

---

# Contents

Preface	xiii
Acknowledgments	xv
Suggested Topics for Courses	xvii
Notation and Symbols	xxi
<b>Part I. Geometry of Submanifolds of Euclidean Space</b>	
Chapter 1. Intuitive Introduction to Submanifolds in Euclidean Space	3
§1.1. The idea of a submanifold as being locally Euclidean	3
§1.2. Euclidean and vector inner product spaces	5
§1.3. Introduction to submanifolds via examples	10
§1.4. Notes and commentary	23
§1.5. Exercises	23
Chapter 2. Differential Calculus of Submanifolds	25
§2.1. Point-set topology and vector calculus	25
§2.2. The implicit function theorem	32
§2.3. Parametrized and higher-codimension submanifolds	37
§2.4. Examples of coordinate charts for submanifolds	41
§2.5. Immersions and embeddings	45
§2.6. Notes and commentary	53
§2.7. Exercises	53

---

Chapter 3. Linearizing Submanifolds: Tangent and Tensor Bundles	57
§3.1. Tangent space	58
§3.2. Tangent bundle	67
§3.3. The normal and cotangent bundles	72
§3.4. Tensors, tensor products, and tensor bundles	77
§3.5. Notes and commentary	82
§3.6. Exercises	82
Chapter 4. Curvature and the Local Geometry of Submanifolds	85
§4.1. Paths in the plane and in space	86
§4.2. First fundamental form of a submanifold	100
§4.3. Euclidean covariant derivative	107
§4.4. Second fundamental form	112
§4.5. Induced covariant derivative on a hypersurface	126
§4.6. Curvature tensor of a submanifold	134
§4.7. Gauss and Codazzi equations	140
§4.8. Theorema Egregium	144
§4.9. Fundamental theorem of hypersurface theory	147
§4.10. Notes and commentary	154
§4.11. Exercises	155
Chapter 5. Global Theorems in the Theory of Submanifolds	161
§5.1. Totally umbilical hypersurfaces	162
§5.2. Closed hypersurfaces	164
§5.3. Convex closed hypersurfaces	168
§5.4. The Hessian, Laplacian, and divergence theorem	173
§5.5. Hypersurface integrals	177
§5.6. Highlights of submanifold theory	181
§5.7. Appendix	189
§5.8. Notes and commentary	191
§5.9. Exercises	192
<b>Part II. Differential Topology and Riemannian Geometry</b>	
Chapter 6. Smooth Manifolds	201
§6.1. Manifolds	202
§6.2. Tangent space	210

---

§6.3.	Maps between manifolds and their derivatives	215
§6.4.	Vector fields	220
§6.5.	Tensors	224
§6.6.	Categories of manifolds	227
§6.7.	Notes and commentary	231
§6.8.	Exercises	231
Chapter 7.	Riemannian Manifolds	233
§7.1.	Riemannian metrics	233
§7.2.	Covariant differentiation	237
§7.3.	Geodesics	242
§7.4.	First variation of arc length	246
§7.5.	The exponential map	249
§7.6.	The completeness condition	254
§7.7.	Curvature	256
§7.8.	Curvature decomposition	262
§7.9.	Constant sectional curvature manifolds	272
§7.10.	Second variation of arc length and stable geodesics	281
§7.11.	Jacobi fields, conjugate points, and the index form	284
§7.12.	The Rauch comparison theorem and its applications	294
§7.13.	Cut locus and injectivity domain and range	302
§7.14.	Statement of the Cheeger–Gromov compactness theorem	310
§7.15.	Holonomy	311
§7.16.	Notes and commentary	319
§7.17.	Exercises	319
Chapter 8.	Differential Forms and the Method of Moving Frames on Manifolds	323
§8.1.	Orientations on manifolds	323
§8.2.	Differential forms and integration	328
§8.3.	Moving frames	335
§8.4.	Cartan structure equations	341
§8.5.	The Hessian and Laplacian	345
§8.6.	Conformally equivalent metrics	347
§8.7.	Hyperbolic 2-space	352
§8.8.	Uniformization of geometric surfaces	352
§8.9.	Submanifolds of Riemannian manifolds	358

