Chapter 1

Introduction

1.1. The square torus

A common first example of a compact manifold is the square torus, $\mathbb{C}/\mathbb{Z}[i]$, where $\mathbb{Z}[i] = \{m + ni : m, n \in \mathbb{Z}\}$ is the set of Gaussian integers. This surface has many different interpretations, each of which lends itself to generalization:

Polygon. First, it can be viewed as the unit square

$$\{z = x + iy \in \mathbb{C} : 0 \leq x, y \leq 1\}$$

with parallel sides identified by translation. Precisely, the vertical sides $\{iy : 0 \leq y \leq 1\}$ and $\{1 + iy : 0 \leq y \leq 1\}$ are identified by the translation $z \mapsto z + 1$ and the horizontal sides $\{x : 0 \leq x \leq 1\}$ and $\{i + x : 0 \leq x \leq 1\}$ are identified by the translation $z \mapsto z + i$. See Figure 1.1.

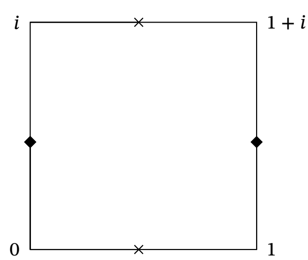

Figure 1.1. A square torus, with side identifications.

Holomorphic 1**-forms.** The square torus such as $\mathbb{C}/\mathbb{Z}[i]$ has the structure of a *Riemann surface*, that is, a surface with a complex structure, since $\mathbb{Z}[i]$ acts on $\mathbb{C}$ by translations, which are holomorphic. Moreover, the family of holomorphic 1-forms $\{e^{i\theta}dz : \theta \in [0, 2\pi)\}$ on $\mathbb{C}$ descends to $\mathbb{C}/\mathbb{Z}[i]$, since $d(z+c) = dz$.

Constant curvature geometric structure. Since translations are Euclidean isometries, dz induces a (flat) Euclidean metric on $\mathbb{C}/\mathbb{Z}[i]$, inherited from the Euclidean metric on $\mathbb{C}$. In particular, this metric has *constant* (zero) Gaussian curvature.

Dynamical systems. In addition to these three interpretations of the square torus, we have a 1-parameter family of dynamical systems, indexed by $S^1 = [0, 2\pi)/(0 \sim 2\pi)$: given $\theta \in S^1$, we have the flow

$$\phi_t^\theta(z) = (z + te^{i\theta}) \mod \mathbb{Z}[i].$$

Generalizations. When we study general translation surfaces, we will have each of these three interpretations, appropriately modified: polygons, with parallel sides identified by translation; Riemann surfaces, together with holomorphic 1-forms; and flat metrics (with singularities), as well as a 1-parameter family of linear flows. Moreover, we will consider *spaces* of translation surfaces—note that in the above constructions, we never really used the fact that the square torus is a square, but rather that we had a parallelogram with parallel sides identified by translation.

1.2. The space of tori

Families of tori. Given any $v, w \in \mathbb{C}^*$ that are not $\mathbb{R}$-collinear, that is, $v/w \notin \mathbb{R}$, we can construct the lattice $\Lambda = \Lambda(v, w) = \{mv + nw : m, n \in \mathbb{Z}\} \subset \mathbb{C}$ and the associated torus $\mathbb{C}/\Lambda$, which can be viewed as the parallelogram defined by v, w with pairs of sides identified by translation. One choice that corresponds to the square torus is $v = 1, w = i$, so $\Lambda(v, w) = \mathbb{Z}[i]$. Note that any other choice of v, w so that $\Lambda(v, w) = \mathbb{Z}[i]$ would yield the same torus, presented in a different way.

1-**forms and flows.** Again the family of 1-forms $\{e^{i\theta}dz : \theta \in [0, 2\pi)\}$ descends to $\mathbb{C}/\Lambda$, and again, there is an associated family of linear flows ϕ_t^θ as above. There is a natural *parameter space* of 1-forms living on the space of tori. We note that each 1-form gives a choice of *vertical direction* on the torus. The vertical direction of the 1-form $e^{i\theta}dz$ on the torus $\mathbb{C}/\Lambda$ can be thought of as the set of tangent vectors v to $\mathbb{C}$ such that $e^{i\theta}dz(v)$ is of the form ci where $c > 0$. The 1-form $e^{i\theta}dz$ on $\mathbb{C}/\Lambda$ can be thought of as the 1-form dz on the torus $\mathbb{C}/e^{-i\theta}\Lambda$. We will denote the space of pairs of tori together with 1-forms by $\Omega(\emptyset)$, where $\emptyset$ refers to the fact that 1-form dz does not have any zeros. That is,

$$\Omega(\emptyset) = \{(\mathbb{C}/\Lambda, dz) : \Lambda \text{ a lattice in } \mathbb{C}\}.$$

