Chapter 1

Intuitive Introduction to Submanifolds in Euclidean Space

This book is about differential geometry, which is the study of the geometry of *manifolds*. In Part I of this book, we study the special class of manifolds that are defined as being contained in some higher-dimensional Euclidean space, which are called *submanifolds*. In Part II we begin the study of manifolds in general. In Part III we study analysis on manifolds and its geometric applications.

Regarding our everyday experience, submanifolds are everywhere. For example, the surface of a ping pong ball is a 2-dimensional manifold; so are the surfaces of doughnuts and Möbius strips. In this chapter, we discuss surfaces in an intuitive manner and through pictures help the reader visualize these spherical, toroidal, and twisted shapes, waiting until later to define them rigorously.

1.1. The idea of a submanifold as being locally Euclidean

What is a submanifold? In this chapter we will start with a more intuitive approach to answering this question:

> The idea of an n-dimensional submanifold is that it is a subset of Euclidean space $\mathbb{R}^N$ that is *locally Euclidean* of dimension n, where $N \geq n$.

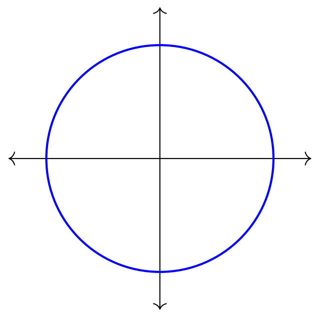

Figure 1.1.1. A **circle** is a 1-dimensional submanifold of $\mathbb{R}^2$.

Our present discussion of submanifolds will largely be based on examples and visualization. In the next chapter, we will introduce the rigorous definition of submanifold. However, in this chapter, we will freely use the term 'submanifold' in a less formal manner before subsequently presenting its definition.

As one of the simplest and yet important examples, a *circle* is a 1-dimensional submanifold of $\mathbb{R}^2$. See Figure 1.1.1.

Take a point on the circle and look at a *neighborhood* of it (informally, we say that we are looking *locally* at the circle). The intersection of the neighborhood and the circle is an arc. If we straighten out an arc, we get an interval, which is locally what 1-dimensional Euclidean space $\mathbb{R}$ looks like. See Figures 1.1.2–1.1.4.

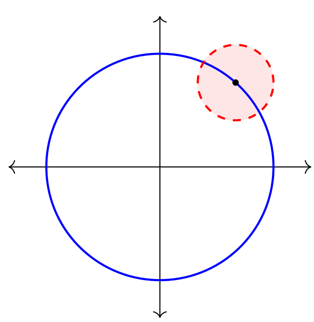

Figure 1.1.2. A neighborhood (the pink disk) in $\mathbb{R}^2$ of a point on a circle.

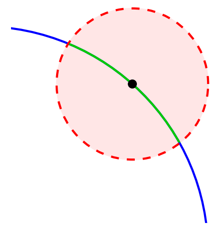

Figure 1.1.3. The green arc represents the intersection of the circle and the neighborhood.

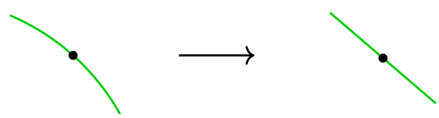

Figure 1.1.4. The green arc in Figure 1.1.3 can be smoothly straightened out to an interval. In this sense, the circle is locally Euclidean.

1.2. Euclidean and vector inner product spaces

The study of the *geometry* of submanifolds begins with considering Euclidean space as an inner product space. The Euclidean inner product yields the notions of *distance*, *length*, and *angle*, and hence geometry.[1]

Euclidean space $\mathbb{R}^n$ is defined to be the n-fold Cartesian product of $\mathbb{R}$ with itself. We may consider elements of $\mathbb{R}^n$ as either **points** or **vectors**. We use lowercase letters to denote elements considered as points and uppercase letters for elements considered as vectors. We may write a point $\mathbf{x} \in \mathbb{R}^n$ as an n-tuple

$$\mathbf{x} = (x^1, x^2, \ldots, x^n), \tag{1.1}$$

where $x^i \in \mathbb{R}$ for each $1 \leq i \leq n$. Note that the components x^i of $\mathbf{x}$ have *upper* indices; this will be our convention. We denote the **origin** in $\mathbb{R}^n$ by

$$\mathbf{0} = (0, 0, \ldots, 0). \tag{1.2}$$

We denote vectors similarly.

[1] In dimension 2, we are thus considering the geometry of Euclid. The notions of length and angle lead to the concepts of similarity and congruence of triangles and the definitions of special types of triangles such as isosceles and equilateral triangles. However, the notion of inner product is a much more modern development, introduced around the 18th or 19th century.

1.2.1. Vector spaces. It is natural to generalize the algebraic operations of addition and multiplication from $\mathbb{R}$ to $\mathbb{R}^n$. Euclidean space $\mathbb{R}^n$ is an example of a *vector space*. We can add vectors $\mathbf{X} = (X^1, X^2, \ldots, X^n)$ and $\mathbf{Y} = (Y^1, Y^2, \ldots, Y^n)$:

$$\mathbf{X} + \mathbf{Y} = (X^1 + Y^1, X^2 + Y^2, \ldots, X^n + Y^n), \tag{1.3}$$

and we have scalar multiplication:

$$a\mathbf{X} = (aX^1, aX^2, \ldots, aX^n). \tag{1.4}$$

Let $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{R}^n$ and $a, b \in \mathbb{R}$. The vector space properties for addition are

(Associative law) $(\mathbf{X} + \mathbf{Y}) + \mathbf{Z} = \mathbf{X} + (\mathbf{Y} + \mathbf{Z})$,

(Commutative law) $\mathbf{Y} + \mathbf{X} = \mathbf{X} + \mathbf{Y}$,

(Additive identity) $\mathbf{X} + \mathbf{0} = \mathbf{0} + \mathbf{X} = \mathbf{X}$,

(Inverse) $\mathbf{X} + (-\mathbf{X}) = (-\mathbf{X}) + \mathbf{X} = \mathbf{0}$.

The vector space properties for scalar multiplication are

(Multiplicative compatibility) $a(b\mathbf{X}) = (ab)\mathbf{X}$,

(Multiplicative identity) $1\mathbf{X} = \mathbf{X}$,

(Vector addition distributivity) $a(\mathbf{X} + \mathbf{Y}) = a\mathbf{X} + a\mathbf{Y}$,

(Scalar addition distributivity) $(a + b)\mathbf{X} = a\mathbf{X} + b\mathbf{X}$.

With the introduction of the Euclidean inner product, the study of Euclidean geometry and trigonometry becomes more algebraic.

In general, when a set $\mathbb{V}$ with a binary operation $+$, called **addition**, and a field $\mathbb{F}$ (such as $\mathbb{R}$) with an action (a.k.a. external binary operation) $\cdot : \mathbb{F} \times \mathbb{V} \to \mathbb{V}$, called **scalar multiplication**, satisfies these axioms for all $a, b \in \mathbb{F}$ and $\mathbf{X}, \mathbf{Y}, \mathbf{Z} \in \mathbb{V}$, some $\mathbf{0} \in \mathbb{V}$, and $1 \in \mathbb{F}$ the field identity,[2] then we say that $\mathbb{V}$ is a **vector space over** $\mathbb{F}$. In this book, in the context of real differential geometry, we will always take $\mathbb{F} = \mathbb{R}$. In complex differential geometry, which has relations with algebraic geometry, one takes $\mathbb{F} = \mathbb{C}$, the field of complex numbers.

Let $\mathbb{V}$ be a real vector space. The dual space $\mathbb{V}^*$ is the space of **linear functions** from $\mathbb{V}$ to $\mathbb{R}$. We denote this as

$$\mathbb{V}^* := \mathrm{Lin}(\mathbb{V}, \mathbb{R}). \tag{1.5}$$

Another notation for the right-hand side is $\mathrm{Hom}(\mathbb{V}, \mathbb{R})$, where Hom is for *homomorphism*.

[2]We denote the additive identity of $\mathbb{V}$ by $\mathbf{0}$.

1.2.2. Inner product vector spaces. We use the notations $\cdot$ or $\langle\,,\rangle$ to denote the **Euclidean inner product** defined by

$$\mathbf{X}\cdot\mathbf{Y} := \langle\mathbf{X},\mathbf{Y}\rangle := X^1Y^1 + X^2Y^2 + \cdots + X^nY^n. \tag{1.6}$$

Let $\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathbb{R}^n$ and $a,b\in\mathbb{R}$. The inner product properties are

(Symmetry) $$\mathbf{Y}\cdot\mathbf{X} = \mathbf{X}\cdot\mathbf{Y},$$

(Bilinearity) $$(a\mathbf{X}+b\mathbf{Y})\cdot\mathbf{Z} = a(\mathbf{X}\cdot\mathbf{Z}) + b(\mathbf{Y}\cdot\mathbf{Z}),$$

(Positive-definite) $$\mathbf{W}\cdot\mathbf{W} > 0 \qquad \text{for } \mathbf{W}\in\mathbb{R}^n\setminus\{\mathbf{0}\}.$$

The second property is linearity in the first argument, but since $\cdot$ is symmetric, it implies *bilinearity*; namely, we also have

$$\mathbf{Z}\cdot(a\mathbf{X}+b\mathbf{Y}) = a(\mathbf{Z}\cdot\mathbf{X}) + b(\mathbf{Z}\cdot\mathbf{Y}).$$

In general, when a vector space $\mathbb{V}$ over a field $\mathbb{F}$ with a map $\cdot : \mathbb{V}\times\mathbb{V}\to\mathbb{F}$ satisfies these axioms for all $a,b\in\mathbb{F}$, $\mathbf{X},\mathbf{Y},\mathbf{Z}\in\mathbb{V}$, and $\mathbf{W}\in\mathbb{V}\setminus\{\mathbf{0}\}$, then we say that $\cdot$ is an **inner product** on $\mathbb{V}$ and that $(\mathbb{V},\cdot)$ is an **inner product space**. In Exercise 1.1 you are asked to give the precise definition of an inner product (vector) space.

1.2.3. General inner products on $\mathbb{R}^n$. The notion of Euclidean inner product extends to general inner products on vector spaces; for simplicity, we consider the special case where the vector spaces are Euclidean spaces. Let

$$A : \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R} \tag{1.7}$$

define an inner product on $\mathbb{R}^n$. That is, we are taking $\mathbb{V}=\mathbb{R}^n$ in the definition of an inner product space, but we are not necessarily taking $\cdot$ to be the Euclidean inner product. By definition, A is a symmetric, positive-definite bilinear function; i.e.,

$$A(c\mathbf{V}_1+\mathbf{V}_2,\mathbf{W}) = cA(\mathbf{V}_1,\mathbf{W}) + A(\mathbf{V}_2,\mathbf{W}), \tag{1.8a}$$

$$A(\mathbf{V},c\mathbf{W}_1+\mathbf{W}_2) = cA(\mathbf{V},\mathbf{W}_1) + A(\mathbf{V},\mathbf{W}_2), \tag{1.8b}$$

$$A(\mathbf{V},\mathbf{W}) = A(\mathbf{W},\mathbf{V}), \tag{1.8c}$$

$$A(\mathbf{V},\mathbf{V}) > 0 \quad \text{if } \mathbf{V}\neq\mathbf{0}. \tag{1.8d}$$

Note that the second equation (1.8b) follows from the first (1.8a) and the third (1.8c). Given A, we have correspondingly the symmetric positive-definite matrix

$$a := (a_{ij})_{i,j=1}^n \tag{1.9}$$

defined by

$$a_{ij} := A(\mathbf{e}_i,\mathbf{e}_j), \tag{1.10}$$

where $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ denotes the standard Euclidean basis of $\mathbb{R}^n$; that is, $\mathbf{e}_i := (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 is in the i-th slot. We have (see Exercise 1.2)

$$A(\mathbf{V}, \mathbf{W}) = \sum_{i,j=1}^{n} a_{ij} V^i W^j. \tag{1.11}$$

Conversely, given a symmetric positive-definite matrix

$$a = (a_{ij})_{i,j=1}^n,$$

the bilinear function $A : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ defined by (1.11) is the unique inner product satisfying

$$A(\mathbf{e}_i, \mathbf{e}_j) = a_{ij}.$$

So we have a one-to-one correspondence between inner products on $\mathbb{R}^n$ and symmetric positive-definite $n \times n$ matrices.

The Euclidean inner product is just one inner product on $\mathbb{R}^n$, namely the one corresponding to the $n \times n$ identity matrix I_n:

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij} = I_n(\mathbf{e}_i, \mathbf{e}_j),$$

where δ_{ij} is the Kronecker delta function defined by

$$\delta_{ij} := \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.12}$$

1.2.4. Euclidean distances and angles. The Euclidean inner product determines distances and angles and hence the geometry of Euclidean space. This principle is fundamental to the study of differential geometry. The **length** of a vector $\mathbf{X} \in \mathbb{R}^n$ is defined by

$$\|\mathbf{X}\| := \sqrt{\mathbf{X} \cdot \mathbf{X}} = \sqrt{(X^1)^2 + (X^2)^2 + \cdots + (X^n)^2}. \tag{1.13}$$

The Euclidean distance $d_{\mathbb{E}}$ between two points $\mathbf{x}$ and $\mathbf{y}$ is defined by

$$d_{\mathbb{E}}(\mathbf{x}, \mathbf{y}) := \|\mathbf{x} - \mathbf{y}\| := \|\mathbf{X} - \mathbf{Y}\| \tag{1.14}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are $\mathbf{x}$ and $\mathbf{y}$ considered as vectors, respectively.[3]

We have for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$:

(1) $d_{\mathbb{E}}(\mathbf{x}, \mathbf{y}) = d_{\mathbb{E}}(\mathbf{y}, \mathbf{x})$.

(2) $d_{\mathbb{E}}(\mathbf{x}, \mathbf{y}) \geq 0$ with $d_{\mathbb{E}}(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$.

(3) (Triangle inequality)

$$d_{\mathbb{E}}(\mathbf{x}, \mathbf{z}) \leq d_{\mathbb{E}}(\mathbf{x}, \mathbf{y}) + d_{\mathbb{E}}(\mathbf{y}, \mathbf{z}). \tag{1.15}$$

[3] In the subsequent discussions, we will often not distinguish between points and vectors, treating them as equivalent.

The proof of this is left as Exercise 1.4. The reason why (1.15) is called the ‘triangle inequality’ is that it implies that the sum of the lengths of any two sides of a triangle is greater than or equal to the length of the remaining side.

Definition 1.1. In general, when a set $\mathfrak{M}$ with a function $d : \mathfrak{M} \times \mathfrak{M} \to \mathbb{R}$ satisfies these axioms for all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathfrak{M}$, we say that $(\mathfrak{M}, d)$ is a **metric space** (see Exercise 1.3).

The **Cauchy–Schwarz inequality** says that for all vectors $\mathbf{V}, \mathbf{W} \in \mathbb{R}^n$,

$$|\mathbf{V} \cdot \mathbf{W}| \leq \|\mathbf{V}\| \, \|\mathbf{W}\|. \tag{1.16}$$

For the proof, see Exercise 1.5. Moreover, the Cauchy–Schwarz inequality is used to prove the triangle inequality (1.15).

We have that $\mathbf{X} \cdot \mathbf{Y} = 0$ if and only if $\mathbf{X}$ and $\mathbf{Y}$ are perpendicular (orthogonal) to each other.

The concept of perpendicularity is a special case of the more general relationship between vectors, which can be expressed through the angle between them. The **angle** $\angle(\mathbf{X}, \mathbf{Y})$ between two nonzero vectors $\mathbf{X}$ and $\mathbf{Y}$ in $\mathbb{R}^n$ is

$$\angle(\mathbf{X}, \mathbf{Y}) := \cos^{-1} \frac{\mathbf{X} \cdot \mathbf{Y}}{\|\mathbf{X}\| \, \|\mathbf{Y}\|}. \tag{1.17}$$

For the proof of this, see Exercise 1.6. Figure 1.2.1 shows the idea of the proof, which involves the vector projection map.

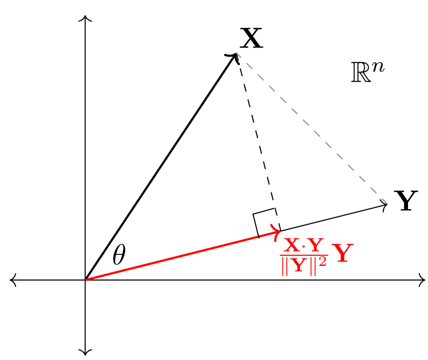

Figure 1.2.1. The vectors $\mathbf{X}$ and $\mathbf{Y}$, making an angle $\theta = \angle(\mathbf{X}, \mathbf{Y})$, and the vector projection of $\mathbf{X}$ onto $\mathbf{Y}$.

1.3. Introduction to submanifolds via examples

As we stated at the beginning of this chapter, *submanifolds*, loosely speaking, are locally Euclidean spaces inside higher-dimensional Euclidean spaces. Before we give the formal definition of submanifolds at the end of the next chapter, we discuss several examples.

1.3.1. Hyperplanes. Hyperplanes are examples of 'linear' submanifolds. As such, we will at first understand them algebraically, via the Euclidean inner product.

1.3.1.1. *Coordinate hyperplanes.* Recall that a **bijection** is an injective and surjective map. Recall that in general a map $f : X \to Y$ is a bijection if and only if it has an inverse $g : Y \to X$. That is, $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. Often, a convenient way to show that a function is a bijection is to show that it has an inverse.

Define the **j-th coordinate hyperplane**

$$\mathbb{R}^n_j := \{\mathbf{X} \in \mathbb{R}^{n+1} \colon X^j = 0\} \tag{1.18}$$

for $1 \leq j \leq n+1$; see Figure 1.3.1. As a vector subspace of $\mathbb{R}^{n+1}$, $\mathbb{R}^n_j$ is an n-dimensional vector space over $\mathbb{R}$. We have the bijective linear map (vector space isomorphism) $\rho_j : \mathbb{R}^n_j \to \mathbb{R}^n$ defined by

$$\rho_j : (X^1, \ldots, X^{j-1}, 0, X^{j+1}, \ldots, X^{n+1}) \mapsto (X^1, \ldots, X^{j-1}, X^{j+1}, \ldots, X^{n+1}). \tag{1.19}$$

This map in effect identifies $\mathbb{R}^n_j$ with $\mathbb{R}^n$. Note that the inverse map $\rho_j^{-1} : \mathbb{R}^n \to \mathbb{R}^n_j$ is also a linear map. Thus, $\mathbb{R}^n_j$ is Euclidean, both globally and locally, making it a submanifold of $\mathbb{R}^{n+1}$.

1.3.1.2. *Affine hyperplanes.* An **affine hyperplane** in $\mathbb{R}^{n+1}$ is a subset of the form

$$\mathbb{A} = \{\mathbf{X} \in \mathbb{R}^{n+1} \colon \mathbf{X} \cdot \mathbf{N} = a\}, \tag{1.20}$$

where $a \in \mathbb{R}$ and $\mathbf{N}$ is a fixed nonzero vector, called a **normal vector** to $\mathbb{A}$. See Figure 1.3.2.

If $a = 0$, then $\mathbb{A}$ passes through the origin $\mathbf{0}$ and we call $\mathbb{A}$ a **linear hyperplane**.

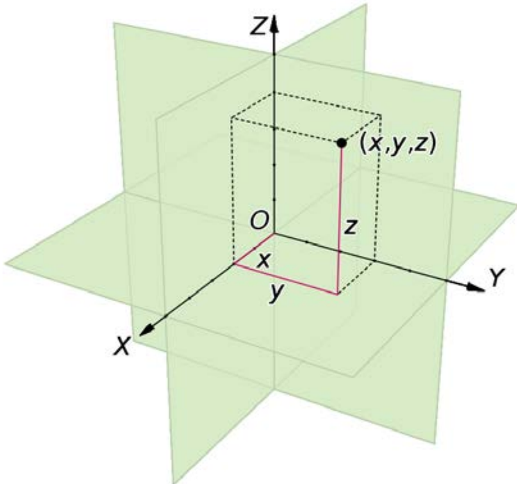

Figure 1.3.1. The coordinate hyperplanes of $\mathbb{R}^3$. Credit: By Jorge Stolfi – Own work, Public Domain, `https://commons.wikimedia.org/w/index.php?curid=6692547`

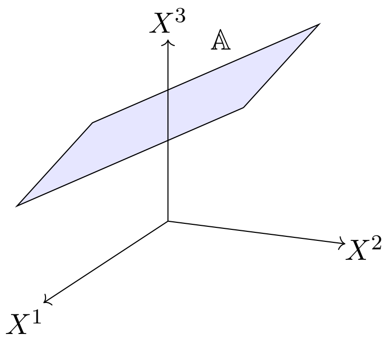

Figure 1.3.2. An affine hyperplane $\mathbb{A}$ in $\mathbb{R}^3$.