§8.10. Notes and commentary	362
§8.11. Exercises	362
Chapter 9. The Gauss–Bonnet and Poincaré–Hopf Theorems	367
§9.1. Morse functions and the Morse lemma	367
§9.2. Intersection number and degree	373
§9.3. Statement of the Gauss–Bonnet formula	379
§9.4. Proof of the Gauss–Bonnet formula	381
§9.5. The Gauss–Bonnet formula on surfaces with boundary	387
§9.6. The Gauss–Bonnet formula on orbifolds	388
§9.7. Notes and commentary	392
§9.8. Exercises	393
Chapter 10. Bundles and the Chern–Gauss–Bonnet Formula	395
§10.1. The tangent bundle revisited	396
§10.2. The geodesic flow on the unit tangent bundle	398
§10.3. Vector bundles	407
§10.4. Lie groups	415
§10.5. Fiber bundles	423
§10.6. Principal bundles	427
§10.7. Connections on vector bundles and principal bundles	429
§10.8. The Chern–Gauss–Bonnet formula	434
§10.9. Index theorems on 4-manifolds	446
§10.10. Einstein manifolds	453
§10.11. Notes and commentary	454
§10.12. Exercises	455
<b>Part III. Elliptic and Parabolic Equations in Geometric Analysis</b>	
Chapter 11. Linear Elliptic and Parabolic Equations	459
§11.1. The fundamental Bochner–Weitzenböck formula and integration by parts	459
§11.2. Eigenvalues of the Laplacian on closed manifolds	465
§11.3. The geometry of geodesic spheres	474
§11.4. The Bishop–Gromov volume comparison theorems	491
§11.5. Splitting theorems	498
§11.6. Harmonic functions on complete noncompact manifolds	504

---

§11.7.	The heat equation	509
§11.8.	Hodge theory	516
§11.9.	Notes and commentary	526
§11.10.	Exercises	528
Chapter 12.	Elliptic Equations and the Geometry of Minimal Surfaces	533
§12.1.	The definition of minimal submanifold	534
§12.2.	The monotonicity formula for minimal submanifolds	534
§12.3.	Minimal graphs and the Bernstein theorem	540
§12.4.	The Simons identity	551
§12.5.	Curvature estimate for stable minimal hypersurfaces	554
§12.6.	Statement of Plateau's problem	560
§12.7.	Minimal surfaces in 3-manifolds with positive scalar curvature	561
§12.8.	Introduction to harmonic maps	567
§12.9.	Notes and commentary	572
§12.10.	Exercises	573
Chapter 13.	Geometric Flows of Curves in the Plane	577
§13.1.	Isoperimetric inequality in the plane	578
§13.2.	The inverse curve shortening flow	582
§13.3.	Introduction to the curve shortening flow	587
§13.4.	Evolution of geometric quantities under the curve shortening flow	592
§13.5.	Preserving embeddedness	596
§13.6.	Gage's monotonicity formula	600
§13.7.	The evolution of the curvature $\kappa$	602
§13.8.	Hamilton's Harnack estimate	607
§13.9.	Huisken's monotonicity formula	609
§13.10.	Arzelà–Ascoli theorem and equicontinuous families of functions	613
§13.11.	Singularity analysis	615
§13.12.	Huisken's chord-arc estimate	623
§13.13.	Convergence to a round point	631
§13.14.	Notes and commentary	637
§13.15.	Exercises	638

---

Chapter 14. Uniformization of Surfaces via Heat Flow	641
§14.1. Families of conformally equivalent metrics on surfaces	642
§14.2. Variation of the curvature under a conformal variation of the metric	643
§14.3. The normalized Ricci flow equation on surfaces	645
§14.4. Short-time existence of the normalized Ricci flow	649
§14.5. A lower bound for the curvature under the normalized Ricci flow	650
§14.6. Estimating the curvature from above under the normalized Ricci flow	654
§14.7. Uniform convergence of the metric $g(t)$ as $t$ approaches the supremal time, when $\chi(M^2) < 0$	659
§14.8. Estimating derivatives of the curvature	661
§14.9. Long-time existence and convergence when $\chi(M^2) < 0$	662
§14.10. Statement of the general convergence theorem	664
§14.11. The Ricci flow on the 2-sphere	664
§14.12. The entropy and Harnack estimates	671
§14.13. Ricci solitons	680
§14.14. Uniformization of 2-dimensional orbifolds	685
§14.15. Appendix: The Kazdan–Warner identity	685
§14.16. Notes and commentary	687
§14.17. Exercises	688
Bibliography	695
Index	717