We emphasize that there will be many v, w for each Λ so that $\Lambda = \Lambda(v, w)$. For example,

$$\Lambda(v, w) = \Lambda(v + w, w) = \Lambda(v, v + w).$$

The $GL^+(2,\mathbb{R})$-action. There is a natural $\mathbb{R}$-linear action of the group $GL^+(2,\mathbb{R})$ consisting of 2×2 invertible matrices with real entries and positive determinant on $\Omega(\emptyset)$, coming from the left $\mathbb{R}$-linear action of $GL(2,\mathbb{R})$ on $\mathbb{C}$ given by viewing $x + iy$ as the column vector $\begin{pmatrix} x & y \end{pmatrix}$; that is,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot (x + iy) = (ax + by) + i(cx + dy), \tag{1.2.1}$$

and

$$g \cdot (\mathbb{C}/\Lambda(v, w)) = \mathbb{C}/\Lambda(g \cdot v, g \cdot w).$$

Note that if $\Lambda(v, w) = \Lambda(v', w')$, then $\Lambda(g\cdot v, g\cdot w) = \Lambda(g\cdot v', g\cdot w')$. This action will generalize to spaces of higher-genus translation surfaces. The embedded action of $\mathbb{R}^+$ by multiplication (corresponding to scalar multiples $tI_2, t \in \mathbb{R}^*$ of the identity matrix I_2) scales the area of tori by t^2. The action of S^1 via rotation matrices

$$r_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.2.2}$$

changes the underlying 1-form by multiplication by $e^{-i\theta}$, or, equivalently, multiplies the lattice Λ by $e^{i\theta}$. Putting these actions together gives a $\mathbb{C}^*$-action on the space $\Omega(\emptyset)$, which can be thought of as preserving the underlying torus but changing the 1-form by scaling; that is, for $\zeta \in \mathbb{C}^*$,

$$\zeta(\mathbb{C}/\Lambda, dz) = (\mathbb{C}/\Lambda, \zeta dz) = (\mathbb{C}/\zeta^{-1}\Lambda, dz).$$

Area 1 **tori.** As many of the properties we are interested in will be independent of scaling, we will pay special attention to the set of *area* 1 tori—that is, tori for which the parallelogram spanned by v, w has area 1. Note that if $v = x + iy, w = u + is$, the area of the parallelogram (assuming v, w are positively oriented) is given by

$$\begin{aligned} A(v, w) &= \det \begin{pmatrix} x & u \\ y & s \end{pmatrix} = xs - yu \\ &= \operatorname{Im}(\overline{v}w) = \frac{i}{2}(v\overline{w} - \overline{v}w). \end{aligned}$$

The subset of unit-area tori

$$\mathcal{H} = \{(\mathbb{C}/\Lambda(v, w), dz) : A(v, w) = 1\} \subset \Omega(\emptyset)$$

is preserved by the action of the subgroup $SL(2,\mathbb{R})$ of $GL^+(2,\mathbb{R})$ consisting of determinant 1 matrices. The next exercise uses this action to help us to identify the space $\mathcal{H}$ with a quotient of $SL(2,\mathbb{R})$.

Exercise 1.1. *Show that any lattice $\Lambda = \Lambda(v, w)$ with $A(v, w) = 1$ can be written as $\Lambda = g \cdot \mathbb{Z}[i]$, $g \in SL(2,\mathbb{R})$, and if $h \in SL(2,\mathbb{Z}), h \cdot \mathbb{Z}[i] = \mathbb{Z}[i]$. Conclude that the $\mathbb{R}$-linear action of $SL(2,\mathbb{R})$ acts* transitively *on the space of pairs $(\mathbb{C}/\Lambda(v, w), dz) \in \mathcal{H}$ and the stabilizer of the pair $(\mathbb{C}/\mathbb{Z}[i], dz)$ is the subgroup*

