n - x_n^{\frac{2}{n+p}}(1 - |x'|^2)^{\frac{n+p-2}{n+p}}$. Show that

(a) u is smooth, convex in Ω with $u = 0$ on $\partial\Omega$, $\det D^2 u \in L^1(\Omega)$, and

(b) $u \notin C^\alpha(\overline{\Omega})$ for any $\alpha > \frac{2}{n+p}$.

It follows that for $1/n < \alpha < 1$, we can choose $\max\{2/\alpha - n, 1\} < p < n$, and the above function shows that the exponent $1/n$ in Lemma 3.18 is sharp.

Problem 3.17. Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose that $u \in W^{2,n}(\Omega)$ satisfies

$$\det D^2 u = 0 \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega.$$

Show that $u \equiv 0$.

Problem 3.18. Let Ω be a bounded convex domain in $\mathbb{R}^n$. Let $u \in W^{2,n}(\Omega)$ be a convex function. Show that $\mu_u = (\det D^2 u)\, dx$.

3.10. Notes

Most materials in this chapter are classical for which we largely rely on Figalli [**F2**] and Gutiérrez [**G2**]. In particular, most results and arguments in Sections 3.2, 3.3, 3.5, and 3.7 can be found in [**G2**, Chapter 1] and [**F2**, Chapter 2]; see also Le–Mitake–Tran [**LMT**, Chapter 3]. Quite different from the real Monge–Ampère case where the Monge–Ampère measure can be defined for all convex functions, the complex Monge–Ampère measure cannot be defined for all plurisubharmonic functions on $\mathbb{C}^n$; see Bedford–Taylor [**BT**] for examples.

In Section 3.3, the sharp form of the Aleksandrov maximum principle in Theorem 3.12 is implicit in Trudinger [**Tr3**]. It can be used to give a proof of the classical isoperimetric inequality; see Theorem 4.16. The proof of Theorem 3.16 is based on Gilbarg–Trudinger [**GT**, Section 9.1]. For the Aleksandrov–Bakelman–Pucci maximum principle in Theorem 3.16, we refer to the original papers [**Al2**, **Al3**, **Ba2**, **Pu**]. Kuo and Trudinger [**KT**] extended the ABP estimates to $a^{ij} D_{ij} u$ being in L^q where $q < n$, and the new estimates depend on restrictions on the eigenvalues of the coefficient matrix $(a^{ij})_{1 \leq i,j \leq n}$. Brendle [**Bre**] recently proved the isoperimetric inequality for a minimal submanifold in Euclidean space, using a method inspired by the ABP maximum principle.

In Section 3.5, Lemma 3.25 is from Caffarelli [**C2**]; see also Trudinger–Wang [**TW6**, Section 3.2]. Lemma 3.26 is from [**C2**]. Lemma 3.29 is from Le [**L13**].

Example 3.32 in Section 3.6 and Proposition 3.42 in Section 3.8 show that many barehanded estimates based on the Aleksandrov maximum principle are essentially sharp.

For the Dirichlet problem in Section 3.7, see Bakelman [**Ba5**, Chapter 11] for a historical account; see also Rauch–Taylor [**RT**] for original motivations. Theorems 3.33, 3.36, and 3.37 were first established by Aleksandrov and Bakelman. Our proofs here follow [**RT**]. Theorem 3.39 is independently due to Błocki [**Bl**] and Hartenstine [**Ha1**]; it is also contained in Trudinger–Wang [**TW6**, Section 2.3] and Han [**H**, Section 8.2] with different proofs.

Theorem 3.43 in Section 3.8 is due to Ozanski [**Oz**]; it was first established for strictly convex domains by Rauch and Taylor [**RT**]. Ozanski [**Oz**] also discusses an interesting application of Theorem 3.43 in the theory of the two-dimensional Navier–Stokes equations.

Problem 3.7 is taken from Donaldson [**D2**]. Problem 3.8 is based on Viterbo [**Vb**] where he discovered that the ABP estimate for the Monge–Ampère equation is essentially equivalent to an isoperimetric inequality in symplectic geometry. Problems 3.12–3.14(b) are taken from Le [**L12**] while Problem 3.14(a) is taken from [**L7**]. Problems 3.11 and 3.16 are taken from [**L13**]. For a different construction of examples showing the sharpness of the exponent $1/n$ in Lemma 3.18, see Jerrard [**Jer**] in the context of Monge–Ampère functions.