Example 1.2. *Any affine hyperplane is a submanifold of dimension n. In fact, it is* globally *Euclidean, not just* locally *Euclidean. We can see this as follows. Since the normal vector $\mathbf{N} = (N^1, \ldots, N^{n+1}) \neq \mathbf{0}$, there exists $1 \leq i \leq n+1$ such that $N^i \neq 0$. Then the map $\pi_i : \mathbb{A} \to \mathbb{R}^n_i$ defined by*

$$\pi_i(\mathbf{X}) = (X^1, \ldots, X^{i-1}, 0, X^{i+1}, \ldots, X^{n+1}) \tag{1.21}$$

is a bijection; see Figure 1.3.3 *and Exercise* 1.7.

To the affine hyperplane $\mathbb{A}$ we can associate the parallel linear subspace

$$\mathbb{L}_{\mathbb{A}} := \{\mathbf{X} \in \mathbb{R}^{n+1} \colon \mathbf{X} \cdot \mathbf{N} = 0\}. \tag{1.22}$$

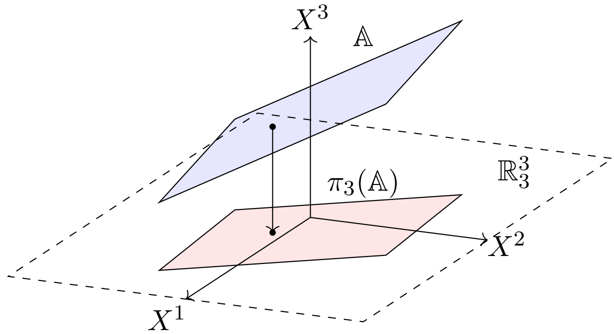

Figure 1.3.3. An affine hyperplane $\mathbb{A}$ in $\mathbb{R}^3$ and its projection under π_3 onto the third coordinate plane $\mathbb{R}^3_3$.

We can identify $\mathbb{A}$ and $\mathbb{L}_{\mathbb{A}}$ by the map $\phi_{\mathbb{A}} : \mathbb{L}_{\mathbb{A}} \to \mathbb{A}$ defined by (see Exercise 1.8)

$$\phi_{\mathbb{A}}(\mathbf{X}) := \mathbf{X} + a\,\frac{\mathbf{N}}{|\mathbf{N}|^2}. \tag{1.23}$$

In general, we say that two affine hyperplanes are **parallel** if their respective nonzero normal vectors are scalar multiples of one another. The relation of two affine hyperplanes being parallel is an equivalence relation.

To recap, affine hyperplanes $\mathbb{A}$ are examples of submanifolds of $\mathbb{R}^{n+1}$, as they can be made Euclidean by the identifications $\pi_i \circ \phi_{\mathbb{A}}$. Since the maps are defined on the *whole* spaces, we say that affine hyperplanes are *globally* Euclidean.

1.3.2. Spheres. Generalizing round circles to all dimensions, we have round spheres. A sphere is the set of points of constant distance to a fixed point in a Euclidean space. Let $\mathbf{c}_0 \in \mathbb{R}^{n+1}$ and $r > 0$. The n-dimensional **sphere** of radius r and center $\mathbf{c}_0$ is

$$S^n(r) := \{\mathbf{x} \in \mathbb{R}^{n+1} \colon \|\mathbf{x} - \mathbf{c}_0\| = r\}. \tag{1.24}$$

To keep track of the dimension in our terminology, we call $S^n(r)$ an n**-sphere** of radius r. See Figure 1.3.4.

Example 1.3. *Any n-sphere is an n-dimensional submanifold of $\mathbb{R}^{n+1}$.*

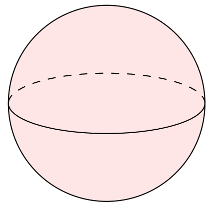

Figure 1.3.4. An n-sphere. This is the boundary sphere and does not include the interior of the $(n+1)$-dimensional ball that it bounds.

1.3.2.1. *Spheres are locally Euclidean.* Let

$$S^n := S^n(1) = \{\mathbf{x} \in \mathbb{R}^{n+1} \colon \|\mathbf{x}\| = 1\}. \tag{1.25}$$

That is, S^n is the unit n-sphere centered at the origin.

We now explicitly explore in what way spheres are locally Euclidean. Define $2(n+1)$ hemispheres of S^n by

$$S^n_{i,\pm} := \{\mathbf{x} \in S^n \colon \pm x^i > 0\}. \tag{1.26}$$

For example,

$$S^n_{n+1,-} = \{\mathbf{x} \in S^n \colon x^{n+1} < 0\}. \tag{1.27}$$

Let $S^n_{i,+}$ and $S^n_{i,-}$ be called the northern and southern i-th hemispheres, respectively. Taking $n = 2$, we have that a model for the southern hemisphere of the Earth is $S^2_{3,-}$.

Let $\mathbf{x} \in S^n$ be any point. Since $\|\mathbf{x}\| = 1 > 0$, there exists i such that $x^i \neq 0$. If $x^i > 0$, then $\mathbf{x} \in S^n_{i,+}$. Otherwise, if $x^i < 0$, then $\mathbf{x} \in S^n_{i,-}$. This shows that

$$S^n = \bigcup_{i=1}^{n+1} S^n_{i,\pm}. \tag{1.28}$$

That is, the collection $\left\{S^n_{i,\pm}\right\}_{i=1}^{n+1}$ of hemispherical subsets of the unit sphere S^n is a **covering** of S^n.

Consider the open unit n-dimensional ball in the i-th coordinate hyperplane $\mathbb{R}^n_i = \{\mathbf{x} \in \mathbb{R}^{n+1} \colon x^i = 0\}$ defined by

$$B_i := \{\mathbf{x} \in \mathbb{R}^n_i \colon \|\mathbf{x}\| < 1\}. \tag{1.29}$$

Each hemisphere $S^n_{i,\pm}$ is similar to (an open subset of) Euclidean space in the sense that we have maps, which we will call 'coordinate charts',

$$\phi_{i,\pm} : S^n_{i,\pm} \to B_i \tag{1.30}$$

defined by the coordinate projections

$$\phi_{i,\pm}(\mathbf{x}) = (x^1, \ldots, x^{i-1}, 0, x^{i+1}, \ldots, x^{n+1}). \tag{1.31}$$

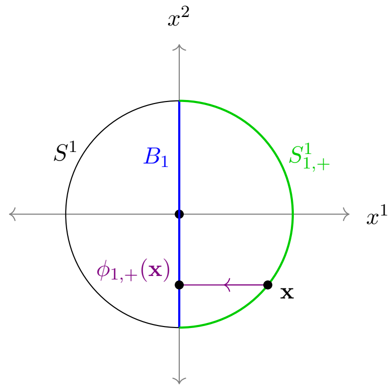

Figure 1.3.5. Visualizing the map $\phi_{1,+} : S^1_{1,+} \to B_1$ in the case $n = 1$.

See Figure 1.3.5. These coordinate charts are bijections and differentiable with injective derivatives; the precise properties of these maps will be explained in detail in the next chapter when we discuss 'embeddings'. In any case, in this sense, S^n is locally Euclidean. So spheres are submanifolds. See Exercise 1.9.

1.3.2.2. *Stereographic projection.* An alternative way to see that spheres are locally Euclidean is via stereographic projection. Let S^n be the unit n-sphere in $\mathbb{R}^{n+1}$. Let $\mathbf{N} := \mathbf{e}_{n+1} := (0, \ldots, 0, 1)$ be the **north pole**. Let $\mathbb{L}$ be the hyperplane defined by

$$\mathbb{L} := \{\mathbf{y} \in \mathbb{R}^{n+1} : y^{n+1} := \mathbf{y} \cdot \mathbf{N} = -1\}. \tag{1.32}$$

Then $\mathbb{L}$ is tangent to S^n at the south pole $\mathbf{S} := -\mathbf{N} = (0, \ldots, 0, -1)$. In Figure 1.3.7, which visualizes stereographic projection, $\mathbb{L}$ is the blue ocean the globe is sitting on.

Stereographic projection is the map G from $S^n \setminus \{\mathbf{N}\}$ to $\mathbb{L}$ which is defined to take a point $\mathbf{x}$ to the intersection of the hyperplane $\mathbb{L}$ and the unique line passing through $\mathbf{N}$ and $\mathbf{x}$. See Figure 1.3.6.

The stereographic projection map is equal to the map $G : S^n \setminus \{\mathbf{N}\} \to \mathbb{L}$ defined by

$$G(\mathbf{x}) = \frac{2\mathbf{x} - (1 + \mathbf{x} \cdot \mathbf{N})\,\mathbf{N}}{1 - \mathbf{x} \cdot \mathbf{N}} \tag{1.33}$$

for $\mathbf{x} \in S^n \setminus \{\mathbf{N}\}$. See Exercise 1.10; Exercise 1.11 provides a formula for the inverse of G.

In Figure 1.3.7, we have $n = 2$ and

$$G(x, y, z) = \left(\frac{2x}{1 - z}, \frac{2y}{1 - z}, -1 \right). \tag{1.34}$$

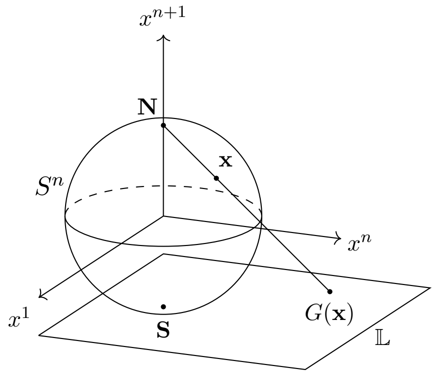

Figure 1.3.6. Picturing the stereographic projection map from the sphere minus the north pole $G : S^n \setminus \{\mathbf{N}\} \to \mathbb{L}$.

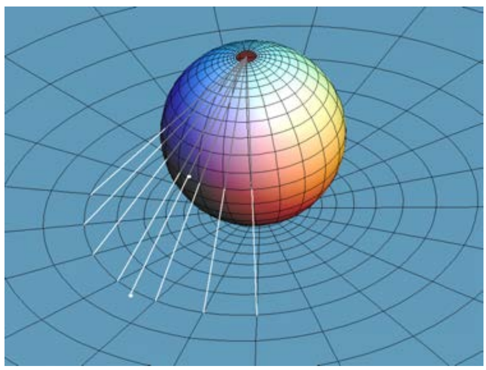

Figure 1.3.7. Stereographic projection. Credit: 3DXM Virtual Mathematics Museum by The 3D-XplorMath Consortium. © Copyright 3DXM Consortium. `https://3d-xplormath.org/`

Finally, we mention that spherical geometry has a long history, having been studied for thousands of years. It has been developed for centuries for various practical applications such as navigation and astronomy.

1.3.3. Graphs. Graphs are 'easy' examples of submanifolds. Moreover, they fully reflect the local nonlinearity of submanifolds and thus present an important class to study.

1.3.3.1. *Graphs of functions.* Let $\mathcal{U}$ be a nonempty open subset of $\mathbb{R}^n$.[4] Let $f : \mathcal{U} \to \mathbb{R}$. The **graph** of f is defined as

$$G^n := \{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \mathcal{U}\}, \tag{1.35}$$

[4]See §2.1.1 below for the definition of *open subset* of $\mathbb{R}^n$. Since we are thinking intuitively for now, its precise definition is not necessary for this discussion.

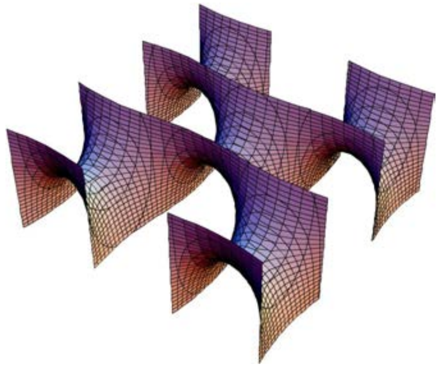

Figure 1.3.8. Scherk's surface. Credit: By Erminia Naccarato (Miriane) – Own work, CC BY-SA 3.0, `https://commons.wikimedia.org/w/index.php?curid=2240083`

which is a submanifold of $\mathbb{R}^{n+1}$. A graph is globally Euclidean in the sense that we have the bijective map $\phi : G^n \to \mathcal{U}$ defined by

$$\phi(\mathbf{x}, f(\mathbf{x})) := \mathbf{x}. \tag{1.36}$$

Indeed, its inverse $\phi^{-1} : \mathcal{U} \to G^n$ is given by

$$\phi^{-1}(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x})). \tag{1.37}$$

Observe that each subset $S^n_{i,\pm}$ of S^n is the graph of a function.

Example 1.4 (Scherk [**Sch35**]). *Define* $f : \mathcal{D} \to \mathbb{R}$ *by*

$$f(x^1, x^2) := \ln \frac{\cos(x^2)}{\cos(x^1)}, \tag{1.38}$$

where $\mathcal{D}$ *is the largest subset of* $\mathbb{R}^2$ *for which the expression on the right-hand side is defined. The graph of* f *is called Scherk's first surface; see Figure* 1.3.8*. Note that the function* f *is smooth on its domain* $\mathcal{D}$*, which is the set of* $(x^1, x^2) \in \mathbb{R}^2$ *for which* $\cos(x^1)\cos(x^2) > 0$*; see Exercise* 1.12*.*

1.3.3.2. *Affine graphs.* In short, affine graphs are a natural generalization of graphs over Euclidean spaces, and we now proceed to describe them. Let $\mathbb{A}$ be an affine hyperplane. Let $\mathbf{N}$ be a unit vector normal to $\mathbb{A}$. Then

$$\mathbb{A} = \{\mathbf{x} \in \mathbb{R}^{n+1} \colon \mathbf{x} \cdot \mathbf{N} = c\} \tag{1.39}$$

for some $c \in \mathbb{R}$. The orthogonal projection map $\pi_{\mathbb{A}} : \mathbb{R}^{n+1} \to \mathbb{A}$ is defined by

$$\mathbf{y} - \pi_{\mathbb{A}}(\mathbf{y}) := a\mathbf{N} \tag{1.40}$$

for some $a \in \mathbb{R}$. Since $\pi_{\mathbb{A}}(\mathbf{y}) \in \mathbb{A}$ implies that $\pi_{\mathbb{A}}(\mathbf{y}) \cdot \mathbf{N} = c$, we obtain

$$\mathbf{y} \cdot \mathbf{N} - c = a. \tag{1.41}$$

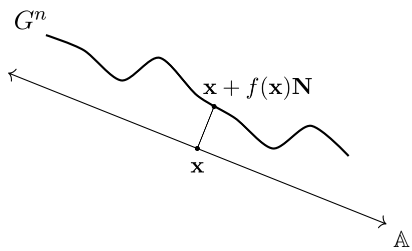

Figure 1.3.9. The affine graph G^n of a function $f : \mathbb{A} \to \mathbb{R}$.

Thus,

$$\pi_{\mathbb{A}}(\mathbf{y}) = \mathbf{y} - (\mathbf{y} \cdot \mathbf{N} - c)\mathbf{N}. \tag{1.42}$$

Let $f : \mathbb{A} \to \mathbb{R}$ be a function. The **affine graph** of f is

$$G^n := \{\mathbf{x} + f(\mathbf{x})\mathbf{N} \colon \mathbf{x} \in \mathbb{A}\}. \tag{1.43}$$

See Figure 1.3.9. The restriction map

$$\pi_{\mathbb{A}}|_{G^n} : G^n \to \mathbb{A} \tag{1.44}$$

is a bijection; see Exercise 1.13.

Given $\mathbf{V} \in \mathbb{R}^n$, denote

$$\mathbb{R}\mathbf{V} := \{t\mathbf{V} \colon t \in \mathbb{R}\}. \tag{1.45}$$

Note that $\mathbb{R}^n = \mathbb{L}_{\mathbb{A}} \oplus \mathbb{R}\mathbf{N}$, where $\mathbb{L}_{\mathbb{A}}$ is defined by (1.22). That is, for each $\mathbf{X} \in \mathbb{R}^n$ there exist unique vectors $\mathbf{Y} \in \mathbb{L}_{\mathbb{A}}$ and $\mathbf{Z} \in \mathbb{R}\mathbf{N}$ such that

$$\mathbf{X} = \mathbf{Y} + \mathbf{Z}. \tag{1.46}$$

Indeed, these unique vectors are given by

$$\mathbf{Y} = \mathbf{X} - (\mathbf{X} \cdot \mathbf{N})\mathbf{N} \quad \text{and} \quad \mathbf{Z} = (\mathbf{X} \cdot vN)\mathbf{N}. \tag{1.47}$$

See Exercise 1.14 for the case where the linear hyperplane $\mathbb{L}_{\mathbb{A}}$ is replaced by an affine hyperplane $\mathbb{A}$.

In the definition of an affine graph, we may replace $\mathbb{A}$ by a subset of $\mathbb{A}$ which is relatively open in $\mathbb{A}$.

Example 1.5. *Consider the case where the affine hyperplane $\mathbb{A}$ is the i-th coordinate hyperplane $\mathbb{R}^n_i$ in $\mathbb{R}^{n+1}$. Let $f : \mathbb{R}^n_i \to \mathbb{R}$. Choosing the unit normal to $\mathbb{R}^n_i$ as $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$, where the 1 in the i-th slot, we have that the affine graph of f is*

$$G^n = \{\mathbf{x} + f(\mathbf{x})\mathbf{e}_i \colon \mathbf{x} \in \mathbb{R}^n_i\}. \tag{1.48}$$

Equivalently,

$$G^n = \{(x^1, \dots, x^{i-1}, f(\mathbf{x}), x^{i+1}, \dots, x^{n+1}) \colon \\ x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^{n+1} \in \mathbb{R}\}. \tag{1.49}$$

For example, consider the unit 2-sphere S^2 in $\mathbb{R}^3$ and take $i = 3$. The equation for S^2 is $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$. Consider the northern third-hemisphere

$$S^2_{3,+} = \{\mathbf{x} \in S^2 \colon x^3 > 0\}. \tag{1.50}$$

We can express the hemisphere $S^2_{3,+}$ as the graph of a function as follows. Define $f : \mathbb{R}^2 \cong \mathbb{R}^3_3 \to \mathbb{R}$ by

$$f(x^1, x^2) := \sqrt{1 - (x^1)^2 - (x^2)^2}. \tag{1.51}$$

Then $S^2_{3,+}$ is the graph of f. For the general case of $S^n_{i,\pm}$, where $1 \le i \le n+1$, see Exercise 1.15.

1.3.3.3. *Radial graphs.* Compact hypersurfaces cannot be graphs over Euclidean spaces, but as will be evident, they can be radial graphs. In short, radial graphs are graphs over the unit sphere, which we now describe beginning with a familiar example.

Consider an ellipse E^1 in the plane, which is the set of all points $(x, y) \in \mathbb{R}^2$ satisfying the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{1.52}$$

where a, b are positive constants. This is a 1-dimensional submanifold of $\mathbb{R}^2$. One can represent the ellipse E^1 in polar coordinates. Namely,

$$E^1 = \{(r(\theta), \theta) \colon \theta \in [0, 2\pi]\}, \tag{1.53}$$

where the formula for the radial function is

$$r(\theta) := \frac{ab}{\sqrt{b^2 \cos^2(\theta) + a^2 \sin^2(\theta)}}. \tag{1.54}$$

See Figure 1.3.10.

Note that, instead of θ being an element of the closed interval $[0, 2\pi]$, we can think of θ as being an element of the **quotient set** $\mathbb{R}/2\pi\mathbb{Z}$. That is, consider the equivalence relation $\sim$ on $\mathbb{R}$ defined by $x \sim y$ if and only if $x - y$ is an integer multiple of 2π. Defining $2\pi\mathbb{Z} := \{2\pi k \colon k \in \mathbb{Z}\}$, we have $x \sim y$ if and only if $x - y \in 2\pi\mathbb{Z}$. For this reason we denote by $\mathbb{R}/2\pi\mathbb{Z}$ the quotient set $\mathbb{R}/\sim$ of equivalence classes $[\theta]$, where $\theta \in \mathbb{R}$.

Note that we can identify $\mathbb{R}/2\pi\mathbb{Z}$ with the unit circle S^1 by the map

$$[\theta] \mapsto (\cos(\theta), \sin(\theta)). \tag{1.55}$$

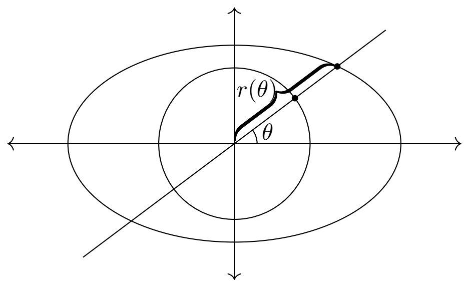

Figure 1.3.10. An ellipse E^1 in polar coordinates is given by $r(\theta) = \frac{ab}{\sqrt{b^2 \cos^2(\theta) + a^2 \sin^2(\theta)}}$.

That is, the element $[\theta] \in \mathbb{R}/2\pi\mathbb{Z}$ is identified with the point in S^1 with polar coordinates $(1, \theta)$, that is, radius 1 and angle θ; see Exercise 1.16. The considerations above form a special case of the notion of *quotient space* in topology (see Definition 2.5 below).