$SL(2,\mathbb{Z})$ of integer determinant 1 *matrices, and thus, by the orbit-stabilzer theorem* (*see, for example, Artin* [**5**, Proposition 6.4]), *we can identify*

$$\mathcal{H} = SL(2,\mathbb{R})/SL(2,\mathbb{Z})$$

via

$$gSL(2,\mathbb{Z}) \mapsto (\mathbb{C}/g \cdot \mathbb{Z}[i], dz).$$

1.3. Dynamics

Returning to the square torus, the dynamics of the linear flow ϕ_t^θ satisfies a natural dichotomy, depending on the angle θ. If $\tan(\theta)$ (that is, the slope of the line at angle θ with the horizontal) is rational, every orbit is periodic. On the other hand, if $\tan(\theta)$ is irrational, every orbit is dense and, moreover, *equidistributed*: if $A \subset \mathbb{C}/\mathbb{Z}[i]$ is an open set, then for all x,

$$\lim_{T\to\infty} \frac{1}{T}|\{0 \le t \le T : \phi_t^\theta(x) \in A\}| = m(A),$$

where m denotes Lebesgue measure on the torus. That is, the proportion of time each orbit spends in a given subset of the torus tends to the measure of the subset. We now show how to prove a discretized version of this using the idea of a *first return map*, which we will explore further in the context of translation surfaces and interval exchange maps in Chapter 4. We identify $I = [0,1]/0 \sim 1 = \mathbb{R}/\mathbb{Z}$ with the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ via the map

$$x \mapsto e^{2\pi i x}.$$

Consider lines that make an angle $0 \le \psi \le \pi/2$ with the horizontal. Note that the horizontal lines ($\psi = 0$) on the square torus are all closed. In the discussion below, we will use $[z]$ to denote the coset $z + \mathbb{Z}[i]$. For $\psi \ne 0$, starting at a point $[x_0] \in \mathbb{C}/\mathbb{Z}[i], x_0 \in [0,1)$, define T_ψ to be the *first return map* of the flow ϕ_t^ψ in direction ψ to the closed curve $S = \{[x] : 0 \le x \le 1\}$ (note that $[0] = [1]$), sitting in our square torus $\mathbb{C}/\mathbb{Z}[i]$. The flow trajectory $\phi_t^\psi([x_0])$ returns to S when it hits the line $y = 1$. See Figure 1.2.

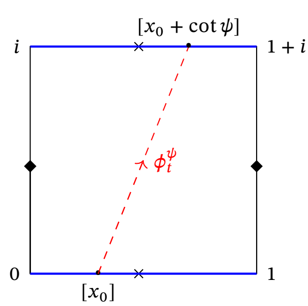

Figure 1.2. The first return map of the flow ϕ_t^ψ to the closed curve S.

Since

$$\phi_t^{\psi}([x_0]) = [x_0 + te^{i\psi}] = [(x_0 + t\cos\psi) + it\sin\psi],$$

this return happens at $t = \csc\psi$, and we see that the point of intersection of the line in direction ψ with the copy of S given by $\{[x+i] : 0 \leq x \leq 1\}$ has x-coordinate $x + \cot(\psi)$, which says that

$$T_\psi([x_0]) = [(x_0 + \cot\psi) \bmod(1)].$$

Via the identification of the interval $[0,1)$ with the circle, the map T_ψ transforms to the map we denote $R_\alpha : S^1 \to S^1$

$$R_\alpha(e^{2\pi i\phi}) = e^{2\pi i(\phi+\alpha)}$$

where $\alpha = \cot\psi$. Depending on whether α is rational or irrational, the map R_α has very different behavior. We leave as an exercise the proof that if α is rational, the map R_α is periodic.

Exercise 1.2. *Prove that if $\alpha \in \mathbb{Q}$ and written in lowest terms as $\alpha = p/q$ with $p, q \in \mathbb{Z}$,* $\gcd(p,q) = 1$, *then every orbit of R_α is periodic with period q.*

Equidistribution. For α irrational, we have a classical equidistribution result, due originally to Weyl [**171**]. We say a sequence of points $\{x_n\}_{n\geq 1}$ in a topological space X *equidistributes* with respect to a (probability) measure η if the sequence of probability measures

$$\sigma_N := \frac{1}{N}\sum_{n=1}^{N} \delta_{x_i}$$

converges in the weak-* sense to η; that is, for every continuous function $f \in C(X)$,

$$\lim_{N\to\infty}\int_X f\,d\sigma_N = \lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(x_i) = \int_X f\,d\eta.$$

Theorem 1.3.1. *If $\alpha \notin \mathbb{Q}$, then every orbit of R_α is dense and in fact every orbit equidistributes in the circle with respect to the Lebesgue probability measure $d\phi$, where we parameterize $S^1 = \{e^{2\pi i\phi} : 0 \leq \phi < 1\}$.*

Proof. Weyl [**171**] showed (see Exercise 1.3) that a sequence $\{x_n\}_{n\geq 1} \subset [0,1)$ becomes equidistributed with respect to Lebesgue measure if and only if for every positive integer k

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi ikx_n} = 0 = \int_{S^1} e^{2\pi ik\phi}\,d\phi.$$

