Preface

Complex manifold theory is one of the most beautiful branches of geometry, in which algebraic topology, differential geometry, algebraic geometry, homological algebra, complex analysis, and partial differential equations (PDEs) come together in deep and surprising ways to create a rich theory. This book is meant to be an introduction to the concepts, techniques, and principal results about complex manifolds (mainly compact ones), aimed primarily at graduate students and researchers who already have a solid background in differential geometry.

What are complex manifolds? They are defined in exactly the same way as smooth manifolds, except the local coordinate charts are required to take their values in $\mathbb{C}^n$ and to overlap holomorphically. This might sound like a minor tweak to the definition of smooth manifolds, but in fact the requirement of holomorphicity changes everything. For example, on a connected compact complex manifold, the only global holomorphic functions are the constants, and the space of holomorphic sections of a holomorphic vector bundle is always finite-dimensional. Whereas every smooth manifold can be smoothly embedded in some Euclidean space, only certain complex manifolds can be holomorphically embedded in $\mathbb{C}^n$ or in complex projective space. There is a deep interplay between differential geometry and complex analysis, especially for Kähler manifolds, the ones on which the metric structure and the holomorphic structure play together nicely.

Complex manifolds have profound applications in many areas of mathematics. Here are a few examples:

- Riemann surfaces (1-dimensional complex manifolds) are essential for understanding global properties of holomorphic functions in one complex variable.
- Complex surfaces (2-dimensional complex manifolds) play a central role in attempts to classify 4-dimensional smooth manifolds.

- Complex manifolds defined by algebraic equations are among the central objects of interest in algebraic geometry, and the study of their differential geometry has contributed important advances in algebraic geometry.

- Calabi–Yau manifolds are complex manifolds that play a crucial role in string theory.

Because complex manifold theory is so intimately connected with algebraic geometry and complex analysis as well as differential geometry, there are many paths one might follow to get to know the subject. This book is about the differential geometry of complex manifolds, so the techniques and results in it are all firmly situated in differential geometry. Although I survey quite a few of the connections with other subjects, especially complex algebraic geometry, my choice of topics is heavily influenced by my desire to focus on techniques that will be familiar to those with a good background in differential geometry but not necessarily in commutative algebra or analysis of several complex variables. I have tried to include enough points of contact with algebraic geometry that the book can serve as a useful jumping-off point for differential geometers who decide to delve into the algebraic geometry literature, while at the same time serving to introduce algebraic geometers and complex analysts to a differential-geometric viewpoint on their subject.

I have chosen to use the Kodaira embedding theorem—which characterizes those compact complex manifolds that admit holomorphic embeddings into projective spaces—as a unifying theme for the book, because it draws on most of the important techniques in complex manifold theory and it illustrates one of the most profound differences between smooth manifolds and complex ones. Many of the definitions, theorems, and techniques introduced in the book are collected together to lay the groundwork for proving that profound theorem. But not everything is here for that purpose—I also hope to offer readers a strong general background that will prepare them for more advanced study in any aspect of complex geometry.

There are many other good introductory books on complex manifolds. Some excellent examples are [**Bal06**, **Dem12**, **Huy05**, **Mor07**, **Wel08**, **Zhe00**] and the first few chapters of [**GH94**]. What distinguishes this book is an approach that readers of my previous graduate texts will find familiar—instead of aiming for comprehensive coverage of all the aspects of the subject (which would be impossible in any case), I aim for two overarching goals: first, to make the explanations of definitions and concepts user-friendly, well motivated, and accessible; and second, to write the proofs with enough detail and rigor that students will hopefully not be left wondering how to bridge the gaps. This approach results in explanations that may be more wordy than some readers are used to; but in my experience it really helps beginners start to feel comfortable with a new subject.

Prerequisites

The main prerequisite is familiarity with the foundational results on topological, smooth, and Riemannian manifolds. Because this subject draws on so many of those results, there would be no point in trying to summarize all the requisite differential-geometric background here. All of the background material on manifolds that a reader needs to understand this book (and more) is contained in my three previous graduate textbooks [**LeeTM**, **LeeSM**, **LeeRM**], and I draw freely on them throughout this book (with specific references whenever appropriate).

Familiarity with elementary complex analysis is also a prerequisite, but only at the level of a typical undergraduate course on complex analysis in one variable. Any decent undergraduate complex analysis textbook will serve as a reference, such as [**BC13**, **MH98**, **Gam01**].

Beyond these subjects, the reader should also have a basic familiarity with algebraic topology—particularly singular homology and cohomology at the level of [**Hat02**] or [**Mun84**]. I give references for the main results that I use in the text.

For readers who are well versed in the prerequisite material, this book should be essentially self-contained, with one major exception: all of the results on Hodge theory rest on a fundamental Fredholm theorem for elliptic partial differential equations (Thm. 9.14), which is stated here without proof because developing the machinery for proving it would carry us too far afield into the weeds of PDE theory. The theorem is easy to state and easy to use, so readers can accept it on faith; or, for those who are curious about the proof, I offer several references where proofs can be found.

Exercises and Problems

Like my other graduate texts, this book includes both exercises (integrated into the text) and problems (collected at the ends of the chapters). The exercises are mostly routine verifications, and are there to provide the reader with opportunities to check how well they have digested the material; while the problems are mostly more difficult (some considerably so), and are designed to challenge the reader to grapple more deeply with the ideas in the text. As was the case with my previous books, I have not and do not intend to prepare written solutions to the problems or the exercises, because I do not want to deprive readers of the opportunity to get "stuck" on a problem and do the productive work of finding their own ways forward. In any case, most of these are not problems that have a single "right answer."

Typographical Conventions

This book generally follows the same typographical conventions as my previous graduate texts. Mathematical terms are typeset in ***bold italics*** when they are officially defined, to make them easy to spot on the page. The exercises in the text are indicated with the symbol ►, and numbered consecutively with the theorems to make them easy to find. The symbol □ marks the ends of proofs, and also marks the ends of statements of corollaries that follow so easily that they do not need proofs. The symbol // marks the ends of numbered examples. End-of-chapter problems are numbered 1-1, 1-2, 1-3, etc., with hyphens instead of dots, to make it easier to distinguish problem references from exercise references.

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I welcome feedback from readers about any aspects of the book, especially if you find mistakes or unclear passages. There will be an updated list of corrections on my website. I hope you enjoy the book.

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