We can parametrize the ellipse by a map $F : S^1 \to E^1$ defined by

$$F(1, \theta) = (r(\theta), \theta) \tag{1.56}$$

for $[\theta] \in \mathbb{R}/2\pi\mathbb{Z}$ and where $r(\theta)$ is defined by (1.54). See Figure 1.3.10.

More generally, we can define a graph over the unit sphere S^n, where $n \geq 1$. Let $f : S^n \to \mathbb{R}_+$. The **radial graph** of f is

$$R^n_f := \{f(\mathbf{x})\mathbf{x} \colon \mathbf{x} \in S^n\}. \tag{1.57}$$

We also say that R^n_f is **starshaped**.

Define the **radial projection map** $\pi : R^n_f \to S^n$ by

$$\pi(f(\mathbf{x})\mathbf{x}) := \mathbf{x}; \tag{1.58}$$

see Figure 1.3.11. This map is well-defined; see Exercise 1.17.

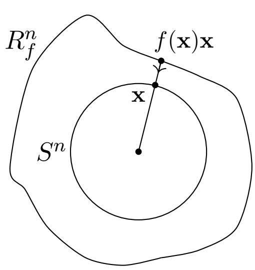

Figure 1.3.11. A radial graph R^n_f projecting onto S^n.

Via the bijection π, we can think of R^n_f as a sphere S^n. This is applicable even if f is not a continuous function. This is thinking of R^n_f intrinsically via the map π. However, if we think of R^n_f extrinsically as a subset of $\mathbb{R}^{n+1}$, then the function f affects the topology and geometry of R^n_f.

1.3.4. Surfaces of revolution. Surfaces of revolution are surfaces that possess rotational symmetry around an axis.

1.3.4.1. *Tori.* Two-dimensional submanifolds of $\mathbb{R}^3$ are easy to visualize since we see them often in everyday life.

A **torus** T^2 is a doughnut shaped surface in $\mathbb{R}^3$. See Figure 1.3.12. In general, 2-dimensional submanifolds are called **surfaces**.

To understand tori beyond the intuitive level, we consider the class of surfaces in the next subsection.

1.3.4.2. *General surfaces of revolution.* Let C^1 be a **curve** in $\mathbb{R}^2$. We can imagine $\mathbb{R}^2$ as a vertical hyperplane in $\mathbb{R}^3$. Then we can rotate C^1 about the x^3-axis to obtain a **surface of revolution**.

Example 1.6 (Torus). *If we are given a circle C^1_{cir} of radius r centered at a point $(c, 0)$, where $0 < r < c$, then the surface of revolution T^2 obtained by rotating C^1_{cir} about the x^3-axis is called a torus.*

Example 1.7 (Catenoid). *Let*

$$C^1_{\text{cat}} := \{(r, x^3) \colon r = \cosh(x^3)\}. \tag{1.59}$$

*The surface of revolution obtained by rotating C^1_{cat} about the x^3-axis is called a **catenoid**; see Figures* 1.3.13 *and* 1.3.14. *Consequently, the catenoid is*

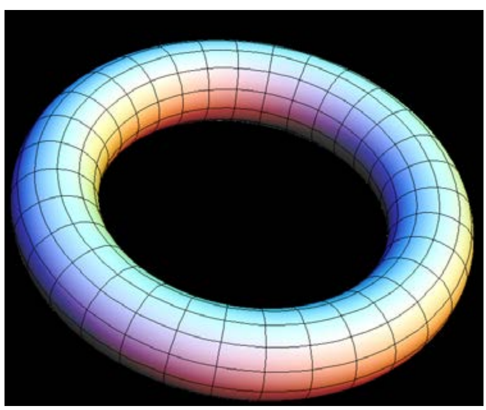

Figure 1.3.12. A torus T^2. Credit: 3DXM Virtual Mathematics Museum by The 3D-XplorMath Consortium. © Copyright 3DXM Consortium. `https://3d-xplormath.org/`

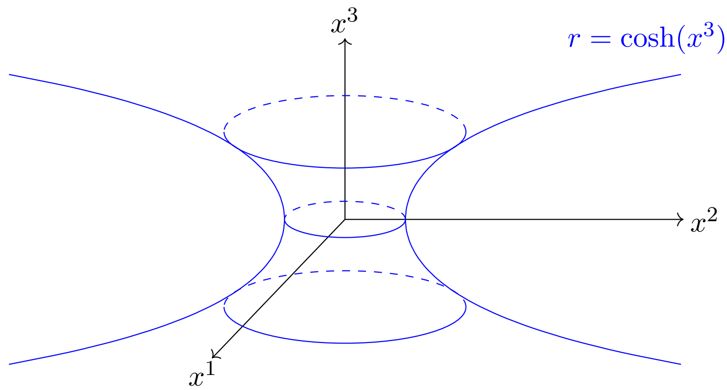

Figure 1.3.13. The catenoid surface is obtained by rotating the curve $r = \cosh(x^3)$ about the x^3-axis.

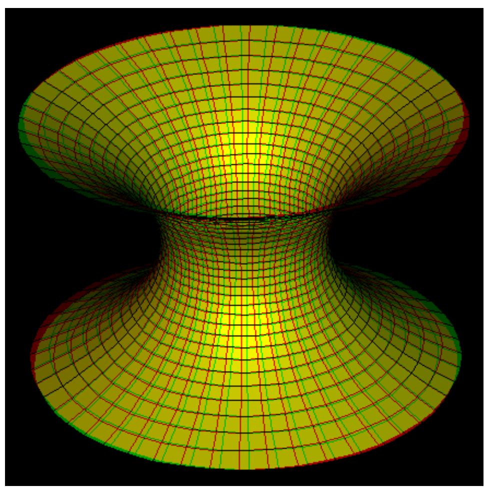

Figure 1.3.14. A catenoid. Credit: 3DXM Virtual Mathematics Museum by The 3D-XplorMath Consortium. © Copyright 3DXM Consortium. `https://3d-xplormath.org/`

given by the equation

$$(x^1)^2 + (x^2)^2 = \cosh^2(x^3). \tag{1.60}$$

For the following, we denote the Euclidean coordinates by x, y, z instead of x^1, x^2, x^3. In general, given a positive function $r = f(z)$ defined for $a \le z \le b$, we can rotate its graph around the z-axis to obtain a surface of revolution M^2. Here, we are treating r as the radial polar coordinate, so that the equation for the surface of revolution M^2 is

$$x^2 + y^2 = f(z)^2. \tag{1.61}$$

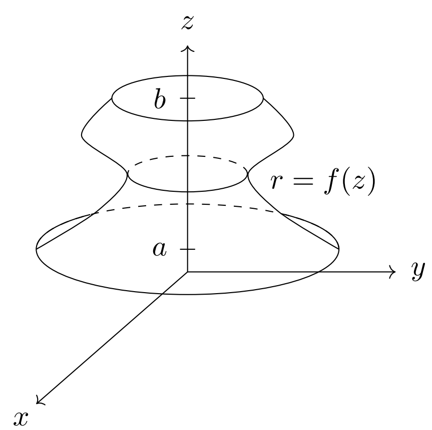

Figure 1.3.15. A surface of revolution.

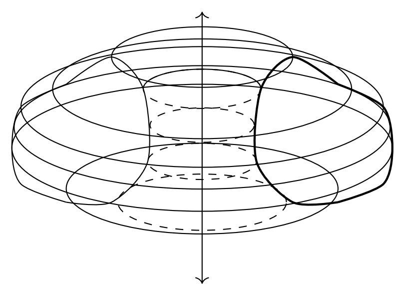

Figure 1.3.16. The surface of revolution obtained by revolving about the z-axis a closed curve (thickened in the diagram) in the rz-plane.

See Figure 1.3.15. We can parametrize the surface of revolution M^2 by the map $F : [a, b] \times [0, 2\pi] \to \mathbb{R}^3$ defined by

$$F(z, \theta) := \big(f(z)\cos(\theta), f(z)\sin(\theta), z\big) \tag{1.62}$$

for $z \in [a, b]$ and $\theta \in [0, 2\pi]$. This is simply because $r = f(z)$ and since $x = r\cos(\theta)$ and $y = r\sin(\theta)$ in polar coordinates.

Now suppose, more generally, that we want to revolve a curve in the plane to obtain a surface. For example, revolve a circle to obtain a torus. How do we parametrize the surface? Let $\gamma : (a, b) \to \mathbb{R}^2$ be an 'embedded' path, where we write $\gamma(t) = (r(t), z(t))$. Assume that the 'radius' $r(t)$ is positive. We may revolve the curve $\gamma\big((a, b)\big)$ to obtain a surface of revolution M^2. See Figure 1.3.16. A parametrization of M^2 minus the curve $\{(r(t), 0, z(t)) \colon t \in (a, b)\}$ is given by the map $F : (a, b) \times (0, 2\pi) \to \mathbb{R}^3$ defined by

$$F(t, \theta) := \big(r(t)\cos(\theta), r(t)\sin(\theta), z(t)\big). \tag{1.63}$$

In general, surfaces of revolution provide a wonderful class of surfaces on which to study geometry.

Level sets are discussed in §2.2 below. We will discuss parametrized submanifolds in §2.3 below. These provide additional important examples of submanifolds.

1.4. Notes and commentary

Euclidean space and its subsets and functions form the setting of real analysis; for a classic book, see Rudin [**Rud76**]. The are many books on the differential geometry of curves and surfaces; for example, do Carmo [**dC16**] is a classic.

1.5. Exercises

Exercise 1.1. *Write down the definition of an inner product (vector) space. Check with some standard source such as Hoffman and Kunze* [**HK71**] *that you have written down the correct definition.*

Exercise 1.2. *Using the bilinearity of the function A in* (1.7)*, prove* (1.11).

Exercise 1.3. *Write down the definition of a metric space $(\mathfrak{M}, d)$. Check with some standard source such as Rudin* [**Rud76**] *or Munkres* [**Mun00**] *that you have written down the correct definition.*

Exercise 1.4. *Prove the following version of the triangle inequality: For all vectors $\mathbf{V}, \mathbf{W} \in \mathbb{R}^n$,*

$$\|\mathbf{V} + \mathbf{W}\| \leq \|\mathbf{V}\| + \|\mathbf{W}\|. \tag{1.64}$$

Prove that (1.15) *follows from this. By definition, this implies that $(\mathbb{R}^n, d_{\mathbb{E}})$ is a metric space.*

Exercise 1.5. *Prove the Cauchy–Schwarz inequality* (1.16).
Hint: Show that for all $\lambda \in \mathbb{R}$,

$$\|\mathbf{V} - \lambda \mathbf{W}\|^2 = \|\mathbf{V}\|^2 + \lambda^2 \|\mathbf{W}\|^2 - 2\lambda\, \mathbf{V} \cdot \mathbf{W}. \tag{1.65}$$

Then take $\lambda = \frac{\mathbf{V} \cdot \mathbf{W}}{\|\mathbf{W}\|^2}$.

Exercise 1.6. *Let $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^n$. Verify that the vectors $\mathbf{Y}$ and $\mathbf{X} - \frac{\mathbf{X} \cdot \mathbf{Y}}{\|\mathbf{Y}\|^2} \mathbf{Y}$ are perpendicular. Then, using the right triangle in Figure* 1.2.1*, prove* (1.17).

Exercise 1.7. *Prove that the map $\pi_i : \mathbb{A} \to \mathbb{R}^n_i$ defined by* (1.21) *is a bijection, where $\mathbb{A}$ is the affine hyperplane defined by* (1.20) *and $N^i \neq 0$. Give the formula for its inverse $\pi_i^{-1} : \mathbb{R}^n_i \to \mathbb{A}$.*

Exercise 1.8. *Show that $\phi_{\mathbb{A}}$ defined by* (1.23) *is a bijection.*

Exercise 1.9. *Show that each map $\phi_{i,\pm} : S^n_{i,\pm} \to B_i$, as defined by* (1.30)*, is a bijection. Give the formula for its inverse $\phi^{-1}_{i,\pm} : B_i \to S^n_{i,\pm}$.*

Exercise 1.10. *Prove that the stereographic projection map is given by* (1.33)*.*

Exercise 1.11. *Let G be the stereographic projection map.*

(i) *Prove that $G(\mathbf{x}) \cdot \mathbf{N} = -1$ for all $\mathbf{x} \in S^n \setminus \{\mathbf{N}\}$. This proves that G is well-defined.*

(ii) *Prove that the map $F : \mathbb{L} \to S^n \setminus \{\mathbf{N}\}$ defined by*

$$F(\mathbf{y}) = \frac{4\mathbf{y} + (\|\mathbf{y}\|^2 - 1)\,\mathbf{N}}{\|\mathbf{y}\|^2 + 3} \tag{1.66}$$

is the inverse of the map G. Thus, the map F parametrizes the sphere minus a point.

Exercise 1.12. *Explain why the domain of the function f defining Scherk's surface in Example* 1.4 *is like the dark squares of an infinite chessboard.*

Exercise 1.13. *Show that the map $\pi_{\mathbb{A}}|_{G^n}$ defined by* (1.44) *is a bijection. Give the formula for its inverse $\pi_{\mathbb{A}}|^{-1}_{G^n} : \mathbb{A} \to G^n$.*

Exercise 1.14. *Let $\mathbb{A}$ be the affine hyperplane given by* (1.39)*. Let $\mathbf{X} \in \mathbb{R}^n$. Find the formulas for the vectors $\mathbf{Y}$ and $\mathbf{Z}$ such that $\mathbf{Y} \in \mathbb{A}$ is the orthogonal projection of $\mathbf{X}$ onto $\mathbb{A}$ and $\mathbf{X} = \mathbf{Y} + \mathbf{Z}$.*

Exercise 1.15. *Let $\phi_{i,\pm} : S^n_{i,\pm} \to B_i$ be as in Exercise* 1.9*. What is the formula for the function, denoted by $f_{i,\pm} : B_i \to \mathbb{R}$, such that the map $\phi^{-1}_{i,\pm}$ parametrizes its graph?*

Exercise 1.16. *Show that the map from $\mathbb{R}/2\pi\mathbb{Z}$ to S^1 defined by* (1.55) *is well-defined and a bijection.*

Exercise 1.17. *Show that the map π given by* (1.58) *is a well-defined bijection. Give the formula for its inverse $\pi^{-1} : S^n \to R^n_f$.*

Chapter 5

Global Theorems in the Theory of Submanifolds

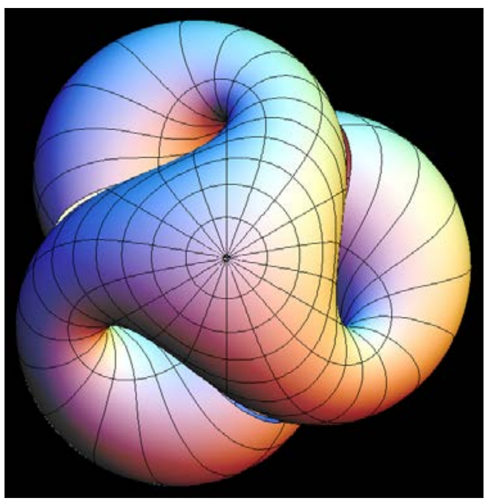

Figure 5.0.1. Boy's Surface (Bryant and Kusner), which is an immersion of the real projective plane in $\mathbb{R}^3$. Credit: 3DXM Virtual Mathematics Museum by The 3D-XplorMath Consortium. © Copyright 3DXM Consortium. `https://3d-xplormath.org/`

In this chapter, we discuss a small selection of global results on hypersurfaces in Euclidean space. In §5.1 we consider totally umbilical hypersurfaces and we classify them as round spheres or hyperplanes. In §5.2 we explore closed hypersurfaces and show that there exist points about which they are locally convex; we also discuss the strict convexity in the case of nonvanishing Gauss curvature. In §5.3 we discuss some global results on convex hypersurfaces. In §5.4 we discuss the Hessian, Laplacian, and divergence operators. Integral identities for hypersurfaces are proved in §5.5. In §5.6 we discuss the rigidity of closed convex surfaces in $\mathbb{R}^3$. §5.7 is a short appendix on calculating the Hessian and Laplacian of the position map in local coordinates.

5.1. Totally umbilical hypersurfaces

Spheres and hyperplanes are the most symmetric Euclidean hypersurfaces. In this section we show that we can characterize these hypersurfaces by a simple property that their second fundamental forms satisfy.

5.1.1. The totally umbilical hypersurface equation. Suppose that $n \geq 2$ and that an embedded hypersurface M^n in $\mathbb{R}^{n+1}$ has the property that its second fundamental form is equal to a function times its first fundamental form; that is,

$$\mathrm{II} = f\,\mathrm{I} \tag{5.1}$$

for some function f on M^n. In this case we say that M^n is **totally umbilical**.

By tracing equation (5.1), we have that the mean curvature of M^n satisfies $H = nf$, so that equation (5.1) becomes

$$\mathrm{II} = \frac{1}{n} H\,\mathrm{I}\,. \tag{5.2}$$

Note that this equation is true for any curve in the plane and this is why we restrict our attention to the case where $n \geq 2$.

The **divergence** of a 2-tensor α on M^n is defined by

$$\operatorname{div}(\alpha)_k := I^{ij} \nabla_i \alpha_{jk}, \tag{5.3}$$

where ∇ denotes the Levi-Civita connection of (M^n, I). Recall that (I^{ij}) is the inverse matrix of (I_{ij}) and that the components of the covariant derivative of the tensor α are denoted in local coordinates by $\nabla_i \alpha_{jk} := (\nabla_{\partial_i} \alpha)(\partial_j, \partial_k)$. More generally, the divergence of a k-tensor is a trace of the covariant derivative of the tensor, which produces a $(k-1)$-tensor. In particular, the divergence of the second fundamental form is the 1-form given by

$$\operatorname{div}(\mathrm{II})_k := I^{ij} \nabla_i \mathrm{II}_{jk}\,. \tag{5.4}$$

(See §5.4.2 below for further discussion of the divergence operator.)

5.1.2. Classification of complete totally umbilical hypersurfaces. The main result of this section is the following.

Proposition 5.1 (Complete totally umbilical hypersurfaces are spheres or hyperplanes)**.** *Any complete (as a metric space), connected, totally umbilical embedded hypersurface M^n in $\mathbb{R}^{n+1}$ is either a round sphere (if $H \neq 0$) or a hyperplane (if $H = 0$).*

Proof. We may take the divergence of equation (5.2) and use the Codazzi equation (4.175) to obtain

$$I^{ij} \frac{1}{n} \nabla_i H \, \mathrm{I}_{jk} = I^{ij} \nabla_i \, \mathrm{II}_{jk} = I^{ij} \nabla_k \, \mathrm{II}_{ij} \, ;$$

that is,

$$\frac{1}{n} \nabla_k H = \nabla_k H. \tag{5.5}$$

Since $n \neq 1$, this implies that $\nabla H = 0$ on M^n and since we assumed that M^n is connected, this implies that the mean curvature H is constant on M^n. Thus, II is equal to a constant times I on all of M^n.

Denote $\kappa := \frac{1}{n} H$, so that our connected hypersurface M^n in $\mathbb{R}^{n+1}$ has the property that $\mathrm{II} = \kappa \, \mathrm{I}$ on M^n, where κ is a constant. Let N be the unit normal vector field to M^n used to define the second fundamental form. Define the map $F : M^n \to \mathbb{R}^{n+1}$ by

$$F(\mathbf{x}) := \kappa \mathbf{x} - N_{\mathbf{x}}. \tag{5.6}$$

We may think of F as a Euclidean vector field on M^n; that is,

$$F = \kappa \, \mathrm{id} - N, \tag{5.7}$$

where $\mathrm{id}(\mathbf{x}) = \mathbf{x}$ is the identity vector field. For any tangent vectors $\mathbf{V}$ and $\mathbf{W}$ on M^n, using that κ is a constant we compute that

$$\begin{aligned} \langle D_{\mathbf{V}} F, \mathbf{W} \rangle &= \langle D_{\mathbf{V}} (\kappa \, \mathrm{id} - N), \mathbf{W} \rangle \\ &= \langle \kappa \mathbf{V} - D_{\mathbf{V}} N, \mathbf{W} \rangle \\ &= \kappa \, \mathrm{I}(\mathbf{V}, \mathbf{W}) - \mathrm{II}(\mathbf{V}, \mathbf{W}) \\ &= 0, \end{aligned}$$

where D denotes the Euclidean covariant derivative. We also have

$$\langle D_{\mathbf{V}} F, N \rangle = \langle \kappa \mathbf{V} - D_{\mathbf{V}} N, N \rangle = 0.$$

Thus, $D_{\mathbf{V}} F = \mathbf{0}$ for all tangent vectors $\mathbf{V}$. Since M^n is connected, we obtain that the vector field F on M^n is equal to a constant vector, which we call $\mathbf{X}_0$.