We use this result to show equidistribution of every orbit of R_α. Writing $e^{2\pi i x_n} = R_\alpha^n(e^{2\pi i x})$ we have

$$\begin{aligned}(1.3.1)\qquad \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} &= \frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k(x+n\alpha)} = \frac{1}{N}e^{2\pi i k x}\sum_{n=1}^{N} e^{2\pi i k n\alpha}\\ &= \frac{e^{2\pi i(kx+k\alpha)}}{N}\frac{1-e^{2\pi i k N\alpha}}{1-e^{2\pi i k\alpha}}.\end{aligned}$$

Since α is irrational, $1 - e^{2\pi i k\alpha} \neq 0$. Since $|e^{2\pi i t}| = 1$ for any real t, the limit of the above quantity is 0 as $N \to \infty$. □

Exercise 1.3. *Prove Weyl's theorem that a sequence $\{x_n\}_{n\geq 1}$ on S^1 becomes equidistributed if and only if for every positive integer k*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} e^{2\pi i k x_n} = 0$$

assuming the Stone-Weierstrass theorem (see, for example, Rudin [**143**, Theorem 1.26]) *which states that the set of trigonometric polynomials*

$$\left\{\sum_{j=1}^{J} a_j f_j \,:\, a_j \in \mathbb{C}\right\}$$

consisting of finite linear combinations of functions f_j given by

$$f_j(\theta) = e^{2\pi i j\theta} = \cos j\theta + i\sin j\theta$$

is dense *in the set $C(S^1)$ of complex-valued continuous functions on S^1.*

1.4. Counting

We've seen above (Exercise 1.2) that rational slopes correspond to (families of) *periodic* trajectories for linear flows on the square torus $\mathbb{C}/\mathbb{Z}[i]$. We can then ask to *count* the periodic trajectories, graded by length. To remove ambiguity, we choose one representative from each family, by considering only periodic trajectories through the point 0. These trajectories are in 1-to-1 correspondence with the set P of *primitive* Gaussian integers,

$$P = \{m + ni \,:\, m, n \in \mathbb{Z}, \gcd(m,n) = 1\}.$$

Defining

$$N_{\text{prim}}(R) = \#\left(P \cap B(0,R)\right),$$

we will argue that

$$\lim_{R\to\infty}\frac{N_{\text{prim}}(R)}{\pi R^2} = \frac{1}{\zeta(2)},$$

where $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. The constant $1/\zeta(2) = \frac{6}{\pi^2}$ is related to the natural volume of the moduli space of tori. We will see similar quadratic asymptotics and relations to volumes of moduli spaces for general translation surfaces in Chapter 6.

Tiling. Without the primitivity condition, the count

$$N(R) = \#\{m + ni \in \mathbb{Z}[i] : m^2 + n^2 \leq R^2\}$$

is seen by a tiling argument to grow asymptotically like πR^2.

Exercise 1.4. *Prove*

$$\lim_{R\to\infty} \frac{N(R)}{\pi R^2} = 1.$$

Counting primitive points. To count primitive vectors, first note that

$$\mathbb{Z}[i] \setminus \{0\} = \bigsqcup_{n\in\mathbb{N}} nP,$$

so if

$$N_{\text{prim}}(R) = \#(P \cap B(0,R))$$

was asymptotic to $c_P \pi R^2$ for some c_P, we would have, for $n \in \mathbb{N}$,

$$\#(nP \cap B(0,R)) = \#(P \cap B(0,R/n)) = N_{\text{prim}}(R/n)$$

is asymptotic to $\frac{c_P}{n^2}\pi R^2$ for all $n \in \mathbb{N}$. Therefore,

$$c_P \sum_{n\in\mathbb{N}} n^{-2} = 1;$$

that is, $c_P = \frac{1}{\zeta(2)} = 6/\pi^2$. Alternately, note that a heuristic for the probability that an element $m + ni \in \mathbb{Z}[i]$ is in P is given by the Euler product over all primes p

$$\prod_p (1 - p^{-2}) = \frac{1}{\zeta(2)},$$

since the probability that m and n are both divisible by a particular prime p is p^{-2}. We leave formalizing this intuition as a (nontrivial) exercise:

Exercise 1.5. *Prove*

$$\lim_{R\to\infty} \frac{N_{\text{prim}}(R)}{\pi R^2} = \frac{1}{\zeta(2)} = 6/\pi^2.$$

Orbits. Finally, we note that $\mathbb{Z}_{\text{prim}}[i] = SL(2,\mathbb{Z}) \cdot i$ consists of a *single* orbit of $SL(2,\mathbb{Z})$ acting on $\mathbb{C}^*$, which will be relevant for us in particular in Chapter 7, where we study particular families of translation surfaces which resemble the torus in many important ways.