
Contents

Preface	xiii
Notation	xv
Chapter 1. Introduction	1
§1.1. The Monge–Ampère Equation	2
§1.2. The Linearized Monge–Ampère Equation	8
§1.3. Plan of the Book	12
§1.4. Notes	16
§1.5. Problems	17
Chapter 2. Geometric and Analytic Preliminaries	19
§2.1. Convex Sets	19
§2.2. The Hausdorff Distance	25
§2.3. Convex Functions and the Normal Mapping	28
§2.4. Boundary Principal Curvatures and Uniform Convexity	42
§2.5. Calculus with Determinant	48
§2.6. John’s Lemma	52
§2.7. Review of Measure Theory and Functional Analysis	59
§2.8. Review of Classical PDE Theory	63
§2.9. Pointwise Estimates and Perturbation Argument	69
§2.10. Problems	74
§2.11. Notes	77

Part 1. The Monge–Ampère Equation

Chapter 3. Aleksandrov Solutions and Maximum Principles	81
§3.1. Motivations and Heuristics	82
§3.2. The Monge–Ampère Measure and Aleksandrov Solutions	85
§3.3. Maximum Principles	91
§3.4. Global Hölder Estimates and Compactness	97
§3.5. Comparison Principle and Global Lipschitz Estimates	99
§3.6. Explicit Solutions	105
§3.7. The Dirichlet Problem and Perron’s Method	106
§3.8. Comparison Principle with Nonconvex Functions	116
§3.9. Problems	119
§3.10. Notes	123
Chapter 4. Classical Solutions	125
§4.1. Special Subdifferential at the Boundary	126
§4.2. Quadratic Separation at the Boundary	129
§4.3. Global Estimates up to the Second Derivatives	134
§4.4. Existence of Classical Solutions	139
§4.5. The Role of Subsolutions	141
§4.6. Pogorelov’s Counterexamples to Interior Regularity	142
§4.7. Application: The Classical Isoperimetric Inequality	144
§4.8. Application: Nonlinear Integration by Parts Inequality	145
§4.9. Problems	146
§4.10. Notes	148
Chapter 5. Sections and Interior First Derivative Estimates	149
§5.1. Sections of Convex Functions	150
§5.2. Caffarelli’s Localization Theorem and Strict Convexity	156
§5.3. Interior Hölder Gradient Estimates	163
§5.4. Counterexamples to Hölder Gradient Estimates	167
§5.5. Geometry of Sections	169
§5.6. Vitali Covering and Crawling of Ink-spots Lemmas	174
§5.7. Centered Sections	176
§5.8. Problems	183
§5.9. Notes	184

Chapter 6. Interior Second Derivative Estimates	185
§6.1. Pogorelov's Second Derivative Estimates	186
§6.2. Second-Order Differentiability of the Convex Envelope	192
§6.3. Density Estimate	195
§6.4. Interior Second Derivative Sobolev Estimates	198
§6.5. Counterexamples of Wang and Mooney	210
§6.6. Interior Second Derivative Hölder Estimates	214
§6.7. Further Remarks on Pogorelov-type Estimates	226
§6.8. Problems	229
§6.9. Notes	231
Chapter 7. Viscosity Solutions and Liouville-type Theorems	233
§7.1. Viscosity Solutions of the Monge–Ampère Equation	233
§7.2. Viscosity Solutions with Boundary Discontinuities	242
§7.3. Liouville-type Theorems	252
§7.4. Application: Connections between Four Important PDEs in Two Dimensions	256
§7.5. Legendre–Lewy Rotation	262
§7.6. Problems	265
§7.7. Notes	267
Chapter 8. Boundary Localization	269
§8.1. Preliminary Localization Property of Boundary Sections	270
§8.2. Savin's Boundary Localization Theorem	278
§8.3. Normalized Altitude of Boundary Sections	283
§8.4. Proof of the Boundary Localization Theorem	294
§8.5. Pointwise Hölder Gradient Estimates at the Boundary	296
§8.6. Boundary Localization for Degenerate and Singular Equations	298
§8.7. Problems	299
§8.8. Notes	300
Chapter 9. Geometry of Boundary Sections	301
§9.1. Maximal Interior Sections and Rescalings	302
§9.2. Global Hölder Gradient Estimates	312
§9.3. Dichotomy, Volume Growth, and Engulfing Properties	316
§9.4. Global Inclusion, Exclusion, and Chain Properties	319

§9.5.	Besicovitch's Covering Lemma, Maximal Function, and Quasi-Distance	325
§9.6.	Problems	333
§9.7.	Notes	334
Chapter 10.	Boundary Second Derivative Estimates	335
§10.1.	Global Second Derivative Sobolev Estimates	335
§10.2.	Wang's Counterexamples	338
§10.3.	Boundary Pogorelov Second Derivative Estimates	341
§10.4.	Pointwise Second Derivative Hölder Estimates at the Boundary	347
§10.5.	Global Second Derivative Hölder Estimates	361
§10.6.	Problems	364
§10.7.	Notes	366
Chapter 11.	Monge–Ampère Eigenvalue and Variational Method	367
§11.1.	The Monge–Ampère Energy	370
§11.2.	Parabolic Monge–Ampère Flow	380
§11.3.	Degenerate Monge–Ampère Equations	387
§11.4.	Monge–Ampère Eigenvalue and Eigenfunctions	392
§11.5.	Global Regularity for Degenerate Equations	401
§11.6.	Problems	404
§11.7.	Notes	405
Part 2. The Linearized Monge–Ampère Equation		
Chapter 12.	Interior Harnack Inequality	409
§12.1.	Caffarelli–Gutiérrez Harnack Inequality	410
§12.2.	Proof of the Interior Harnack Inequality	412
§12.3.	Interior Hölder Estimates	424
§12.4.	Application: The Affine Bernstein Problem	427
§12.5.	Problems	435
§12.6.	Notes	437
Chapter 13.	Boundary Estimates	439
§13.1.	Global Hölder Estimates	440
§13.2.	Application: Solvability of Affine Mean Curvature and Abreu's Equations	443
§13.3.	Boundary Gradient Estimates	445

§13.4. Boundary Hölder Gradient Estimates	448
§13.5. Boundary Harnack Inequality	459
§13.6. Application: Minimizers of Linear Functionals with Prescribed Determinant	468
§13.7. Problems	471
§13.8. Notes	472
Chapter 14. Green's Function	475
§14.1. Bounds and Higher Integrability	478
§14.2. Local Integrability Estimates for the Gradient	486
§14.3. Monge–Ampère Sobolev Inequality	489
§14.4. Global Higher Integrability Estimates	494
§14.5. Global Integrability Estimates for the Gradient	501
§14.6. Application: Hölder Estimates for Equations with Critical Inhomogeneity	505
§14.7. Problems	511
§14.8. Notes	512
Chapter 15. Divergence Form Equations	513
§15.1. Interior Uniform Estimates via Moser Iteration	514
§15.2. Interior Hölder estimates	521
§15.3. Global Hölder Estimates	524
§15.4. Application: The Dual Semigeostrophic Equations	528
§15.5. Application: Singular Abreu Equations	533
§15.6. Problems	547
§15.7. Notes	550
Bibliography	553
Index	571