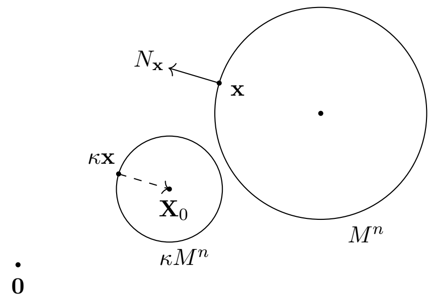

Figure 5.1.1. Visualizing the equation $\kappa\mathbf{x} - N_{\mathbf{x}} = \mathbf{X}_0$. This is an example of rescaling by curvature, which in this case is κ.

(1) Suppose that $\kappa \neq 0$. We then conclude that for all $\mathbf{x} \in M^n$, we have that $\kappa\mathbf{x} - N_{\mathbf{x}} = F(\mathbf{x}) = \mathbf{X}_0$ (see Figure 5.1.1); that is,

$$\mathbf{x} = \frac{1}{\kappa}\big(\mathbf{X}_0 + N_{\mathbf{x}}\big).$$

Since each $N_{\mathbf{x}}$ is a unit vector, we obtain from this that M^n is a relatively open subset of the sphere of radius $\frac{1}{|\kappa|}$ centered at the point $\frac{1}{\kappa}\mathbf{X}_0$. Since M^n is assumed to be complete, we conclude that M^n must be equal to the entire sphere.

(2) Suppose that $\kappa = 0$; that is, M^n is totally geodesic. Since M^n is connected, in this case we have that N is a constant vector field on M^n, which we call $\mathbf{N}_0$. For any tangent vector $\mathbf{V}$ of M^n we have that

$$\mathbf{V}(\mathrm{id}\cdot N) = \mathbf{V}\cdot N = \mathbf{V}\cdot\mathbf{N}_0 = 0.$$

Since M^n is assumed to be connected, this implies that the function $\mathrm{id}\cdot N$ is equal to a constant on M^n; see Exercise 5.10. That is, there exists $C \in \mathbb{R}$ such that $\mathbf{x}\cdot\mathbf{N}_0 = C$ for all $\mathbf{x} \in M^n$. Thus, M^n is a relatively open subset of a hyperplane. Since M^n is complete, we have that M^n is equal to the entire hyperplane. This completes the proof of the proposition. □

5.2. Closed hypersurfaces

Hypersurfaces can be categorized by whether they are compact or noncompact. In this section we demonstrate some nice properties of closed (i.e., compact without boundary) hypersurfaces.

5.2.1. Existence of a locally convex neighborhood. Let M^n be a connected closed hypersurface embedded in $\mathbb{R}^{n+1}$. By the Jordan–Brouwer Separation Theorem (see [**GP10**, Chapter 2, §5]), we have that $\mathbb{R}^{n+1} \setminus M^n$ consists of two components, exactly one of which has compact closure; we

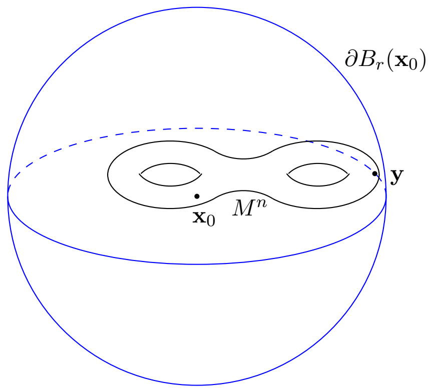

Figure 5.2.1. A closed embedded hypersurface M^n enclosed in the smallest ball $\overline{B_r(\mathbf{x}_0)}$. The point $\mathbf{y}$ is a point on M^n farthest from $\mathbf{x}_0$.

call this component the **interior region**. Let N be the unit **outward normal** to M^n; that is, N points in the direction opposite of the interior region. Note that the unit normal field N is a smooth vector field. In this case, we say that M^n is **oriented** and that N defines an orientation on M^n (see §8.1 below for a more general definition).

Proposition 5.2 (Existence of a locally convex neighborhood in any closed hypersurface)**.** *If M^n is a closed hypersurface embedded in $\mathbb{R}^{n+1}$, then there exists a point on M^n for which the second fundamental form* II *is positive-definite in a neighborhood of this point.*

Proof. Fix a point $\mathbf{x}_0 \in \mathbb{R}^{n+1}$; e.g., we can take $\mathbf{x}_0$ to be the origin. Since the distance function to $\mathbf{x}_0$ is continuous on M^n and since M^n is compact, there exists a point $\mathbf{y} \in M^n$ that maximizes the distance to $\mathbf{x}_0$. We have

$$M^n \subset \overline{B_r(\mathbf{x}_0)},$$

where $r := \|\mathbf{y} - \mathbf{x}_0\| > 0$ and $\mathbf{y} \in \partial B_r(\mathbf{x}_0)$. See Figure 5.2.1.

Observe that the unit outward normal to both M^n and $\partial B_r(\mathbf{x}_0)$ at $\mathbf{y}$ is given by the single vector $N_{\mathbf{y}} = \frac{\mathbf{y}-\mathbf{x}_0}{\|\mathbf{y}-\mathbf{x}_0\|}$. Let $\kappa_i(\mathbf{y})$ be any principal curvature of M^n at $\mathbf{y}$ and let e_i be an associated principal vector, so that

$$\mathrm{L}(e_i) = \kappa_i(\mathbf{y}) e_i,$$

where $\mathrm{L} : T_{\mathbf{y}}M \to T_{\mathbf{y}}M$ is the Weingarten map.

Let P be the 2-plane containing the points $\mathbf{x}_0$ and $\mathbf{y}$ and spanned by the vectors $N_{\mathbf{y}}$ and e_i. Consider the curve C in M^n defined by $M^n \cap P$. Since $\mathbf{y}$ is a farthest point from $\mathbf{x}_0$, we have that C is contained in the 2-disk $\bar{D}_r(\mathbf{x}_0) = \overline{B_r(\mathbf{x}_0)} \cap P$. Near $\mathbf{y}$, parametrize C by the graphical path $\beta : I \to P$ with $\beta'(0) = e_i$ and $\beta(t) = te_i + f(t)N_{\mathbf{y}}$, where $f : I \to \mathbb{R}$,

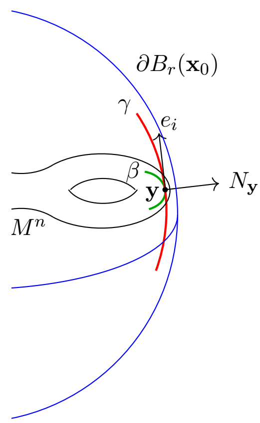

Figure 5.2.2. The path β in M^n and the comparison path γ in $\partial B_r(\mathbf{x}_0)$.

$f(0) = r$, and $f'(0) = 0$. We have

$$\kappa_i(\mathbf{y}) = -\left.\langle D_{e_i}\beta'(t), N_{\mathbf{y}}\rangle\right|_{t=0} = -f''(0). \tag{5.8}$$

On the other hand, in comparison, near $\mathbf{y}$ the round circle $\partial B_r(\mathbf{x}_0) \cap P$ is parametrized by the graphical path $\gamma : I \to P$ defined by $\gamma(t) = te_i + g(t)N_{\mathbf{y}}$, where $g(t) = \sqrt{r^2 - t^2}$. See Figure 5.2.2.

Since $C \subset \bar{D}_r(\mathbf{x}_0)$, we have $f(t) \leq g(t)$ and $f(0) = g(0)$. Thus, by the second derivative test in the sense of calculus, we have

$$\kappa_i(\mathbf{y}) = -f''(0) \geq -g''(0) = r^{-1} > 0.$$

In conclusion, we have shown that $\mathrm{II} \geq r^{-1}\,\mathrm{I}$ at the point $\mathbf{y}$. In particular, $H(\mathbf{y}) \geq nr^{-1}$ and $K(\mathbf{y}) \geq r^{-n}$. Moreover, since the second fundamental form is continuous, this implies that II is positive-definite in a neighborhood of $\mathbf{y}$. □

For the proof of the following, see Exercise 5.11.

Corollary 5.3. *There do not exist closed immersed minimal hypersurfaces in* $\mathbb{R}^{n+1}$.

5.2.2. The Gauss map. Let $F : M^n \to \mathbb{R}^{n+1}$ be an immersed hypersurface. Assume that there exists a smooth unit normal vector field $\mathbf{x} \mapsto N_{F(\mathbf{x})}$, so that $N_{F(\mathbf{x})}$ is perpendicular to $T_{F(\mathbf{x})}M$ for $\mathbf{x} \in M^n$. In particular, M^n is oriented.

Definition 5.4. The **Gauss map** of an oriented immersed hypersurface is the map of a point to its normal; that is, $G : M^n \to S^n \subset \mathbb{R}^{n+1}$ is defined by

$$G(\mathbf{x}) := N_{F(\mathbf{x})}. \tag{5.9}$$

Locally, we may consider M^n as an embedded hypersurface, so that $M^n \subset \mathbb{R}^{n+1}$ and the parametrization $F : M^n \to \mathbb{R}^{n+1}$ is the inclusion map. Then we have that $T_{\mathbf{x}}M$ and $T_{G(\mathbf{x})}S^n$ are both perpendicular to $N_{\mathbf{x}}$. The derivative of the Gauss map is the Weingarten map:

$$dG_{\mathbf{x}} = \mathrm{L} : T_{\mathbf{x}}M \to T_{G(\mathbf{x})}S^n. \tag{5.10}$$

Hence, the Gauss curvature may be expressed as

$$K(\mathbf{x}) = \det(\mathrm{L}) = \det(dG_{\mathbf{x}}). \tag{5.11}$$

Let $d\mu$ denote the volume form on M^n with respect to the first fundamental form $g := \mathrm{I}$ and the orientation on M^n induced by N. Let $d\bar{\mu}$ denote the volume form of the unit-sphere metric $\bar{g}$ on S^n. See §8.2.6 below for a discussion of the volume form of an oriented Riemannian manifold.

Lemma 5.5. *We have*

$$K\, d\mu = G^*(d\bar{\mu}), \tag{5.12}$$

where $G : M^n \to S^n$ is the Gauss map.

Proof. Let $\{e_i\}_{i=1}^n$ be an orthonormal frame at a point $\mathbf{x} \in M^n$ such that $e_i \in T_{F(\mathbf{x})}M$ and $\{N_{F(\mathbf{x})}, e_1, \ldots, e_n\}$ is positively oriented with respect to the standard orientation on $\mathbb{R}^{n+1}$.[1] (Denoting $e_0 = N_{F(\mathbf{x})}$, by definition we have that $\langle e_i, e_j\rangle = \delta_{ij}$ for $0 \le i, j \le n$.) Since we may identify $T_{F(\mathbf{x})}M$ with $T_{G(\mathbf{x})}S^n$ (as both of these hyperplanes are normal to $N_{F(\mathbf{x})}$), we may consider $\{e_i\}_{i=1}^n$ as an orthonormal tangent frame at $G(\mathbf{x}) \in S^n$. We also have that $\{e_i\}_{i=1}^n$ is positively oriented with respect to the standard orientation on S^n since $N_{F(\mathbf{x})}$ is equal to the unit outward normal to S^n at $G(\mathbf{x})$. Let $\{\omega^j\}_{j=1}^n$ be the dual orthonormal coframe, which is the basis for the cotangent space $T^*_{F(\mathbf{x})}M = T^*_{G(\mathbf{x})}S^n$ satisfying $\omega^j(e_i) = \delta_{ij}$ for $1 \le i, j \le n$. Then

$$d\mu_{\mathbf{x}} = \omega^1 \wedge \cdots \wedge \omega^n \qquad \text{on } M^n \tag{5.13}$$

and

$$d\bar{\mu}_{G(\mathbf{x})} = \omega^1 \wedge \cdots \wedge \omega^n \qquad \text{on } S^n. \tag{5.14}$$

---

[1] That is, the $(n+1)\times(n+1)$ matrix $\begin{pmatrix} N_{F(\mathbf{x})} & e_1 & \cdots & e_n \end{pmatrix}$ has positive determinant.

We compute using (5.14) that

$$\begin{aligned} G^*(d\bar{\mu})(e_1,\ldots,e_n) &= d\bar{\mu}\big(dG(e_1),\ldots dG(e_n)\big) \\ &= d\bar{\mu}\big(\,\mathrm{L}(e_1),\ldots,\mathrm{L}(e_n)\big) \\ &= \det(\mathrm{L})\, d\bar{\mu}(e_1,\ldots,e_n) \\ &= K, \end{aligned}$$

where $\mathrm{L} : T_{F(\mathbf{x})}M \to T_{F(\mathbf{x})}M$ is the Weingarten map. By (5.13), this implies that $K\, d\mu = G^*(d\bar{\mu})$. □

By integrating (5.12) and using the naturality of the integration of differential forms (see (8.22) below), we have:

Corollary 5.6. *If $G : M^n \to S^n$ is a diffeomorphism (which is necessarily orientation preserving), then*

$$\int_{M^n} K\, d\mu = n\omega_{n+1}, \tag{5.15}$$

where $n\omega_{n+1}$ is the volume of the unit n-sphere.

5.3. Convex closed hypersurfaces

Let M^n be an oriented immersed complete hypersurface in $\mathbb{R}^{n+1}$. We say that M^n is **locally uniformly convex** if its second fundamental form is positive-definite; i.e., $\mathrm{II} > 0$ on M^n.

Let M^n be an embedded closed hypersurface and let Ω be its interior region, which has compact closure $\overline{\Omega}$. We say that M^n is **weakly convex** if for every $x, y \in M^n$, the line segment $\overline{xy}$ is contained in $\overline{\Omega}$. We say that M^n is **strictly convex** if for every pair of distinct points $x, y \in M^n$, the interior of the line segment $\overline{xy}$ is contained in Ω. For example, the boundary of a cube, with its edges and vertices rounded off so that the surface is smooth, is weakly convex. On the other hand, the boundary of an ellipsoid is strictly convex.

5.3.1. Convexity of closed hypersurfaces with nonvanishing Gauss curvature.

In this subsection we discuss a result of Hadamard [**Had97**] and Stoker [**Sto36**], which gives a nice characterization of embedded closed convex hypersurfaces.

Lemma 5.7. *If $F : M^n \to \mathbb{R}^{n+1}$ is an orientable immersed closed hypersurface with Gauss curvature never vanishing and $n \geq 1$, then there exists a smooth choice of unit normal field such that $\mathrm{II} > 0$ on all of M^n; that is, M^n is locally uniformly convex.*

Proof. By Exercise 5.11, there exists an open subset $\mathcal{U}$ of M^n, with a choice of a smooth unit normal vector field N on $\mathcal{U}$, on which $\mathrm{II} > 0$. Since M^n is orientable, we can extend N to all of M^n as a smooth unit normal vector field. Since $K \neq 0$ everywhere, by continuity we have that

$$\mathrm{II} > 0 \quad \text{on all of } M^n. \tag{5.16}$$

Indeed, suppose for a contradiction that II were not positive-definite on all of M^n. Let $\kappa_1 : M^n \to \mathbb{R}$ be the smallest principal curvature function. Then

$$\kappa_1(\mathbf{x}) = \inf_{\mathbf{V} \in T_{F(\mathbf{x})}M,\, \|\mathbf{V}\|=1} \mathrm{II}(\mathbf{V}, \mathbf{V}),$$

so that κ_1 would be continuous and change sign. Therefore, by the intermediate value theorem, there would exist a point $\mathbf{x}_0 \in M^n$ such that $\kappa_1(\mathbf{x}_0) = 0$, which would imply that $K(\mathbf{x}_0) = 0$, which is a contradiction. □

Lemma 5.8. *If M^n is an orientable closed hypersurface immersed in $\mathbb{R}^{n+1}$ with $\mathrm{II} > 0$ and $n \geq 2$, then there exists a smooth choice of a unit normal field such that the Gauss map $G : M^n \to S^n$ is a diffeomorphism.*

Proof. Since $\det(dG) = \det(\mathrm{L}) = K > 0$, we have that the Gauss map G is an immersion. This implies that G is locally a diffeomorphism. Hence, for each $\mathbf{y} \in S^n$, its preimage $G^{-1}(\mathbf{y})$ is a finite subset of M^n. So we may write $G^{-1}(\mathbf{y}) =: \{\mathbf{x}_1, \dots, \mathbf{x}_k\}$, where $k := |G^{-1}(\mathbf{y})|$. Let $r := \frac{1}{2}\min_{i \neq j} \|\mathbf{x}_i - \mathbf{x}_j\|$ and define $\mathcal{U}_i := M^n \cap B_r(\mathbf{x}_i)$. Then $\{\mathcal{U}_i\}_{i=1}^k$ is a collection of disjoint open subsets of M^n and $\mathbf{x}_i \in \mathcal{U}_i$ for each i. We claim that there exists $\epsilon > 0$ such that

$$G^{-1}(B_\epsilon(\mathbf{y}) \cap S^n) \subset \bigcup_{i=1}^k \mathcal{U}_i =: \mathcal{U}. \tag{5.17}$$

If not, then for each positive integer j there would exist $\mathbf{w}_j \in M^n \setminus \mathcal{U}$ such that $G(\mathbf{w}_j) \in B_{1/j}(\mathbf{y})$. Since M^n is compact, by passing to a subsequence, we would have that $\mathbf{w}_j$ converges to some point $\mathbf{w}_\infty \in M^n$. Since $M^n \setminus \mathcal{U}$ is a closed set, we would have that $\mathbf{w}_\infty \in M^n \setminus \mathcal{U}$. By the continuity of G, we would also have that $G(\mathbf{w}_\infty) = \lim_{j\to\infty} G(\mathbf{w}_j) = \mathbf{y}$, which is a contradiction since $\mathbf{w}_\infty \notin \mathcal{U} \supset G^{-1}(\mathbf{y})$. Thus we have proved the existence of $\epsilon > 0$ satisfying (5.17).

Now, since G is a local diffeomorphism, there exists $\epsilon' \in (0, \epsilon)$ such that $G^{-1}(\mathcal{V}')$, where $\mathcal{V}' := B_{\epsilon'}(\mathbf{y}) \cap S^n$, is equal to the disjoint union of sets $\mathcal{U}_i'$, $1 \leq i \leq k$, such that G maps each $\mathcal{U}_i'$ diffeomorphically to $\mathcal{V}'$ and $\mathbf{x}_i \in \mathcal{U}_i'$ for each i. This proves that G is a smooth covering map (see Definition 6.33 below for the definition). Furthermore, because the image of G is S^n, which is simply connected since $n \geq 2$, we conclude that the Gauss map $G : M^n \to S^n$ is a diffeomorphism (see [**Mas77**] for covering space theory). □

Theorem 5.9 (Hadamard and Stoker). *If M^n is an orientable closed hypersurface immersed in $\mathbb{R}^{n+1}$ with Gauss curvature never vanishing and $n \geq 2$, then M^n is strictly convex, embedded, and* $\mathrm{II} > 0$ *on all of M^n.*

Proof. We first show that for each $\mathbf{x} \in M^n$, the immersed hypersurface M^n lies on one side of the tangent hyperplane $T_{F(\mathbf{x})}M^n$. To this end and for a contradiction, suppose that there exists $\mathbf{x}_0 \in M^n$ such that M^n lies on both sides of $T_{F(\mathbf{x}_0)}M^n$. Let $\mathbf{N}_0$ be a choice of unit normal vector at $\mathbf{x}_0$. Define the 'height' function $f : M^n \to \mathbb{R}$ by

$$f(\mathbf{x}) = \langle F(\mathbf{x}), \mathbf{N}_0 \rangle.$$

Observe that for all $\mathbf{V} \in T_{F(\mathbf{x}_0)}M$, we have

$$\mathbf{V}(f) = \langle \mathbf{V}, \mathbf{N}_0 \rangle = 0.$$

Thus the function f has at least three critical points on M^n: one at a maximum point $\mathbf{x}_+$ of f, one at a minimum point $\mathbf{x}_-$ of f, and one at $\mathbf{x}_0$, where the value of f is strictly between its minimum and its maximum since M^n lies on both sides of $T_{F(\mathbf{x}_0)}M^n$. We then have that the unit normal vectors $G(\mathbf{x}_+)$, $G(\mathbf{x}_-)$, and $G(\mathbf{x}_0)$ are each plus or minus one another and hence two of these vectors must be equal, which is a contradiction to the injectivity of the Gauss map from Lemma 5.8. Let us take stock of what we have proven so far. We have shown that for each $\mathbf{x} \in M^n$, the tangent hyperplane $T_{F(\mathbf{x})}M$ divides $\mathbb{R}^{n+1}$ into two open half-spaces $H^+_{\mathbf{x}}$ and $H^-_{\mathbf{x}}$ so that M^n is contained in the closed half-space $\bar{H}^-_{\mathbf{x}} = H^-_{\mathbf{x}} \cup T_x M$.

