# Preface

This book is intended for a second- or third-year graduate topics course on the subject of *translation surfaces* and their moduli spaces, with a focus on associated dynamical systems, counting problems, and group actions.

In recent years, translation surfaces and their moduli spaces have been the objects of extensive study and interest, with connections to widely varied fields including (but not limited to) geometry and topology; Teichmüller theory; low-dimensional dynamical systems; homogeneous dynamics and Diophantine approximation; and algebraic and complex geometry. Contributions to the field have been recognized by major awards, invited addresses at international conferences, and multiple excellent survey articles on various aspects of the subject. However, there has not yet been a textbook written about translation surfaces and the various viewpoints that have been developed that a student or researcher who is interested in the field can use as an accessible introduction.

This book aims to fill that gap, with attention to definitions and an introduction to some of the big ideas in the field, centered on the ergodic properties of translation flows and counting problems for saddle connections, and associated renormalization techniques, without attempting to reach the full state of the art (an aim that is in any case impossible given the speed at which the field is evolving).

In particular, we do not discuss in great detail the subject of the Kontsevich-Zorich cocycle and its Lyapunov exponents and the subject of Rauzy induction, instead giving a brief survey of some of these ideas and mostly referring the reader to the detailed surveys by Forni [**70**], Zorich [**177**], Forni-Matheus [**71**], and Yoccoz [**174**]. We only scratch the surface, no pun intended, of the subject of *square-tiled surfaces*, which have inspired a lot of interesting connections to arithmetic and combinatorics. We do not discuss in any detail the deep connections to algebraic geometry that have driven some of the exciting recent developments in the field, referring the reader to the survey of Bud-Chen [**30**].

So what *do* we cover in this book? As a kind of overture, we discuss in Chapter 1 the important motivating example of the flat torus, exploring its geometry, and its associated dynamical and counting problems. The linear flow on the torus and its associated first return map, a rotation of a circle, are amongst the first dynamical systems ever studied. The counting of closed orbits is intricately tied to number theory. We discuss, as motivation, the moduli space of translation surfaces on a torus, a bundle over the well-known modular curve and the action of $GL^+(2,\mathbb{R})$ on this space of translation surfaces.

Translation surfaces are higher-genus generalizations of flat tori. In Chapter 2, we define translation surfaces from three perspectives (Euclidean geometry, complex analysis, and geometric structures) and prove the equivalence of these definitions, something that is often hard to find in the literature. We define when two translation surfaces are equivalent, leading to the definition of moduli spaces of translation surfaces. We show how some translation surfaces arise from *unfolding* billiards in rational polygons. We introduce the subject of half-translation surfaces, or quadratic differentials.

In Chapter 3, we begin with a short introduction to Teichmüller theory and its relation to the study of translation surfaces. We then define moduli spaces and strata of translation surfaces and introduce the $GL^+(2,\mathbb{R})$-action on strata of translation surfaces. We introduce natural period coordinates on strata and use these coordinates to construct the canonical $SL(2,\mathbb{R})$-invariant measure on strata of area 1 translation surfaces. We show the existence of *Delaunay triangulations*, following the work of Masur-Smillie [**121**]. Understanding these triangulations allows us to show that the measure of strata is finite.

In Chapter 4, we discuss the natural dynamical systems associated to translation surfaces, namely, *linear flows* and their first return maps, *interval exchange transformations*. We explore their ergodicity and mixing properties. Along the way we study an important example of a translation surface flow for which every orbit is dense but not every orbit is equidistributed with respect to Lebesgue measure, a phenomenon that does not occur in the case of linear flows on the torus, where a theorem of Weyl implies that dense orbits are always equidistributed.

In Chapter 5, we show how information about the recurrence properties of an orbit of a translation surface under the positive diagonal subgroup of $SL(2,\mathbb{R})$ (the *Teichmüller geodesic flow*) can be used to get information about the ergodic properties of the associated linear flow on an individual translation surface. This is an example of a major theme in the subject, the interplay between properties of the orbit of a translation surface under the $SL(2,\mathbb{R})$-action and properties of the translation surface itself. We show (§5.2) that the $SL(2,\mathbb{R})$-action on each stratum is ergodic. At the end of Chapter 5, in §5.4, we discuss quantitative versions of renormalization ideas, including our brief discussion of Lyapunov exponents.

As another example of the strength of renormalization ideas, we show in Chapter 6 how the *ergodic properties* of the $SL(2,\mathbb{R})$-action can be used to obtain *counting results* for *saddle connections* and, subsequently, to compute the intrinsic volumes of strata. Finally, in Chapter 7, we discuss examples, characterizations, and properties of surfaces with large affine symmetry groups, known as *lattice* or *Veech* surfaces.

Broadly speaking, the first four and a half chapters of the book are quite detailed and relatively self-contained, while from the middle of Chapter 5 we attempt to communicate some of the big recent ideas in the field, with somewhat less detail, but hopefully a sufficient set of precise references and suggestions for further reading.

**Intended uses.** This book is intended to be used for a one-semester graduate course introducing students to the subject of translation surfaces. As such, we have included exercises throughout. Attempting these exercises is a crucial part of following along with this book. This book can also be used for researchers interested in getting a big-picture overview of the subject together with concrete details of important definitions and concepts.

**Prerequisites and suggested reading.** We have attempted to keep this book as self-contained as possible. Recommended prerequisites include first-year graduate courses in complex analysis (at the level of, for example, Conway [**38**]), measure theory (Royden [**142**]), and manifolds (do Carmo [**47**]). Ideally, the reader will have some knowledge of Riemann surfaces and ergodic theory, at the level of, for example, Jost [**95**] for Riemann surfaces and Walters [**170**] or Einsiedler-Ward [**52**] for ergodic theory. A couple of nice books targeted at undergraduates that may be useful to keep alongside our book are Schwartz's *Mostly Surfaces* [**150**], which introduces the basics of translation surfaces and covers the topology of surfaces in a very accessible fashion, and Davis's [**42**] recent book *Billiards, Surfaces and Geometry: A Problem-Centered Approach*.

**Black Boxes.** In order to keep this book relatively self-contained, at several junctures we have to assume certain results whose proofs would be beyond the scope of our book. We have tried to be as clear as possible in indicating we are doing this, by labeling such results using the header *Black Box*.

**Black Box.** *A* Black Box *is a result we require but whose proof is beyond the scope of the book.*

For example, we will treat the Riemann-Roch (Black Box 2.3.2) and Gauss-Bonnet (Black Box 2.2.1) theorems as Black Boxes.

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