Preface

Manifolds are essential objects of study in both differential geometry and algebraic topology. In algebraic topology, we define algebraic structures — through homology and homotopy in particular — using the topological structures of the manifolds. In differential geometry, we consider geometric-analytic structures defined on smooth manifolds in ways that are compatible with the smooth structures of manifolds. Notably, these geometric structures are not unique and are in fact quite flexible.

Of the many possible questions, it is natural and pertinent to ask under what conditions there exist canonical geometric structures on manifolds. More generally, one may ask if all smooth closed manifolds admit decompositions into pieces that admit canonical geometric structures. In dimension 3, this question was posed by Thurston, who made substantial progress and finally answered affirmatively by the Hamilton–Perelman proof of the geometrization conjecture. The existence of geometric decompositions has topological consequences, and in particular, the Poincaré conjecture follows. The proof employed the analysis of the Ricci flow, which is a central equation in geometric analysis. In Kähler geometry, the analytic minimal model program asks a similar question. Regarding canonical geometric structures *in* manifolds, minimal submanifolds are important objects of study. Other important canonical geometric objects of study include Einstein metrics; Yang–Mills, self-dual, and Seiberg–Witten connections (which are in the setting of bundles); constant scalar curvature metrics; harmonic maps; and solutions to the Einstein field equations in general relativity.

The goal of this book is to provide an introduction to differential geometry with a focus on Riemannian geometry at the level of a beginning graduate course, where the capstone is an introduction to modern geometric analysis. Motivations for the study of differential geometry come from many sources. For example, in theoretical physics, Einstein's theory of general relativity is based on Lorentzian geometry, while gauge theories and string theory are discussed in the language of differential geometry.

Our goals for readers include the following objectives: to ease the transition from an undergraduate-level to a graduate-level course in differential geometry; to introduce fundamental concepts in differential geometry that are beneficial for applications to fields beyond mathematics such as physics, engineering, biology, and computer science; and to equip them with essential material for further study in geometric analysis. While our treatment of the basics of differential geometry is not intended to be extensive or comprehensive, we aim to provide a clear and engaging pathway toward the study of related or more advanced topics.

Differential geometry is connected to a wide range of subjects in mathematics. As such, we recall basic concepts in topology, real analysis, and abstract algebra to make the book more self-contained, although some prior exposure to these subjects on the reader's part is ideal. Since our treatment is superficial in this respect, we provide some references at the undergraduate level for readers interested in learning more about the background material in these subjects. At a minimum, the reader is assumed to have at least a background in vector calculus and linear algebra. Readers should also have a good understanding of what a mathematical proof is, at the level of an introductory upper-division undergraduate course.

The idea of this book is that differential geometry, as well as geometric analysis, is just supercharged vector calculus. As we will see, the vector calculus aspect is familiar ground, whereas the supercharging aspect is actually quite approachable. In particular, familiarity with partial differential equations is not a necessary prerequisite for the geometric analysis part of this book. In fact, for the most part, one only needs real analysis at the level of Rudin's classic text *Principles of Mathematical Analysis* [**Rud76**], informally known as 'Baby Rudin', and predominantly the key a priori estimates stem from the first and second derivative tests as well as integration by parts in calculus.

We strive for this book to be readable, although at certain points the book is not wholly self-contained in that it assumes certain standard results whose statements are relatively easy to understand even if their proofs are beyond the scope of this book. The hope and expectation, at the cost of such minor inconveniences, are that new topics are opened up for the reader. For those results we do not prove in the book, we provide references that contain examples and accessible details.

Some of the topics we cover are as follows.

Part I on the Geometry of Submanifolds of Euclidean Space. Chapters 1 and 2: Submanifolds in Euclidean space; we study this notion with an eye toward its generalization to differentiable manifolds in Part II. Chapter 3: Tangent, cotangent, normal, and tensor spaces and bundles. Chapter 4: The local extrinsic and intrinsic geometry of submanifolds in Euclidean space. Chapter 5: We discuss some classical theorems on the global extrinsic geometry of submanifolds.

Part II on Differential Topology and Riemannian Geometry. Chapter 6: We introduce abstract smooth manifolds. Chapter 7: We introduce some basic concepts and results in Riemannian geometry, with an emphasis on comparison geometry. Chapter 8: We introduce differential forms and Cartan's method of moving frames. Chapter 9: Using the set-up of the previous chapter, we prove the Gauss–Bonnet formula for surfaces. Chapter 10: We discuss fiber, vector, and principal bundles and Lie groups. Using the unit tangent bundle, we discuss the Chern–Gauss–Bonnet formula.

Part III on Elliptic and Parabolic Equations in Geometric Analysis. Chapter 11: We discuss Bochner formulas, the Bishop–Gromov volume comparison theorem, and linear elliptic and parabolic equations in Riemannian geometry. Chapter 12: We study the use of nonlinear elliptic equations in differential geometry, with the focus on minimal surfaces. Chapter 13: We begin our study of geometric evolution equations, which are parabolic equations in differential geometry. Here, geometric objects evolve according to their curvature and become more uniform. In this chapter our focus is on the curve shortening flow, where embedded plane curves shrink to 'round' points. Chapter 14: We consider the uniformization of surfaces by heat flow techniques. More specifically, we use Hamilton's Ricci flow on surfaces to prove that arbitrary metrics evolve to constant curvature metrics.

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