Since M^n lies on one side of each tangent plane, we have that M^n must be embedded. For simplicity, now assume that F is the inclusion map, so that $F(\mathbf{x}) = \mathbf{x}$. Let Ω denote the compact region that M^n bounds. From the above, one can show that for each pair of distinct points $\mathbf{x}$ and $\mathbf{y}$ in M^n, the line segment $\overline{\mathbf{xy}}$ is contained in Ω. Furthermore, by using that $\mathrm{II} > 0$, one can prove that the open line segment from $\mathbf{x}$ to $\mathbf{y}$ is contained in the interior of Ω. This completes the proof of the theorem (see Exercise 5.12 for a hint of how to fill in the details above). □

5.3.2. Convex hypersurfaces and the support function. Let M^n be an embedded closed hypersurface in $\mathbb{R}^{n+1}$ and let N be the unit outward normal vector field. Its **support function** $\sigma : M^n \to \mathbb{R}$ is defined by

$$\sigma(\mathbf{x}) = \langle \mathbf{x}, N_{\mathbf{x}} \rangle, \tag{5.18}$$

where $N_{\mathbf{x}}$ denotes the unit normal at $\mathbf{x}$. Since the tangent hyperplane $TP_{\mathbf{x}}M$ passes through $\mathbf{x}$ and is perpendicular to $N_{\mathbf{x}}$, this is a signed distance from the origin to the tangent hyperplane $TP_{\mathbf{x}}M$.

The normal component of the position vector is

$$\mathbf{x}^\perp = \langle \mathbf{x}, N_{\mathbf{x}} \rangle N_{\mathbf{x}} = \sigma(\mathbf{x}) N_{\mathbf{x}}, \tag{5.19}$$

whereas the tangential component of the position vector is

$$\mathbf{x}^\top = \mathbf{x} - \langle \mathbf{x}, N_\mathbf{x} \rangle N_\mathbf{x} = \mathbf{x} - \sigma(\mathbf{x}) N_\mathbf{x}. \tag{5.20}$$

5.3.3. The support function as a function on the unit n-sphere. Now let M^n be a strictly convex embedded closed hypersurface in $\mathbb{R}^{n+1}$. Since the Gauss map $G : M^n \to S^n$, defined by $G(\mathbf{x}) = N_\mathbf{x}$, is a diffeomorphism, we may parametrize M^n by its inverse $G^{-1} : S^n \to M^n$. So the support function may be considered as a function on S^n:

$$\sigma := \sigma \circ G^{-1} : S^n \to \mathbb{R}. \tag{5.21}$$

That is,

$$\sigma(\mathbf{n}) := \langle G^{-1}(\mathbf{n}), \mathbf{n} \rangle \quad \text{for } \mathbf{n} \in S^n. \tag{5.22}$$

We can recover a convex closed embedded hypersurface M^n from its support function $\sigma : S^n \to \mathbb{R}$ because the compact region Ω that M^n bounds is equal to the intersection of the support half-spaces; that is,

$$\Omega = \bigcap_{\mathbf{y} \in S^n} \bar{H}_\mathbf{y}^- \tag{5.23}$$

(see Exercise 5.13), where

$$\bar{H}_\mathbf{y}^- := \{\mathbf{x} \in \mathbb{R}^{n+1} \colon \langle \mathbf{x}, N_\mathbf{y} \rangle \leq \sigma(\mathbf{y})\}. \tag{5.24}$$

We can show the set equality (5.23) by using the fact that M^n lies on one side of each of its tangent hyperplanes. Consequently, we can recover the hypersurface by $M^n := \partial\Omega$.

Let $\overline{g}$ denote the standard metric on S^n and let $\overline{\nabla}$ denote its associated covariant derivative. Recall that the Weingarten map $\mathrm{L} : T_\mathbf{x}M \to T_\mathbf{x}M$ is a self-adjoint vector space endomorphism for each $\mathbf{x} \in M^n$. We also may naturally identity $T_\mathbf{x}M$ with $T_{G(\mathbf{x})}S^n$. Thus, we may consider the Weingarten map as an endomorphism $\mathrm{L} : T_\mathbf{n}S^n \to T_\mathbf{n}S^n$ for each $\mathbf{n} \in S^n$. Since M^n is strictly convex, L is invertible.

Proposition 5.10. *For any $\mathbf{n} \in S^n$ and $\mathbf{V}, \mathbf{W} \in T_\mathbf{n}S^n$, we have that*

$$A(\mathbf{V}, \mathbf{W}) := (\overline{\nabla}^2 \sigma + \sigma \overline{g})(\mathbf{V}, \mathbf{W}) = \overline{g}(\mathrm{L}^{-1}(\mathbf{V}), \mathbf{W}). \tag{5.25}$$

Proof. The first derivative of σ is given by

$$\mathbf{W}(\sigma) = \mathbf{W}(\langle G^{-1}(\mathbf{n}), \mathbf{n} \rangle) = \langle G^{-1}(\mathbf{n}), \mathbf{W} \rangle,$$

where the second equality is because $\langle D_\mathbf{W}(G^{-1}(\mathbf{n})), \mathbf{n} \rangle = 0$, which in turn is due to $D_\mathbf{W}(G^{-1}(\mathbf{n})) \in T_{G^{-1}(\mathbf{n})}M \cong T_\mathbf{n}S^n$. We compute that the second directional derivative of σ is given by

$$\begin{aligned}\mathbf{V}(\mathbf{W}(\sigma)) &= \mathbf{V}(\langle G^{-1}(\mathbf{n}), \mathbf{W}\rangle) \\ &= \langle d(G^{-1})_{\mathbf{n}}(\mathbf{V}), \mathbf{W}\rangle + \langle G^{-1}(\mathbf{n}), D_{\mathbf{V}}\mathbf{W}\rangle.\end{aligned}$$

We also have that

$$(\overline{\nabla}_{\mathbf{V}}\mathbf{W})(\sigma) = \langle G^{-1}(\mathbf{n}), \overline{\nabla}_{\mathbf{V}}\mathbf{W}\rangle.$$

Thus,

$$\begin{aligned}\overline{\nabla}^2_{\mathbf{V},\mathbf{W}}\sigma &= \mathbf{V}(\mathbf{W}(\sigma)) - (\overline{\nabla}_{\mathbf{V}}\mathbf{W})(\sigma) \\ &= \langle (dG)^{-1}_{\mathbf{n}}(\mathbf{V}), \mathbf{W}\rangle + \langle G^{-1}(\mathbf{n}), D_{\mathbf{V}}\mathbf{W} - \overline{\nabla}_{\mathbf{V}}\mathbf{W}\rangle \\ &= \langle \mathrm{L}^{-1}(\mathbf{V}), \mathbf{W}\rangle + \langle G^{-1}(\mathbf{n}), -\overline{\mathrm{II}}(\mathbf{V}, \mathbf{W})\,\mathbf{n}\rangle\end{aligned}$$

since $dG = \mathrm{L}$ is the Weingarten map and where $\overline{\mathrm{II}}$ is the second fundamental form of $(S^n, \overline{g})$. Since S^n is totally umbilical with principal curvature equal to 1, we have $\overline{\mathrm{II}} = \overline{\mathrm{I}} = \overline{g}$. Therefore

$$(\overline{\nabla}^2_{\mathbf{V},\mathbf{W}}\sigma)(\mathbf{n}) = \overline{g}(\mathrm{L}^{-1}(\mathbf{V}), \mathbf{W}) - \overline{g}(\mathbf{V}, \mathbf{W})\,\sigma(\mathbf{n}). \tag{5.26}$$

The proposition follows. □

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of principal directions with associated principal curvatures $\{\kappa_1, \ldots, \kappa_n\}$. By Proposition 5.10, we have that

$$(\overline{\nabla}^2\sigma + \sigma\overline{g})(e_i, e_i) = A(e_i, e_i) = \overline{g}(\mathrm{L}^{-1}(e_i), e_i) = \kappa_i^{-1}.$$

Therefore, the Gauss curvature may be expressed via

$$\begin{aligned}K^{-1} &= \kappa_1^{-1}\cdots\kappa_n^{-1} \\ &= \prod_{i=1}^{n}\big(\overline{\nabla}^2\sigma + \sigma\overline{g}\big)(e_i, e_i) \\ &= \frac{\det\big(\overline{\nabla}^2\sigma + \sigma\overline{g}\big)}{\det(\overline{g})}.\end{aligned} \tag{5.27}$$

For the expression in the last line, by $\frac{\det(\alpha)}{\det(\beta)}$ for symmetric 2-tensors α and β, we mean $\frac{\det(\alpha_{ij})}{\det(\beta_{ij})}$, where α_{ij} and β_{ij} are the components of α and β with respect to any basis, respectively. It is easy to see that this expression is independent of the choice of basis.

Lemma 5.11.

$$G^*\big(\overline{\nabla}^2\sigma + \sigma\overline{g}\big) = \mathrm{II}\,. \tag{5.28}$$

Proof. We compute for $X, Y \in TM$ that

$$\begin{aligned} G^*\big(\overline{\nabla}^2\sigma + \sigma\overline{g}\big)(X,Y) &= \big(\overline{\nabla}^2\sigma + \sigma\overline{g}\big)\big(dG(X), dG(Y)\big) \\ &= \overline{g}\big(\mathrm{L}^{-1}(\mathrm{L}(X)), \mathrm{L}(Y)\big) \\ &= g(X, \mathrm{L}(Y)) \\ &= \mathrm{II}(X,Y), \end{aligned}$$

where we used that $\overline{g} = g$ via the natural isomorphism between $T_{G(\mathbf{x})}S^n$ and $T_{\mathbf{x}}M$.[2] □

We also observe that

$$G^*(\overline{g})(X,Y) = g(\mathrm{L}(X), \mathrm{L}(Y)) =: \mathrm{II}^2(X,Y), \tag{5.29}$$

so that

$$G^*(\overline{g}) = \mathrm{II}^2\,. \tag{5.30}$$

We call II^2 the square of the second fundamental form. With respect to an orthonormal frame $\{e_i\}_{i=1}^n$,

$$\mathrm{II}^2(e_i, e_j) = \delta_{ij}(\mathrm{II}(e_i,e_i))^2.$$

With respect to local coordinates $\{x^i\}_{i=1}^n$,

$$(\mathrm{II}^2)_{ij} = \sum_{k,\ell=1}^n \mathrm{II}_{ik}\, g^{k\ell}\, \mathrm{II}_{\ell j}\,.$$

We remark that the pointwise norm of the 2-tensor II is defined by

$$|\,\mathrm{II}\,|^2 := \sum_{i,j=1}^n g^{ij}(\mathrm{II}^2)_{ij} = \sum_{i,j,k,\ell=1}^n g^{ij} g^{k\ell}\, \mathrm{II}_{ik}\, \mathrm{II}_{\ell j}\,. \tag{5.31}$$

Equivalently, with respect to an orthonormal frame $\{e_i\}_{i=1}^n$,

$$|\,\mathrm{II}\,|^2 = \sum_{i,j=1}^n \big(\mathrm{II}(e_i,e_j)\big)^2. \tag{5.32}$$

See §5.7 below for the generalization of the notion of pointwise norm to arbitrary covariant tensors.

5.4. The Hessian, Laplacian, and divergence theorem

In this section we introduce two important differential operators: the Hessian and the Laplacian. We consider them both on Euclidean space and for submanifolds.

[2] More formally, if $\iota : T_{G(\mathbf{x})}S^n \to T_{\mathbf{x}}M$ is the natural isomorphism, then $\overline{g} = \iota^* g$.

5.4.1. The Laplacian and divergence theorem on Euclidean space. Given a smooth function $f : \mathcal{U} \to \mathbb{R}$, where $\mathcal{U}$ is an open subset of $\mathbb{R}^n$, its (**Euclidean**) **Laplacian** is defined by

$$\Delta_{\mathbb{E}} f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial (u^i)^2}. \tag{5.33}$$

The Laplacian is the trace of the **Hessian**, which is the $n \times n$ matrix of second derivatives:

$$\operatorname{Hess}(f) := \left(\frac{\partial^2 f}{\partial u^i \partial u^j} \right). \tag{5.34}$$

We say that f is a **harmonic function** if f satisfies the **Laplace equation**:

$$\Delta_{\mathbb{E}} f = 0. \tag{5.35}$$

Rotationally symmetric examples of harmonic functions defined on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ are the **Newtonian potentials** (a.k.a. global **Green's functions**) defined as follows. For $n \geq 3$,

$$G(\mathbf{x}) := c_n \|\mathbf{x}\|^{2-n}, \tag{5.36}$$

where $c_n := \frac{1}{n(n-1)\omega_n}$, ω_n is the volume of the Euclidean unit n-ball, and for $n = 2$,

$$G(\mathbf{x}) := -\frac{1}{2\pi} \ln \|\mathbf{x}\|. \tag{5.37}$$

One calculates that

$$\Delta_{\mathbb{E}} G = 0 \quad \text{on } \mathbb{R}^n \setminus \{\mathbf{0}\}. \tag{5.38}$$

On all of $\mathbb{R}^n$, **in the sense of distributions**, the Laplacian of the Newtonian potential function is equal to the Dirac delta function. So the Newtonian potential function is called a **fundamental solution** to the Laplace equation.

Let $V : \mathcal{U} \to \mathbb{R}^{n+1}$ be a smooth vector field on an open subset $\mathcal{U}$ of $\mathbb{R}^{n+1}$. The (**Euclidean**) **divergence** of $V =: (V^1, \ldots, V^{n+1})$ is defined by

$$\operatorname{div}_{\mathbb{E}}(V)(\mathbf{x}) := \sum_{i=1}^{n+1} \frac{\partial V^i}{\partial u^i}(\mathbf{x}) \tag{5.39}$$

for $\mathbf{x} \in \mathcal{U}$. Observe that the divergence of the gradient of a function is equal to the Laplacian of the function. Namely, recall that if $f : \mathcal{U} \to \mathbb{R}$ is a function, then its Euclidean gradient is given by

$$\nabla_{\mathbb{E}} f = \left(\frac{\partial f}{\partial u^1}, \ldots, \frac{\partial f}{\partial u^{n+1}} \right). \tag{5.40}$$

Thus,

$$\Delta_{\mathbb{E}} f = \operatorname{div}_{\mathbb{E}}(\nabla_{\mathbb{E}} f). \tag{5.41}$$

The **divergence theorem** says the following.

Proposition 5.12 (Euclidean divergence theorem). *Let Ω be a compact domain in $\mathbb{R}^{n+1}$ with smooth boundary $\partial\Omega =: M^n$. If V is a C^1 vector field defined in a neighborhood of Ω, then*

$$\int_\Omega \operatorname{div}_{\mathbb{E}}(V) = \int_{M^n} V \cdot N, \tag{5.42}$$

where N is the unit outward normal vector field to M^n.

For the proof, see Corollary 8.9 below, which is stated in more generality.

By taking V to be equal to the gradient of a function f, we obtain that

$$\int_\Omega \Delta_{\mathbb{E}} f = \int_{M^n} N(f). \tag{5.43}$$

By taking V to be equal to $f\nabla_{\mathbb{E}} h$ for some functions f, h, we obtain that

$$\int_\Omega (f\Delta_{\mathbb{E}} h + \langle \nabla_{\mathbb{E}} f, \nabla_{\mathbb{E}} h\rangle) = \int_{M^n} f N(h). \tag{5.44}$$

A consequence of this is the formula

$$\int_\Omega (f\Delta_{\mathbb{E}} h - h\Delta_{\mathbb{E}} f) = \int_{M^n} (f N(h) - hN(f)). \tag{5.45}$$

5.4.2. The Hessian and Laplacian on a submanifold. Let M^n be a submanifold of $\mathbb{R}^N$. Recall from (4.123) that the Levi-Civita connection ∇ defines for a tangent vector $\mathbf{V}$ on a vector field W on M^n a 'directional derivative' of W along the direction $\mathbf{V}$ by

$$\nabla_{\mathbf{V}} W = (D_{\mathbf{V}} W)^T, \tag{5.46}$$

where $\cdot^T$ denotes the tangential component of a vector in $\mathbb{R}^N$.

Let $f : M^n \to \mathbb{R}$ be a smooth function. Recall that its derivative df is a 1-form on M^n; that is, for each $\mathbf{x} \in M^n$, $df_{\mathbf{x}} : T_{\mathbf{x}} M \to \mathbb{R}$ is a linear function. We can define the covariant derivative of the 1-form df by imposing the product rule:

$$\mathbf{V}(df(W)) =: (\nabla_{\mathbf{V}} df)(W) + df(\nabla_{\mathbf{V}} W); \tag{5.47}$$

that is,

$$(\nabla_{\mathbf{V}} df)(W) := \mathbf{V}(W(f)) - (\nabla_{\mathbf{V}} W)(f). \tag{5.48}$$

For this definition of $\nabla_{\mathbf{V}} df$ to make sense, we need to show that the right-hand side of (5.48) at a point $\mathbf{x} \in M^n$ depends only on the vector field

W at $\mathbf{x}$. To see this, we first note that the right-hand side is linear in W. Secondly, let h be a function on M^n. We compute that

(5.49)
$$\begin{aligned}\mathbf{V}((hW)(f)) - (\nabla_{\mathbf{V}}(hW))(f) &= \mathbf{V}(hW(f)) - \mathbf{V}(h)W(f) - h(\nabla_{\mathbf{V}}W)(f) \\ &= h\mathbf{V}(W(f)) - h(\nabla_{\mathbf{V}}W)(f).\end{aligned}$$

By Corollary 4.39 (the proof applies equally well to the present situation), we conclude that the right-hand side depends only on W at $\mathbf{x}$.

The (covariant) **Hessian** $\nabla^2 f$ of f on M^n is defined by

$$(\nabla^2 f)(\mathbf{V}, \mathbf{W}) := (\nabla_{\mathbf{V}} df)(\mathbf{W}). \tag{5.50}$$

The Hessian $\nabla^2 f$ is a symmetric covariant 2-tensor on M^n. Since the Levi-Civita connection ∇ is torsion-free, we have that $\nabla^2 f$ is symmetric. Namely, we compute that for any tangent vectors $\mathbf{V}, \mathbf{W}$ at a point $\mathbf{x}$ and extensions to vector fields V, W in a neighborhood of $\mathbf{x}$,

$$\begin{aligned}(\nabla^2 f)(\mathbf{V}, \mathbf{W}) - (\nabla^2 f)(\mathbf{W}, \mathbf{V}) &= \mathbf{V}(W(f)) - (\nabla_{\mathbf{V}}W)(f) \\ &\quad - \mathbf{W}(V(f)) + (\nabla_{\mathbf{W}}V)(f) \\ &= [V, W](f) - [V, W](f) \\ &= 0,\end{aligned} \tag{5.51}$$

where we used (4.25) for the penultimate equality.

The **Laplacian** is the trace of the Hessian; that is,

$$\Delta_g f(\mathbf{x}) := \sum_{i=1}^{n} (\nabla^2 f)_{\mathbf{x}}(e_i, e_i), \tag{5.52}$$

where $\{e_i\}$ is any orthonormal basis of tangent vectors at $\mathbf{x}$.

The **divergence** of a tangent vector field V on M^n is defined by

$$\operatorname{div}_g(V) := \sum_{i=1}^{n} \langle \nabla_{e_i} V, e_i \rangle. \tag{5.53}$$

The divergence of a 1-form α on M^n is defined by

$$\operatorname{div}_g(\alpha) := \sum_{i=1}^{n} (\nabla_{e_i} \alpha)(e_i). \tag{5.54}$$

The Laplacian is equal to the divergence of the gradient or derivative:

$$\Delta_g f = \operatorname{div}_g(\nabla f) = \operatorname{div}_g(df). \tag{5.55}$$

With respect to local coordinates $(x^1, \dots, x^n) = F^{-1}$ induced by a parametrization F, we have that

$$\nabla_i f := df(\partial_i F) = \frac{\partial f}{\partial x^i} = \frac{\partial (f \circ F)}{\partial u^i}. \tag{5.56}$$

The components of the Hessian are defined by

$$\nabla_i \nabla_j f := (\nabla^2 f)(\partial_i F, \partial_j F). \tag{5.57}$$

By definition, we may write the components of the Hessian, in terms of the second partial derivatives and the Christoffel symbols, as

$$\begin{aligned}\nabla_i \nabla_j f &= \partial_i F(\partial_j F(f)) - (\nabla_{\partial_i F} \partial_j F)(f) \\ &= \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial f}{\partial x^k}.\end{aligned} \tag{5.58}$$

With respect to local coordinates, the Laplacian is given by

$$\Delta_g f = g^{ij} \nabla_i \nabla_j f = \sum_{i,j=1}^{n} g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial f}{\partial x^k} \right). \tag{5.59}$$

For closed submanifolds, we have the following, which is a special case of Corollary 8.9 below.

Proposition 5.13 (Divergence theorem for closed submanifolds)**.** *Let M^n be a compact hypersurface without boundary in $\mathbb{R}^{n+1}$. Let V be a C^1 vector field on M^n. Then*

$$\int_{M^n} \operatorname{div}_g(V) = 0. \tag{5.60}$$

In particular, by taking $V = \nabla f$ to be a gradient vector field, we obtain

$$\int_{M^n} \Delta_g f = 0. \tag{5.61}$$

5.5. Hypersurface integrals

Using that the Laplacian of the position map of a hypersurface is equal to the mean curvature vector, we derive integral formulas relating the area of a hypersurface, the mean and scalar curvatures, and the support function.

5.5.1. The Hessian and Laplacian of the position map. Let M^n be an embedded submanifold in $\mathbb{R}^N$. Define the distance-squared function $f : M^n \to \mathbb{R}$ by

$$f : \mathbf{x} \mapsto \|\mathbf{x}\|^2.$$

Let V and W be tangent vector fields on M^n. The first derivative of f is given by

$$\frac{1}{2} V(f) = \frac{1}{2} V(\|\mathbf{x}\|^2) = \langle V, \mathbf{x} \rangle = \langle V, \mathbf{x}^\top \rangle, \tag{5.62}$$

where the last equality is because the vector field V is tangential to M^n. In other words, the gradient on M^n of f is given by

$$(\nabla f)_{\mathbf{x}} = 2\mathbf{x}^\top. \tag{5.63}$$

Now let M^n be an embedded hypersurface in $\mathbb{R}^{n+1}$. The second derivative of f is

$$\frac{1}{2}W(V(f)) = W(\langle V, \mathbf{x}\rangle) = \langle D_W V, \mathbf{x}\rangle + \langle V, W\rangle.$$

We also have

$$\frac{1}{2}(\nabla_W V)(f) = \langle \nabla_W V, \mathbf{x}\rangle.$$

Therefore, one-half times the Hessian of f is given by

$$\begin{aligned}\frac{1}{2}(\nabla^2 f)(W,V) &= \frac{1}{2}W(V(f)) - \frac{1}{2}(\nabla_W V)(f)\\ &= \langle D_W V - \nabla_W V, \mathbf{x}\rangle + \langle V, W\rangle\\ &= -\,\mathrm{II}(W,V)\langle N, \mathbf{x}\rangle + \langle V, W\rangle.\end{aligned} \tag{5.64}$$

That is,

$$\frac{1}{2}\nabla^2 f = -\sigma\,\mathrm{II} + \mathrm{I}\,. \tag{5.65}$$

Tracing this, we obtain

$$\frac{1}{2}\Delta f = -\sigma H + n, \tag{5.66}$$

where Δ is the Laplacian of (M^n, I).

We can also perform a similar calculation for the inclusion map $\mathbf{x} : M^n \to \mathbb{R}^{n+1}$. We compute that

$$V(\mathbf{x}) = V$$

and

$$W(V(\mathbf{x})) = D_W V.$$

Furthermore,

$$\nabla_W V(\mathbf{x}) = \nabla_W V.$$

The Hessian of the vector-valued function $\mathbf{x} =: (x^1, \ldots, x^{n+1})$ is defined componentwise:

$$\nabla^2\mathbf{x} := (\nabla^2 x^1, \ldots, \nabla^2 x^{n+1}). \tag{5.67}$$

Therefore,

$$\begin{aligned}\nabla^2_{W,V}\mathbf{x} := (\nabla^2\mathbf{x})(W,V) &= W(V(\mathbf{x})) - (\nabla_W V)(\mathbf{x})\\ &= D_W V - \nabla_W V\\ &= -\,\mathrm{II}(W,V)N.\end{aligned} \tag{5.68}$$

We leave it as an exercise that one can use this formula to give an alternate derivation of (5.65). Note also that by tracing, we have that

$$\Delta\mathbf{x} = -HN \tag{5.69}$$

is the mean curvature vector.

5.5.2. Integral formulas on closed hypersurfaces. Assume now that M^n is a closed embedded hypersurface. Integrating (5.119) while using the divergence theorem (Proposition 5.13) on M^n yields the **Minkowski formula**:

$$n\,\mathrm{Area}(M^n) = \int_{M^n} \sigma H \, d\mu_g, \tag{5.70}$$

where $d\mu_g$ denotes the (n-dimensional) volume form of (M^n, g).[3] For example, if $n = 1$, then we obtain the length formula:

$$\mathrm{L} = \int_{M^1} \sigma \kappa \, ds. \tag{5.71}$$

We have

$$\frac{1}{2}\Delta_{\mathbb{R}^{n+1}} \|\mathbf{x}\|^2 = n + 1. \tag{5.72}$$

Let Ω be the compact region bounded by M^n. By (5.72) and the divergence theorem on $\mathbb{R}^{n+1}$, we have that

$$\begin{aligned}(n+1)\,\mathrm{Vol}(\Omega) &= \int_\Omega \frac{1}{2}\Delta_{\mathbb{R}^{n+1}} \|\mathbf{x}\|^2 \, d\mu_{\mathbb{R}^{n+1}}(\mathbf{x}) \\ &= \int_{M^n} \frac{1}{2} N(\|\mathbf{x}\|^2) \, d\mu_g(\mathbf{x}) \\ &= \int_{M^n} \langle \mathbf{x}, N_{\mathbf{x}} \rangle \, d\mu_g(\mathbf{x}),\end{aligned}$$

where $d\mu_g = d\mu$ is the volume form of (M^n, I). That is,

$$\mathrm{Vol}(\Omega) = \frac{1}{n+1} \int_{M^n} \sigma(\mathbf{x}) \, d\mu_g(\mathbf{x}). \tag{5.73}$$

We now compute the Hessian and Laplacian of the support function σ. Differentiating σ yields

$$V(\sigma) = \langle \mathbf{x}, D_V N \rangle = \langle \mathbf{x}, \mathrm{L}(V) \rangle = \langle \mathbf{x}^\top, \mathrm{L}(V) \rangle = \mathrm{II}(V, \mathbf{x}^\top),$$

where we used $\langle V, N \rangle = 0$ for the first equality. Since the second derivative of the support function σ is

$$W(V(\sigma)) = (\nabla_W \mathrm{II})(V, \mathbf{x}^\top) + \mathrm{II}(\nabla_W V, \mathbf{x}^\top) + \mathrm{II}(V, \nabla_W(\mathbf{x}^\top))$$

by the product rule, we obtain that the covariant Hessian of σ on M^n is given by

$$\begin{aligned}\nabla^2_{W,V}\sigma &= W(V(\sigma)) - (\nabla_W V)(\sigma) \\ &= (\nabla_W \mathrm{II})(V, \mathbf{x}^\top) + \mathrm{II}\left(V, \nabla_W(\mathbf{x}^\top)\right).\end{aligned} \tag{5.74}$$

[3] We use Area to denote the n-dimensional volume of M^n to distinguish it from the $(n+1)$-dimensional volume of the region interior to M^n.

On the other hand, using (4.125) and the product rule, we compute that

$$\begin{aligned}\nabla_W(\mathbf{x}^\top) - \mathrm{II}(W,\mathbf{x}^\top)N &= D_W(\mathbf{x}^\top)\\ &= D_W(\mathbf{x} - \langle \mathbf{x}, N\rangle N)\\ &= W - \langle \mathbf{x}, \mathrm{L}(W)\rangle N - \langle \mathbf{x}, N\rangle\, \mathrm{L}(W),\end{aligned}$$

so that by taking the tangent components we arrive at the formula

$$\nabla_W(\mathbf{x}^\top) = W - \langle \mathbf{x}, N\rangle\, \mathrm{L}(W). \tag{5.75}$$

Therefore, (5.74) yields

$$\nabla^2_{W,V}\sigma = (\nabla_{\mathbf{x}^\top}\mathrm{II})(V,W) + \mathrm{II}(V,W) - \langle \mathbf{x}, N\rangle\, \mathrm{II}^2(V,W), \tag{5.76}$$

where we also used the Codazzi equation (4.175).

Tracing this equation (i.e., taking $W = V = e_i$, where $\{e_i\}_{i=1}^n$ is an orthonormal frame tangent to M^n, and summing over i) yields

$$\begin{aligned}(\Delta_g\sigma)(\mathbf{x}) &= \langle \nabla H(\mathbf{x}), \mathbf{x}^\top\rangle + H(\mathbf{x}) - \langle \mathbf{x}, N_{\mathbf{x}}\rangle |\,\mathrm{II}\,|^2(\mathbf{x})\\ &= \mathrm{div}(H\mathbf{x}^\top) - H(\mathbf{x})\,\mathrm{div}(\mathbf{x}^\top) + H(\mathbf{x}) - \langle \mathbf{x}, N_{\mathbf{x}}\rangle |\,\mathrm{II}\,|^2(\mathbf{x}),\end{aligned} \tag{5.77}$$

where $|\,\mathrm{II}\,|^2$ is defined by (5.31). On the other hand, by (5.75),

$$\mathrm{div}(X^\top)(\mathbf{x}) = \sum_{i=1}^n (\nabla_{e_i}(\mathbf{x}^\top)\cdot e_i) = n - \langle \mathbf{x}, N_{\mathbf{x}}\rangle H(\mathbf{x}). \tag{5.78}$$

Therefore, by integrating (5.77), applying the divergence theorem, and (5.78), we obtain

$$\begin{aligned}0 &= \int_{M^n} (\Delta_g\sigma)(\mathbf{x})\, d\mu(\mathbf{x})\\ &= \int_{M^n} \big((1-n)H(\mathbf{x}) + \langle \mathbf{x}, N_{\mathbf{x}}\rangle (H^2 - |\,\mathrm{II}\,|^2)(\mathbf{x})\big)\, d\mu(\mathbf{x});\end{aligned}$$

that is, we have the Minkowski formula

$$\int_{M^n} R(\mathbf{x})\sigma(\mathbf{x})\, d\mu(\mathbf{x}) = (n-1)\int_{M^n} H(\mathbf{x})\, d\mu(\mathbf{x}), \tag{5.79}$$

where $R = H^2 - |\,\mathrm{II}\,|^2$ is the scalar curvature of M^n. In particular, if $n = 2$, then we have

$$2\int_{M^2} K(\mathbf{x})\sigma(\mathbf{x})\, d\mu(\mathbf{x}) = \int_{M^2} H(\mathbf{x})\, d\mu(\mathbf{x}), \tag{5.80}$$

where K is the Gauss curvature of M^2.

5.5.3. Integral formulas for convex embedded hypersurfaces. Let $A_0 \in \mathbb{R}^{n+1}$ be a constant vector field on Ω. By the Euclidean divergence theorem, we have

$$\int_{M^n} \langle N, A_0 \rangle \, d\mu_g = \int_{\Omega} \operatorname{div}_{\mathbb{R}^{n+1}}(A_0) = 0. \tag{5.81}$$

Now assume that M^n is a strictly convex closed hypersurface. Then the Gauss map G is a diffeomorphism and, by the change of variables formula for the Gauss map transformation, we obtain

$$0 = \int_{M^n} \langle N, A_0 \rangle \, d\mu_g = \int_{S^n} \langle N, A_0 \rangle \, K^{-1} \, d\mu_{S^n}(N), \tag{5.82}$$

where K is the Gauss curvature at $X = G^{-1}(N)$. Since this is true for any $A_0 \in \mathbb{R}^{n+1}$, as a vector equation, this says that

$$\int_{S^n} \frac{1}{K(\mathbf{n})} \mathbf{n} \, d\mu_{S^n}(\mathbf{n}) = \mathbf{0}. \tag{5.83}$$

By Lemma 5.5, the Jacobian determinant of the inverse of the Gauss map is equal to K^{-1}. Therefore

$$\operatorname{Area}(M^n) = \int_{S^n} (G^{-1})^*(d\mu_g) = \int_{S^n} \frac{1}{K(\mathbf{n})} \, d\mu_{S^n}(\mathbf{n}). \tag{5.84}$$

Moreover, by (5.73), we have that

$$\operatorname{Vol}(\Omega) = \frac{1}{n+1} \int_{S^n} \frac{\langle G^{-1}(\mathbf{n}), \mathbf{n} \rangle}{K(\mathbf{n})} \, d\mu_{S^n}(\mathbf{n}). \tag{5.85}$$

5.6. Highlights of submanifold theory

This section discusses some well-known classical results on submanifolds of Euclidean spaces. It requires some knowledge of abstract smooth manifolds, Riemannian manifolds, and moving frames. As such, the readers who are not familiar with those topics may wish to skip this section until after they have learned those prerequisites, which are in Part II of this book.

5.6.1. Rigidity of convex surfaces. In this subsection we discuss the rigidity theorem for convex surfaces of Cohn-Vossen [**CV27**, **CV36**]. As with most expositions, such as Chern [**Che51**], Spivak [**Spi79e**], and Han and Hong [**HH06**], the proof of Cohn-Vossen's theorem that we give is due to Herglotz [**Her43**].

First, we prove a rigidity theorem for isometries of hypersurfaces (not necessarily convex) that preserve the second fundamental form.

Lemma 5.14. *Let M^n and M'^n be two connected hypersurfaces without boundary in $\mathbb{R}^{n+1}$. Let* I, II *and* I', II' *denote their first, second fundamental forms, respectively. If $f : M^n \to M'^n$ is a diffeomorphism such that*

$$f^* \mathrm{I}' = \mathrm{I} \quad \textit{and} \quad f^* \mathrm{II}' = \mathrm{II}, \tag{5.86}$$

then f extends to an ambient isometry from $\mathbb{R}^{n+1}$ to $\mathbb{R}^{n+1}$.

Proof. Let N and N' denote the unit normal vector fields to M^n and M'^n inducing the second fundamental forms II and II', respectively. In general, geometric quantities on M'^n and M^n will be denoted with and without primes, respectively. Let $\mathbf{x} \in M^n$. We have that $df := df_{\mathbf{x}} : T_{\mathbf{x}}M \to T_{f(\mathbf{x})}M'$ is a linear isometry.

By (5.86), we have

$$\big\langle \mathrm{L}'(df(V)), df(W) \big\rangle = \langle \mathrm{L}(V), W \rangle = \big\langle df(\mathrm{L}(V)), df(W) \big\rangle$$

for all $V, W \in T_{\mathbf{x}}M$, so that the Weingarten maps L and L' commute with the map df:

$$\mathrm{L}' \circ df = df \circ \mathrm{L} : T_{\mathbf{x}}M \to T_{f(\mathbf{x})}M'. \tag{5.87}$$

Define $\ell : M^n \to \mathrm{O}(n+1)$ such that

$$\mathrm{O}(n+1) \ni \ell_{\mathbf{x}} := \ell(\mathbf{x}) : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$$

is the unique linear isometry satisfying

$$\ell_{\mathbf{x}}|_{T_{\mathbf{x}}M} = df_{\mathbf{x}}, \quad \ell_{\mathbf{x}}(N_{\mathbf{x}}) = N'_{f(\mathbf{x})}. \tag{5.88}$$

We will show that $V(\ell) = D_V \ell = 0$ for $V \in TM$, so that the map $\mathbf{x} \mapsto \ell_{\mathbf{x}}$ is constant.

Let $Y : M^n \to \mathbb{R}^{n+1}$ be a smooth vector-valued function. Corresponding to the decomposition $M^n \times \mathbb{R}^{n+1} = TM \oplus NM$ given by (3.67), we may write

$$Y = W + aN,$$

where W is a tangential vector field on M^n and $a : M^n \to \mathbb{R}$ is a smooth function. We compute for $V \in T_{\mathbf{x}}M$ that

$$\begin{aligned} D_V Y &= D_V W + V(a)N + a(\mathbf{x}) D_V N \\ &= \nabla_V W - \langle \mathrm{L}(V), W \rangle N + V(a)N + a(\mathbf{x})\, \mathrm{L}(V). \end{aligned}$$

Using that f preserves the first and second fundamental forms, and hence preserves the Levi-Civita connection and Weingarten map, and using (5.87), we compute that

$$\begin{aligned}\ell(D_V Y) &= df(\nabla_V W) - \langle \mathrm{L}(V), W\rangle N' + V(a)N' + a\,df(\mathrm{L}(V))\\ &= \nabla'_{df(V)} df(W) - \langle \mathrm{L}'(df(V)), df(W)\rangle N'\\ &\quad + df(V)(a\circ f^{-1})N' + (a\circ f^{-1})\,\mathrm{L}'(df(V))\\ &= D_{df(V)}df(W) + D_{df(V)}((a\circ f^{-1})N')\\ &= D_{df(V)}(\ell(Y)).\end{aligned}$$

Therefore

$$(D_V\ell)(Y) = D_{df(V)}(\ell(Y)) - \ell(D_V Y) = 0.$$

We have proved that $\ell := \ell_{\mathbf{x}} \in \mathrm{O}(n+1)$ is independent of $\mathbf{x} \in M^n$.

Now choose any point $\mathbf{x}_0 \in M^n$. Consider the Euclidean isometry $G : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ given by translation from $\mathbf{x}_0$ to $f(\mathbf{x}_0)$, followed by the orthogonal transformation ℓ. We have for $\mathbf{x} \in \mathbb{R}^{n+1}$ that $dG_{\mathbf{x}} = \ell$. By this and (5.88), for $\mathbf{x} \in M^n$, we have that

$$dG_{\mathbf{x}}|_{T_{\mathbf{x}}M} = df_{\mathbf{x}} : T_{\mathbf{x}}M \to \mathbb{R}^{n+1}$$

for all $\mathbf{x} \in M^n$. Since $G(\mathbf{x}_0) = f(\mathbf{x}_0)$ and M^n is connected, we conclude from this that G restricted to M^n is equal to f. □

Recall from Example 4.44 that a half-cylinder M^2_{cyl} and a flat slab M^2_{slab} in $\mathbb{R}^3$ are isometric via a map $\Phi : M^2_{\mathrm{slab}} \to M^2_{\mathrm{cyl}}$. While these specific spaces are isometric in this way, it is notable that Φ does not extend to an isometry from $\mathbb{R}^3$ to itself.

This brings us to an interesting distinction: The main result of this section presents a rigidity result for closed strictly convex surfaces embedded in $\mathbb{R}^3$.

Theorem 5.15 (Cohn-Vossen: Rigidity of convex surfaces)**.** *Let M^2 and M'^2 be two closed surfaces in $\mathbb{R}^3$ with positive Gauss curvature. Let* I *and* I' *denote their first fundamental forms, respectively. If there exists an isometry from (M^2, I) to (M'^2, I'), then there exists an isometry from $\mathbb{R}^3$ to $\mathbb{R}^3$ which maps M^2 onto M'^2.*

Proof. Let $J : TM \to TM$ be the almost complex structure of M^2; see (10.103) below.[4] Let $\{e_1, e_2\}$ be a positively oriented local orthonormal tangential frame field defined on an open subset $\mathcal{U}$ of M^2. Then $J(e_1) = e_2$ and $J(e_2) = -e_1$. In particular, $J^2 = -\operatorname{id}_{TM}$. Define the 1-form α on M^2 by

$$\alpha(V) = \langle J(V), \mathbf{x}\rangle \quad \text{for } V \in T_{\mathbf{x}}M. \tag{5.89}$$

[4]For now, we can just think of J as counterclockwise rotation by 90 degrees.

Observe that on $\mathcal{U}$,

$$\alpha(V) = \langle \mathbf{x}, e_2\rangle\langle V, e_1\rangle - \langle \mathbf{x}, e_1\rangle\langle V, e_2\rangle.$$

Let $\{\omega^1, \omega^2\}$ be the local orthonormal coframe dual to $\{e_1, e_2\}$. We have

$$\alpha = \langle \mathbf{x}, e_2\rangle\omega^1 - \langle \mathbf{x}, e_1\rangle\omega^2. \tag{5.90}$$

Let D be the Euclidean covariant derivative. Let ω_i^j denote the connection 1-forms associated to the orthonormal frame field $\{\omega^1, \omega^2\}$; see (8.55) below. For $i = 1, 2$ and $V \in T_{\mathbf{x}}M$, we have

$$\begin{aligned} V\langle \mathbf{x}, e_i\rangle &= \langle D_V \mathbf{x}, e_i\rangle + \langle D_V e_i, \mathbf{x}\rangle \\ &= \langle V, e_i\rangle + \langle \nabla_V e_i, \mathbf{x}\rangle + \langle D_V e_i, N_{\mathbf{x}}\rangle\langle \mathbf{x}, N_{\mathbf{x}}\rangle \\ &= \langle V, e_i\rangle + \omega_i^j(V)\langle \mathbf{x}, e_j\rangle - \mathrm{II}(V, e_i)\sigma(\mathbf{x}), \end{aligned} \tag{5.91}$$

where we used $\sigma(\mathbf{x}) = \langle \mathbf{x}, N_{\mathbf{x}}\rangle$ is the support function; cf. (5.18). Thus

$$d\langle \mathbf{x}, e_i\rangle = \omega^i + \langle \mathbf{x}, e_j\rangle\omega_i^j - \sigma(\mathbf{x})\,\mathrm{L}(e_i)^*,$$

where L is the Weingarten map.

We compute that

$$\begin{aligned} d\alpha &= d\langle \mathbf{x}, e_2\rangle \wedge \omega^1 - d\langle \mathbf{x}, e_1\rangle \wedge \omega^2 + \langle \mathbf{x}, e_2\rangle\, d\omega^1 - \langle \mathbf{x}, e_1\rangle\, d\omega^2 \\ &= \big(-2 + \sigma(\mathbf{x})(\mathrm{II}(e_1, e_1) + \mathrm{II}(e_2, e_2))\big)\,\omega^1 \wedge \omega^2 \end{aligned}$$

since by the first Cartan structure equation (8.71) below, $d\omega^1 - \omega_1^2 \wedge \omega^2 = 0$ and $d\omega^2 - \omega_2^1 \wedge \omega^1 = 0$. Therefore we have the global equation

$$d\alpha = (\sigma H - 2)\, d\mu, \tag{5.92}$$

where H is the mean curvature of M^2. Note that by integrating this formula, we obtain the $n = 2$ case of (5.70); namely,

$$\int_{M^2} \sigma H\, d\mu = 2\,\mathrm{Area}(M^2).$$

Define the 1-form β on M^2 by

$$\beta(W) = \alpha(\mathrm{L}(W)). \tag{5.93}$$

Claim.

$$d\beta = (2\sigma K - H)\, d\mu, \tag{5.94}$$

where K is the Gauss curvature of M^2.

Differentiating (5.90), we have

$$\nabla_V \alpha = V(\langle \mathbf{x}, e_2\rangle)\omega^1 - V(\langle \mathbf{x}, e_1\rangle)\omega^2 + \langle \mathbf{x}, e_2\rangle\, \nabla_V \omega^1 - \langle \mathbf{x}, e_1\rangle\, \nabla_V \omega^2,$$

so by applying (5.91) and (8.61), i.e., $\nabla_V \omega^i = -\omega^i_j(V)\,\omega^j$, we obtain

$$\begin{aligned}(5.95)\quad \nabla_V \alpha &= \big(\langle V, e_2\rangle + \omega^1_2(V)\langle \mathbf{x}, e_1\rangle - \mathrm{II}(V, e_2)\sigma(\mathbf{x})\big)\,\omega^1 \\ &\quad - \big(\langle V, e_1\rangle + \omega^2_1(V)\langle \mathbf{x}, e_2\rangle - \mathrm{II}(V, e_1)\sigma(\mathbf{x})\big)\,\omega^2 \\ &\quad - \langle \mathbf{x}, e_2\rangle \omega^1_2(V)\omega^2 + \langle \mathbf{x}, e_1\rangle \omega^2_1(V)\omega^1 \\ &= \big(\langle V, e_2\rangle - \mathrm{II}(V, e_2)\sigma(\mathbf{x})\big)\,\omega^1 - \big(\langle V, e_1\rangle - \mathrm{II}(V, e_1)\sigma(\mathbf{x})\big)\,\omega^2.\end{aligned}$$

Note that we may rederive (5.92) from this formula.

Now we compute

$$\begin{aligned}2d\beta(V, W) &= (\nabla_V \beta)(W) - (\nabla_W \beta)(V) \\ &= (\nabla_V \alpha)(\mathrm{L}(W)) - (\nabla_W \alpha)(\mathrm{L}(V))\end{aligned}$$

since $\alpha((\nabla_V \mathrm{L})(W)) = \alpha((\nabla_W \mathrm{L})(V))$. For any tangential vector field V, denote $V_i = \langle V, e_i\rangle$ for $i = 1, 2$. By applying (5.95), we obtain

$$\begin{aligned}2d\beta(V, W) &= \big(V_2 - \mathrm{II}(V, e_2)\sigma(\mathbf{x})\big)\ \mathrm{L}(W)_1 - \big(V_1 - \mathrm{II}(V, e_1)\sigma(\mathbf{x})\big)\ \mathrm{L}(W)_2 \\ &\quad - \big(W_2 - \mathrm{II}(W, e_2)\sigma(\mathbf{x})\big)\ \mathrm{L}(V)_1 + \big(W_1 - \mathrm{II}(W, e_1)\sigma(\mathbf{x})\big)\ \mathrm{L}(V)_2.\end{aligned}$$

In particular,

(5.96)

$$\begin{aligned}2d\beta(e_1, e_2) &= \big(-\mathrm{II}(e_1, e_2)\sigma(\mathbf{x})\big)\ \mathrm{L}(e_2)_1 - \big(1 - \mathrm{II}(e_1, e_1)\sigma(\mathbf{x})\big)\ \mathrm{L}(e_2)_2 \\ &\quad - \big(1 - \mathrm{II}(e_2, e_2)\sigma(\mathbf{x})\big)\ \mathrm{L}(e_1)_1 + \big(-\mathrm{II}(e_2, e_1)\sigma(\mathbf{x})\big)\ \mathrm{L}(e_1)_2 \\ &= -H + 2\sigma(\mathbf{x})K.\end{aligned}$$

This proves (5.94).

Observe that the integral of (5.94) yields (5.80); namely,

$$2\int_{M^2} K\sigma d\mu = \int_{M^2} H d\mu.$$

Let $\varphi : (M^2, \mathrm{I}) \to (M'^2, \mathrm{I}')$ be an isometry. Let L and L′ be the Weingarten maps of M^2 and M'^2, respectively. Define

$$\mathrm{L}'' := d(\varphi^{-1}) \circ \mathrm{L}' \circ d\varphi : T_pM \to T_pM.$$

We have

$$\det \mathrm{L}''(\mathbf{x}) = \det \mathrm{L}'(\varphi(\mathbf{x})) = K'(\varphi(\mathbf{x})) = K(\mathbf{x}) = \det \mathrm{L}(\mathbf{x}).$$

Define the 1-form β' on M^2 by

$$(5.97)\qquad \beta'(W) = \alpha(\mathrm{L}''(W)).$$

One computes that

$$(5.98)\qquad d\beta' = \big(\sigma(2K - \det(\mathrm{L} - \mathrm{L}'')) - H'\big)d\mu,$$

where H' is the mean curvature of M'^2. Indeed, by (5.96), we have that

$$\begin{aligned}2d\beta'(e_1, e_2) = &-\sigma(\mathbf{x})\,\mathrm{L}(e_1)_2\,\mathrm{L}''(e_2)_1 + \sigma(\mathbf{x})\,\mathrm{L}(e_1)_1\,\mathrm{L}''(e_2)_2 \\ &+ \sigma(\mathbf{x})\,\mathrm{L}(e_2)_2\,\mathrm{L}''(e_1)_1 - \sigma(\mathbf{x})\,\mathrm{L}(e_2)_1\,\mathrm{L}''(e_1)_2 \\ &- \mathrm{L}''(e_2)_2 - \mathrm{L}''(e_1)_1.\end{aligned}$$

This yields (5.98) since φ being an isometry implies that $K = K'$, the Gauss curvature of M'^2, so that

$$\mathrm{L}(e_1)_1\,\mathrm{L}(e_2)_2 - (\mathrm{L}(e_1)_2)^2 = \mathrm{L}''(e_1)_1\,\mathrm{L}''(e_2)_2 - (\mathrm{L}''(e_1)_2)^2.$$

Now, by Stokes's theorem (see Theorem 8.7 below), we obtain the **Herglotz integral formula**:

$$(5.99)\qquad \begin{aligned}\int_{M^2} H' d\mu &= \int_{M^2} \sigma(2K - \det(\mathrm{L} - \mathrm{L}''))d\mu \\ &= \int_{M^2} H d\mu - \int_{M^2} \sigma \det(\mathrm{L} - \mathrm{L}'')d\mu.\end{aligned}$$

Without loss of generality, we may assume that the origin is in the region interior to M^2, so that $\sigma > 0$. By the first statement of Lemma 5.16 below, we have $\det(\mathrm{L} - \mathrm{L}'') \leq 0$. Hence

$$(5.100)\qquad \int_{M^2} H' d\mu \geq \int_{M^2} H d\mu.$$

By symmetry and since φ is an isometry, we have the reverse inequality. Therefore

$$(5.101)\qquad \int_{M^2} H' d\mu = \int_{M^2} H d\mu,$$

which implies that

$$(5.102)\qquad \int_{M^2} \sigma \det(\mathrm{L} - \mathrm{L}'')\, d\mu = 0.$$

Since $\det(\mathrm{L} - \mathrm{L}'') \leq 0$, we obtain that $\det(\mathrm{L} - \mathrm{L}'') = 0$, from which we conclude that $\mathrm{L} = \mathrm{L}''$ by the second statement in Lemma 5.16 below. Theorem 5.15 now follows from Lemma 5.14. □

Lastly, we prove the algebraic lemma that we used in the proof above.

Lemma 5.16. *If* L *and* L'' *are positive-definite symmetric* 2×2 *matrices satisfying* $\det \mathrm{L} = \det \mathrm{L}''$*, then* $\det(\mathrm{L} - \mathrm{L}'') \leq 0$*. If, in addition,* $\det(\mathrm{L} - \mathrm{L}'') = 0$*, then* $\mathrm{L} = \mathrm{L}''$*.*

Proof. By a change of basis, we may assume that

$$\mathrm{L} = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix}, \quad \mathrm{L}'' = \begin{pmatrix} a & b \\ b & d \end{pmatrix},$$

where $\kappa_1, \kappa_2, a, d > 0$ and $b \in \mathbb{R}$. Since $\det \mathrm{L} = \det \mathrm{L}''$, we have

$$\kappa_1 \kappa_2 = ad - b^2 \leq ad. \tag{5.103}$$

We have

$$\det(\mathrm{L} - \mathrm{L}'') = \kappa_1(\kappa_2 - d) + \kappa_2(\kappa_1 - a) \tag{5.104a}$$

$$= (a - \kappa_1)(d - \kappa_2) - b^2. \tag{5.104b}$$

If $a \leq \kappa_1$, then (5.103) implies that $d \geq \kappa_2$. Here, (5.104b) implies that $\det(\mathrm{L} - \mathrm{L}'') \leq 0$. Similarly, we obtain the same result if $d \leq \kappa_2$. Finally, if $a > \kappa_1$ and $d > \kappa_2$, then (5.104a) implies that $\det(\mathrm{L} - \mathrm{L}'') < 0$. We conclude that $\det(\mathrm{L} - \mathrm{L}'') \leq 0$ in all cases.

Now, suppose that $\det(\mathrm{L} - \mathrm{L}'') = 0$. Then by (5.104a) there exists $c \in \mathbb{R}$ such that

$$d - \kappa_2 = c\kappa_2, \quad a - \kappa_1 = -c\kappa_1,$$

which implies that

$$ad = (1 - c^2)\kappa_1\kappa_2 \leq \kappa_1\kappa_2.$$

We conclude that $b = c = 0$ and hence $\mathrm{L} = \mathrm{L}''$. □

5.6.2. Existence of isometric embeddings. In the next chapter we will define abstract smooth manifolds, and in particular this includes smooth surfaces. Generalizing the first fundamental form of a submanifold, we can define the notion of a Riemannian metric on a smooth manifold. The Weyl embedding problem asks whether every closed oriented Riemannian surface with positive Gauss curvature can be isometrically embedded in $\mathbb{R}^3$.

Theorem 5.17 (Weyl embedding problem)**.** *Let (M^2, g) be a closed oriented Riemannian surface with positive Gauss curvature.*

(1) *(Existence) There exists a smooth isometric embedding of (M^2, g) into $\mathbb{R}^3$.*
(2) *(Uniqueness) The isometric embedding is unique up to the composition with an isometry of $\mathbb{R}^3$.*

Major progress on the proof of the Weyl embedding problem was made by Weyl [**Wey16**], including a fundamental C^2 estimate in the form of a geometric Bochner-type estimate. Lewy [**Lew38**] solved the case where g is real analytic. Notable work includes that of Aleksandrov [**Ale48**]. The proof was finally completed by Nirenberg [**Nir53**] and Pogorelov [**Pog52**] independently. Further work is by Heinz [**Hei59**, **Hei62**] and Guan and Li [**GL94**].

The uniqueness part of Theorem 5.17 is Theorem 5.15.

The existence part of Theorem 5.17 is proved by the *continuity method* in partial differential equations. From Theorem 8.27 below, there exists a function u on M^2 such that the pointwise conformally related metric $\mathrm{e}^{2u}g$ has constant positive Gauss curvature equal to 1. Moreover, since g and $\mathrm{e}^{2u}g$ both have positive Gauss curvature, the metrics

$$g_t := \mathrm{e}^{2tu}g \tag{5.105}$$

have positive Gauss curvature for $0 \leq t \leq 1$. Indeed, by Lemma 8.21 below, we have that

$$\begin{aligned} K(\mathrm{e}^{2tu}g) &= \mathrm{e}^{-2tu}\left(K(g) - t\Delta_g u\right) \\ &= \mathrm{e}^{-2tu}\left(tK(g) - t\Delta_g u\right) + \mathrm{e}^{-2tu}(1-t)K(g) \\ &= t\mathrm{e}^{2(1-t)u}K(\mathrm{e}^{2u}g) + \mathrm{e}^{-2tu}(1-t)K(g). \end{aligned}$$

Now, since (M^2, g_0) has constant positive curvature and hence admits an isometric embedding in $\mathbb{R}^3$, the existence part of Theorem 5.17 is a consequence of the following two propositions.

Proposition 5.18 (Openness)**.** *The set of $t \in [0,1]$ such that (M^2, g_t) admits an isometric embedding in $\mathbb{R}^3$ is open.*

Proposition 5.19 (Closedness)**.** *The set of $t \in [0,1]$ such that (M^2, g_t) admits an isometric embedding in $\mathbb{R}^3$ is closed.*

The proof of these propositions is beyond the scope of this book. We refer the reader to Han and Hong [**HH06**].

5.6.3. The Minkowski problem.

The Minkowski problem asks under what condition is a positive function on S^n equal to the Gauss curvature of some convex embedded hypersurface. The following supplies a necessary and sufficient condition.

Theorem 5.20 (Minkowski problem)**.** *Let κ be a positive smooth function on the unit n-sphere S^n satisfying the integral condition*

$$\int_{S^n} \frac{1}{\kappa(\mathbf{n})}\mathbf{n}\, d\mu_{S^n}(\mathbf{n}) = \mathbf{0}. \tag{5.106}$$

Then there exists a closed strictly convex smooth hypersurface M^n embedded in $\mathbb{R}^{n+1}$ with Gauss curvature satisfying

$$K(G^{-1}(\mathbf{n})) = \kappa(\mathbf{n}) \quad \textit{for all } \mathbf{n} \in S^n, \tag{5.107}$$

where $G : M^n \to S^n$ is the Gauss map and G^{-1} is its inverse. Moreover, M^n is unique up to translation.

By (5.83), condition (5.106) is a necessary condition to solve (5.107).

5.6.4. The Nash C^∞ isometric embedding theorem. The Nash isometric embedding [**Nas56**] is the following.

Theorem 5.21 (Nash isometric embedding)**.** *For any n there exists N such that if M^n is a closed C^k manifold, where $3 \leq k \leq \infty$, then there exists a C^k isometric embedding of M^n into $\mathbb{R}^N$.*

Nash proved this result for ambient dimension $N = \frac{n(3n+11)}{2}$. Gromov [**Gro70**] later improved the dimension to $N = \frac{(n+2)(n+3)}{2}$ for $k \geq 4$. Günther [**G89a**, **G89b**, **G91**] discovered that one can circumvent some of the difficulties which Nash encountered. In particular, the remarkable Nash–Moser iteration method is not required. Moreover, Günther improved the dimension to $N = \max\left\{\frac{n(n+3)}{2} + 5, \frac{n(n+5)}{2}\right\}$.

5.7. Appendix

In this appendix we collect a few supplementary materials related to the discussion in this chapter.

5.7.1. A useful algebraic inequality. We recall an elementary algebraic inequality which is frequently used in geometric analysis. For any symmetric 2-tensor $T = T_{ij} dx^i \otimes dx^j$ on a submanifold M^n of $\mathbb{R}^N$, we have the inequality

$$|T|^2 \geq \frac{1}{n}\big(\operatorname{trace}(T)\big)^2. \tag{5.108}$$

Here, the pointwise norm of a 2-tensor is defined by

$$|T|^2 := \sum_{i,j,k,l=1}^{n} \mathrm{I}^{ik}\, \mathrm{I}^{jl}\, T_{ij} T_{kl}. \tag{5.109}$$

Equivalently, $|T|^2 = \sum_{i,j=1}^n \big(T(e_i, e_j)\big)^2$, where $\{e_i\}_{i=1}^n$ is an orthonormal frame. Note that these definitions generalize (5.31) and (5.32).

We can see inequality (5.108) by diagonalizing the matrix (T_{ij}), i.e., choosing local coordinates $\{x^i\}_{i=1}^n$ so that at a point we have $\mathrm{I}_{ij} = \delta_{ij}$ and $T_{ij} = T_{ii}\delta_{ij}$ (diagonal), where I is the first fundamental form. Then

$$\begin{aligned} |T|^2 - \frac{1}{n}\big(\operatorname{trace}(T)\big)^2 &= \sum_i (T_{ii})^2 - \frac{1}{n}\Big(\sum_j T_{jj}\Big)^2 \\ &= \sum_i \Big(T_{ii} - \frac{1}{n}\sum_j T_{jj}\Big)^2 \\ &\geq 0. \end{aligned}$$

Observe by this that we have equality in (5.108) at a point if and only if at that point $T = \frac{1}{n}\operatorname{trace}(T)\,\mathrm{I}$; cf. Exercise 5.14.

5.7.2. Inner products on spaces of tensors at a point. More generally, given covariant p-tensors α and β on M^n, their pointwise inner product is defined by

$$\langle \alpha, \beta \rangle := \mathrm{I}^{i_1 j_1} \cdots \mathrm{I}^{i_p j_p} \, \alpha_{i_1 \cdots i_p} \beta_{j_1 \cdots j_p}. \tag{5.110}$$

Equivalently, with respect to an orthonormal frame,

$$\langle \alpha, \beta \rangle = \alpha(e_{i_1}, \ldots, e_{i_p}) \beta(e_{i_1}, \ldots, e_{i_p}). \tag{5.111}$$

The pointwise norm of a p-tensor α is defined by

$$|\alpha| := \sqrt{\langle \alpha, \alpha \rangle}. \tag{5.112}$$

Note that definition (5.109) of the norm of a 2-tensor is a special case of this.

5.7.3. Calculating the Hessian and the Laplacian of the position map using local coordinates. In this subsection we give alternate calculations for some of the formulas in §5.5.2. Let X denote the position vector field (a.k.a. position map) on M^n; that is, X is the identity vector field id of $\mathbb{R}^{n+1}$ restricted to M^n. (We can also think of X as the inclusion map of M^n into $\mathbb{R}^{n+1}$.) By definition, $X_{\mathbf{x}} = \mathbf{x}$ for all $\mathbf{x} \in M^n$. We compute, using local coordinates $\{x^i\}_{i=1}^n$, that

$$\frac{1}{2} \nabla_i \|X\|^2 = \frac{1}{2} \frac{\partial}{\partial x^i} \langle X, X \rangle = \left\langle X, \frac{\partial X}{\partial x^i} \right\rangle, \tag{5.113}$$

where $\frac{\partial X}{\partial x^i} = \frac{\partial F}{\partial u^i}$ (F is the local parametrization inducing the local coordinates), so that

$$\frac{1}{2} \nabla \|X\|^2 = \frac{1}{2} \sum_{i=1}^n \nabla_i \|X\|^2 g^{ij} \frac{\partial X}{\partial x^j} = \sum_{i,j=1}^n \left\langle X, \frac{\partial X}{\partial x^i} \right\rangle g^{ij} \frac{\partial X}{\partial x^j} = X^\top;$$

that is,

$$\frac{1}{2} \nabla \|X\|^2 = X^\top. \tag{5.114}$$

Observe that for all $1 \leq i, j, l \leq n$,

$$\frac{\partial^2 X}{\partial x^i \partial x^j} \cdot \frac{\partial X}{\partial x^l} = \sum_{k=1}^n \Gamma_{ij}^k g_{kl} = \sum_{k=1}^n \Gamma_{ij}^k \nabla_k X \cdot \frac{\partial X}{\partial x^l}, \tag{5.115}$$

where the first equality follows from (4.140). Thus, the tangential component of $\frac{\partial^2 X}{\partial x^i \partial x^j}$ is equal to $\sum_{k=1}^n \Gamma_{ij}^k \nabla_k X$. Using this fact, we calculate that

$$\begin{aligned}
\nabla_i \nabla_j X := (\nabla^2 X)(\partial_i, \partial_j) &= \frac{\partial^2 X}{\partial x^i \partial x^j} - \sum_{k=1}^{n} \Gamma_{ij}^k \nabla_k X \\
&= \left(\frac{\partial^2 X}{\partial x^i \partial x^j} \right)^{\perp} \\
&= -\, \mathrm{II}_{ij}\, N_X,
\end{aligned} \tag{5.116}$$

where for the last inequality we used (4.96). That is, the Hessian of the position map (with respect to the first fundamental form I) is equal to minus the second fundamental form times the unit normal. Tracing this yields

$$\Delta X = -H\, N_X. \tag{5.117}$$

That is, the Laplacian of the position map (with respect to I) is equal to the mean curvature vector.

We further compute that

$$\begin{aligned}
\frac{1}{2} \nabla_i \nabla_j \|X\|^2 &= \nabla_i X \cdot \nabla_j X + X \cdot \nabla_i \nabla_j X \\
&= g_{ij} - \mathrm{II}_{ij} \langle X, N_X \rangle.
\end{aligned} \tag{5.118}$$

Tracing this yields the formula

$$\frac{1}{2} \Delta_g \|X\|^2 = n - H \langle X, N_X \rangle = n - H\sigma \tag{5.119}$$

on M^n.

5.8. Notes and commentary

A well-known area of study in submanifold theory regards the Willmore conjecture [**Wil65**]. Given an oriented closed surface M^2 and an immersion $F : M^2 \to \mathbb{R}^3$, the **Willmore energy** of F is defined as

$$W(F) = \int_{M^2} H^2 \, d\mu. \tag{5.120}$$

The **Willmore conjecture** posits that

$$W(F) \geq 2\pi^2. \tag{5.121}$$

This conjecture was proved by Marques and Neves [**MN14**]. Their proof employs the min-max theory for minimal surfaces of Almgren and Pitts. Notable earlier works on the Willmore conjecture were done by Li and Yau [**LY82**] and Simon [**Sim93**].

Another well-known area of study in submanifold theory regards convex bodies. See the book by Schneider [**Sch14b**].

5.9. Exercises

Exercise 5.1. *Let M^n be an embedded hypersurface in $\mathbb{R}^{n+1}$. Extending the definitions of the first and second fundamental forms, the* k***-th fundamental form*** *is the bilinear map on the tangent space $T_{\mathbf{x}}M$ defined by*

$$(X, Y) \mapsto \langle \mathrm{L}^k(X), Y\rangle. \tag{5.122}$$

We denote the third fundamental form by $\mathrm{III}(X,Y) = \langle \mathrm{L}^3(X), Y\rangle$. Show that for a surface M^2 embedded in $\mathbb{R}^3$,

$$\mathrm{L}^2 - H\,\mathrm{L} + K\,\mathrm{id} = 0, \tag{5.123}$$

and hence

$$\mathrm{III} - H\,\mathrm{II} + K\,\mathrm{I} = 0. \tag{5.124}$$

Exercise 5.2. *Let M^n be an embedded closed hypersurface in $\mathbb{R}^{n+1}$. Suppose that $\mathrm{III} = f\,\mathrm{II}$ for some function f on M^n. Show that M^n is a round sphere.*

Exercise 5.3. *Let M^2 be a surface embedded in $\mathbb{R}^3$. Let $\kappa_1 \le \kappa_2$ be the principal curvatures at a point in M^2. Show that*

$$\kappa_1 = \frac{H - \sqrt{H^2 - 4K}}{2}, \qquad \kappa_2 = \frac{H + \sqrt{H^2 - 4K}}{2}, \tag{5.125}$$

where H and K are the mean and Gauss curvatures at that point, respectively.

Exercise 5.4. *Let M^n be a hypersurface embedded in $\mathbb{R}^{n+1}$. Prove that for any vector fields X and Y on M^n,*

$$\nabla_X\big(\mathrm{L}(Y)\big) - \nabla_Y\big(\mathrm{L}(X)\big) = \mathrm{L}([X,Y]), \tag{5.126}$$

where L denotes the Weingarten map of M^n.

Exercise 5.5. *Let M^2 be a surface embedded in $\mathbb{R}^3$. Let U denote the set of umbilical points in M^2. Show that the principal curvature functions $\kappa_i : M^2 \to \mathbb{R}$, $i = 1, 2$, are continuous on M^2. Show that they are smooth on the open subset $M^2 \setminus U$.*

Exercise 5.6. *Let M^2 be a hypersurface embedded in $\mathbb{R}^3$.*

(1) *Prove that on $M^2 \setminus U$ there exist functions a and b such that*

$$\nabla_{e_1} e_2 = a e_1, \qquad \nabla_{e_2} e_1 = b e_2. \tag{5.127}$$

Show that

$$\nabla_{e_1} e_1 = -a e_2, \qquad \nabla_{e_2} e_2 = -b e_1. \tag{5.128}$$

(2) *Show that on $M^2 \setminus U$,*

$$\nabla_{e_1}\big(\mathrm{L}(e_2)\big) - \nabla_{e_2}\big(\mathrm{L}(e_1)\big) = a\kappa_1 e_1 - b\kappa_2 e_2. \tag{5.129}$$

(3) *Show that on* $M^2 \setminus U$,

$$a = -\frac{e_2(\kappa_1)}{\kappa_1 - \kappa_2}, \qquad b = \frac{e_1(\kappa_2)}{\kappa_1 - \kappa_2}. \tag{5.130}$$

Exercise 5.7. *Continuing with the previous exercise:*

(1) *Using the definition* (4.154) *of* Rm, *show that*

$$\mathrm{Rm}(e_1, e_2)e_2 = \big(-e_1(b) - e_2(a) - a^2 - b^2\big)e_1. \tag{5.131}$$

From this we conclude that the Gauss curvature at a nonumbilic point is given by

$$K = -e_1(b) - e_2(a) - a^2 - b^2. \tag{5.132}$$

(2) *Let* $\mathbf{x} \in M^2$ *be a nonumbilic point. Suppose* $\mathbf{x}$ *is a critical point of both* κ_1 *and* κ_2. *Show that*

$$K = -e_1(b) - e_2(a) = -\frac{e_1(e_1(\kappa_2))}{\kappa_1 - \kappa_2} + \frac{e_2(e_2(\kappa_1))}{\kappa_1 - \kappa_2}. \tag{5.133}$$

Exercise 5.8. *Prove that if* M^2 *is a connected embedded minimal surface in* $\mathbb{R}^3$ *with constant Gauss curvature, then* M^2 *is subset of a plane.*

Exercise 5.9.

(1) *Let* M^2 *be an embedded surface in* $\mathbb{R}^3$ *with constant Gauss curvature. Show that if* $\mathbf{x} \in M^2$ *has the property that* $\kappa_1(\mathbf{x}) \neq \kappa_2(\mathbf{x})$, $\mathbf{x}$ *is a relative minimum for* κ_1, *and* $\mathbf{x}$ *is a relative maximum for* κ_2, *then* $K(\mathbf{x}) \leq 0$.

(2) *Show that if* M^2 *is a closed surface with constant positive Gauss curvature, then* M^2 *is a round sphere.*

Exercise 5.10. *Prove the fact that if a function* $f : M^n \to \mathbb{R}$ *satisfies* $\nabla f = \mathbf{0}$ *on* M^n, *where* M^n *is connected, then* f *is constant on* M^n.

Exercise 5.11 (Locally convex neighborhoods in immersed hypersurfaces)**.** *Prove that if* M^n *is a closed immersed hypersurface in* $\mathbb{R}^{n+1}$, *then there exists a point* $\mathbf{y}$ *on* M^n *and a choice of a smooth unit normal vector field in a neighborhood of* $\mathbf{y}$ *such that the second fundamental form* II *of* M^n *is positive definite in this neighborhood. In particular, the mean curvature* H *at* $\mathbf{y}$ *is not equal to zero.*

Exercise 5.12. *Fill in the details for the assertions in the last paragraph of the proof of Theorem* 5.9.
Hint: For example, to show that M^n *is embedded, suppose for a contradiction that it were not and let* $\mathbf{X}_0 \in \mathbb{R}^{n+1}$ *be a point for which there would exist distinct points* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *in* M^n *such that* $F(\mathbf{x}_1) = F(\mathbf{x}_2) = \mathbf{X}_0$. *Obtain a contradiction about whether* $T_{F(\mathbf{x}_1)}M$ *and* $T_{F(\mathbf{x}_2)}M$ *are transverse or not.*

Exercise 5.13. *Prove* (5.23).
Hint: Given $\mathbf{x} \notin \Omega$*, let* $\mathbf{y}$ *be the closest point on* M^n*. Show that* $\mathbf{x} \notin \bar{H}_{\mathbf{y}}^{-}$*.*

Exercise 5.14. *Show that if* T *is a symmetric* 2*-tensor on a submanifold* M^n *of* $\mathbb{R}^N$*, then*

$$\left| T - \frac{1}{n}\operatorname{trace}(T)\, \mathrm{I} \right|^2 = |T|^2 - \frac{1}{n}\big(\operatorname{trace}(T)\big)^2.$$

Exercise 5.15. *Prove that if* T *is a symmetric* 2*-tensor on a submanifold* M^n *of* $\mathbb{R}^N$*, then*

$$|\nabla T|^2 \geq \frac{1}{n}|\operatorname{div}(T)|^2. \tag{5.134}$$

Exercise 5.16 (Huisken [**Hui84**, Lemma 2.2])**.** *Let* M^n *be a hypersurface embedded in* $\mathbb{R}^{n+1}$*. Define the covariant* 3*-tensor* E *by*

$$E_{ijk} = \frac{1}{n+2}\big(\nabla_i H\, g_{jk} + \nabla_j H\, g_{ik} + \nabla_k H\, g_{ij}\big). \tag{5.135}$$

Define the covariant 3*-tensor* F *by* $\nabla_i \mathrm{II}_{jk} = E_{ijk} + F_{ijk}$*. Show that*

$$\langle E, F\rangle = 0 \tag{5.136}$$

and

$$|E|^2 = \frac{3}{n+2}|\nabla H|^2. \tag{5.137}$$

Conclude that

$$\left| \nabla\, \mathrm{II} - \frac{1}{n}\nabla H \otimes g \right|^2 \geq \frac{2(n-1)}{n(n+2)}|\nabla H|^2. \tag{5.138}$$

Give another proof that if M^n *is connected, totally umbilical, and* $n \neq 1$*, then* H *is constant; cf.* (5.5)*.*

Exercise 5.17. *Let* M^n *be a smooth embedded submanifold in* $\mathbb{R}^N$*. Define* $f : M^n \to \mathbb{R}$ *by* $f(\mathbf{x}) = \|\mathbf{x}\|^2$*. Prove that for* $V, W \in T_{\mathbf{x}}M$*,*

$$\frac{1}{2}(\nabla^2 f)(V, W) = \langle B(V, W), \mathbf{x}^{\perp}\rangle + \langle V, W\rangle, \tag{5.139}$$

where ∇ *is the Levi-Civita connection of* M^n *and where* $B = B_{\mathbf{x}} : T_{\mathbf{x}}M \times T_{\mathbf{x}}M \to N_{\mathbf{x}}M$ *is the second fundamental form of* M^n *defined by* (4.115)*. This extends formula* (5.64) *for hypersurfaces.*

Moreover, show that

$$\frac{1}{2}\Delta_{M^n} f = \langle \vec{H}, \mathbf{x}^\perp \rangle + n. \tag{5.140}$$

Exercise 5.18. *Let $M^2 \subset \mathbb{R}^3$ be an embedded minimal surface with a smooth unit normal vector field N. Let $G : M^2 \to S^2$ be its Gauss map. Assume that the principal curvatures $\kappa_1 = -\kappa_2$ are nonzero everywhere on M^2. Show that there exists a positive function $f : M^2 \to \mathbb{R}$ such that*

$$\langle dG(X), dG(Y)\rangle = f(\mathbf{x})\langle X, Y\rangle$$

for $X, Y \in T_{\mathbf{x}}M$, $\mathbf{x} \in M^2$.

Show that if $\{e_1, e_2\}$ is an orthonormal tangent frame at $\mathbf{x} \in M^2$, then $\{dG(e_1), dG(e_2), N_{\mathbf{x}}\}$ has the opposite orientation from $\{e_1, e_2, N_{\mathbf{x}}\}$.

Exercise 5.19. *The m**-th mean curvature** H_m, $0 \leq m \leq n$, is defined to be the m-th elementary symmetric polynomial in the principal curvatures:*

$$H_m := \sum_{1\leq i_1 < i_2 < \cdots < i_m \leq n} \kappa_{i_1} \cdots \kappa_{i_m} \tag{5.141}$$

for $1 \leq m \leq n$, and $H_0 = 1$. Show H_1 is the mean curvature, H_2 is the scalar curvature, and H_n is the Gauss curvature.

Exercise 5.20. *Let II^k be the symmetric 2-tensor corresponding to the k-th power of the Weingarten map L^k; in other words,*

$$\mathrm{II}^k_{ij} := \mathrm{II}_{i\ell_1}\,\mathrm{II}_{\ell_1\ell_2} \cdots \mathrm{II}_{\ell_{k-1}j} \quad \text{(there are } k \text{ factors of II)}$$

for $k \geq 1$, and $\mathrm{II}^0 = g$. Define the symmetric 2-tensor A_m by

$$A_m = \sum_{k=1}^{m} (-1)^{k-1} H_{m-k}\, \mathrm{II}^{k-1} \tag{5.142}$$

for $1 \leq m \leq n$, and $A_0 = 0$. Prove, for $1 \leq m \leq n$, that we have the algebraic identities

$$\mathrm{trace}_g(A_m) = (n - m + 1)H_{m-1} \tag{5.143}$$

and

$$\langle A_m, \mathrm{II}\rangle = mH_m. \tag{5.144}$$

Exercise 5.21.

(1) *Prove the differential identity*

$$\mathrm{div}(A_m) = 0. \tag{5.145}$$

(2) *Prove the variation formula*

$$\frac{\partial}{\partial \mathrm{II}} H_m = A_m, \tag{5.146}$$

where the symmetric 2-tensor $\frac{\partial}{\partial \mathrm{II}} H_m$ *is defined by*

$$\left\langle \frac{\partial}{\partial \mathrm{II}} H_m, \alpha \right\rangle = \frac{d}{dt}\bigg|_{t=0} H_m(\mathrm{II} + t\alpha, g^{-1}) \tag{5.147}$$

for symmetric 2-tensors α.

Exercise 5.22. *Prove the Minkowski formulas*

$$\int_{M^n} H_m \langle X, N \rangle \, d\mu = \frac{n-m+1}{m} \int_{M^n} H_{m-1} \, d\mu \tag{5.148}$$

for $1 \leq m \leq n$.
Hint: Consider

$$\int_{M^n} \mathbf{x} (A_m)_{ij} \nabla_i \nabla_j \mathbf{x} \, d\mu(\mathbf{x})$$

and calculate alternatively by integration by parts and by using the formula $\nabla_i \nabla_j \mathbf{x} = -\mathrm{II}_{ij} N$.

Exercise 5.23. *Let* M^n *be an embedded strictly convex hypersurface. Let* II^{ij} *denote the components of the inverse of the second fundamental form* II^{-1}, *so that* $\sum_{j=1}^{n} \mathrm{II}^{ij} \mathrm{II}_{jk} = \delta_{ik}$. *Define*

$$\mathcal{L} = K \, \mathrm{II}^{ij} \, \nabla_i \nabla_j,$$

which is an elliptic operator since $\mathrm{II} > 0$. *Prove that*

$$\nabla_p K = K \, \mathrm{II}^{ij} \, \nabla_p \mathrm{II}_{ij} \,. \tag{5.149}$$

Hint: Given any $\mathbf{x} \in M^n$, *choose a local parametrization so that* $\Gamma_{ij}^k(\mathbf{x}) = 0$. *Prove the formula in these coordinates and use that the formula is tensorial.*

Exercise 5.24.

(1) *Prove that*

$$\Delta K = K \, \mathrm{II}^{ij} \, \Delta \mathrm{II}_{ij} - K \, \mathrm{II}^{ik} \, \mathrm{II}^{j\ell} \, \nabla_p \mathrm{II}_{k\ell} \, \nabla_p \mathrm{II}_{ij} + \frac{1}{K} |\nabla K|^2.$$

(2) *Using Simons's identity (see* (12.71) *below)*

$$\nabla_i \nabla_j H = \Delta \mathrm{II}_{ij} - \mathrm{II}_{ij}^2 \, H + |\,\mathrm{II}\,|^2 \, \mathrm{II}_{ij},$$

prove that

$$\begin{aligned} \Delta K = {} & K \, \mathrm{II}^{ij} \, \nabla_i \nabla_j H - K \, \mathrm{II}^{ik} \, \mathrm{II}^{j\ell} \, \nabla_p \mathrm{II}_{k\ell} \, \nabla_p \mathrm{II}_{ij} + \frac{1}{K} |\nabla K|^2 \\ & - nK \left(|\,\mathrm{II}\,|^2 - \frac{1}{n} H^2 \right). \end{aligned} \tag{5.150}$$

Exercise 5.25. *Prove that if $M^2 \subset \mathbb{R}^3$ is a strictly convex surface, then*

$$\Delta K = \mathcal{L}H - (\nabla_p \mathrm{II}_{12})^2 + \nabla_p \mathrm{II}_{11} \nabla_p \mathrm{II}_{22} + K(4K - H^2). \tag{5.151}$$

This formula was discovered by Weyl and used in his approach to the Weyl embedding problem; see the next exercise for its application to obtain an a priori estimate.

Exercise 5.26. *Prove that if $M^2 \subset \mathbb{R}^3$ is a strictly convex surface, then*

$$\max_M H^2 \leq \max_M \left(4K - \frac{\Delta K}{K}\right). \tag{5.152}$$

This is Weyl's estimate. Observe that the right-hand side is an intrinsic quantity; i.e., it depends only on the Riemannian metric g. This estimate yields an intrinsic bound for the second fundamental form since the positivity of the Gauss curvature implies that $|\,\mathrm{II}\,|^2 = H^2 - 2K < H^2$.

Exercise 5.27. *Let $A = \overline{\nabla}^2 \sigma + \sigma \overline{g}$ as in Proposition 5.10. Prove, using the Codazzi equation, that*

$$\overline{\nabla}_i A_{jk} = \overline{\nabla}_j A_{ik}. \tag{5.153}$$

Exercise 5.28. *Prove the commutator formula*

$$\begin{aligned} &\overline{\nabla}_i \overline{\nabla}_j A_{k\ell} + \overline{\nabla}_j \overline{\nabla}_i A_{k\ell} \\ &\quad = \overline{\nabla}_k \overline{\nabla}_\ell A_{ij} + \overline{\nabla}_\ell \overline{\nabla}_k A_{ij} + 2\overline{g}_{ij} A_{k\ell} - 2\overline{g}_{k\ell} A_{ij}. \end{aligned} \tag{5.154}$$

Exercise 5.29.

(1) *Prove that*

$$\overline{\nabla}_i \overline{\nabla}_j (H_{-1})^{-1} = \overline{\Delta} A_{ij} + (H_{-1})^{-1} \overline{g}_{ij} - n A_{ij}, \tag{5.155}$$

where

$$(H_{-1})^{-1} := \kappa_1^{-1} + \cdots + \kappa_n^{-1} = \overline{\Delta}\sigma + n\sigma \tag{5.156}$$

is the inverse of the ***harmonic mean curvature****.*

(2) *Prove that*

$$\begin{aligned} &K^{-1} A_{ij}^{-1} \overline{\nabla}_i \overline{\nabla}_j (H_{-1})^{-1} \\ &\quad = K^{-1} A_{ij}^{-1} \overline{\Delta} A_{ij} + K^{-1}((H_{-1})^{-1} H - n^2). \end{aligned} \tag{5.157}$$

Exercise 5.30. *Let β be a positive real number. Prove that*

$$\begin{aligned} \overline{\Delta} K^{-\beta} &= \beta K^{-\beta} A_{ij}^{-1} \overline{\Delta} A_{ij} + \beta \overline{g}^{k\ell} \overline{\nabla}_\ell (K^{-\beta} A_{ij}^{-1}) \overline{\nabla}_k A_{ij} \\ &= \beta K^{-\beta} A_{ij}^{-1} \overline{\Delta} A_{ij} + \beta^2 K^{-\beta} \overline{g}^{k\ell} A_{pq}^{-1} A_{ij}^{-1} \overline{\nabla}_\ell A_{pq} \overline{\nabla}_k A_{ij} \\ &\quad - \beta K^{-\beta} \overline{g}^{k\ell} A_{ip}^{-1} A_{jq}^{-1} \overline{\nabla}_\ell A_{pq} \overline{\nabla}_k A_{ij}. \end{aligned} \tag{5.158}$$

Exercise 5.31. *Let* $\mathcal{L} = K^{-1/n} A_{ij}^{-1} \overline{\nabla}_i \overline{\nabla}_j$.

(1) *Prove that*

$$\mathcal{L}(H_{-1})^{-1} \geq n\overline{\Delta} K^{-1/n} + K^{-1/n}((H_{-1})^{-1} H - n^2). \tag{5.159}$$

(2) *Let* $\mathbf{x}_0 \in S^n$ *be a point at which* $(H_{-1})^{-1}$ *attains its maximum. Prove that at* $\mathbf{x}_0$

$$(H_{-1})^{-1} \leq nK^{-1/n} - \overline{\Delta}(K^{-1/n}). \tag{5.160}$$

Since $\mathbf{x}_0$ *is a maximum point of* $(H_{-1})^{-1}$*, we have obtained an estimate for* $(H_{-1})^{-1}$ *in terms of* K *and its derivatives.*

This is a key estimate for the solution to the Minkowski problem.