

Dirac Operators in Riemannian Geometry

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Graduate Studies
in Mathematics

Volume 25



American Mathematical Society
Providence, Rhode Island

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Introduction

It is well-known that a smooth complex-valued function $f : \mathcal{O} \rightarrow \mathbb{C}$ defined on an open subset $\mathcal{O} \subset \mathbb{R}^2$ is holomorphic if and only if it satisfies the Cauchy-Riemann equation

$$\frac{\partial f}{\partial \bar{z}} = 0 \quad \text{with} \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2} \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

Geometrically, we consider \mathbb{R}^2 here as flat Euclidean space with fixed orientation. Changing this orientation results in replacing the operator $\frac{\partial}{\partial \bar{z}}$ by the differential operator $\frac{\partial}{\partial z} = \frac{1}{2} \left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y} \right)$. Taking both operators together we obtain a differential operator $P : C^\infty(\mathbb{R}^2; \mathbb{C}^2) \rightarrow C^\infty(\mathbb{R}^2; \mathbb{C}^2)$ acting via

$$P \begin{pmatrix} f \\ g \end{pmatrix} = 2i \begin{pmatrix} \frac{\partial g}{\partial z} \\ \frac{\partial f}{\partial \bar{z}} \end{pmatrix}$$

on pairs of complex-valued functions. An easy calculation leads to the following alternative formula for P :

$$P = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \frac{\partial}{\partial x} + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial y}.$$

Denoting the matrices occurring in this formula by γ_x and γ_y ,

$$\gamma_x = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \gamma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

yields

$$P = \gamma_x \frac{\partial}{\partial x} + \gamma_y \frac{\partial}{\partial y}$$

as well as

$$\gamma_x^2 = -E = \gamma_y^2, \quad \gamma_x \gamma_y + \gamma_y \gamma_x = 0.$$

The square of the operator P coincides with the Laplacian Δ on \mathbb{R}^2 :

$$P^2 = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} = \Delta.$$

Thus we have found a square root $P = \sqrt{\Delta}$ of the Laplacian within the class of first order differential operators, and its kernel is, moreover, the space of holomorphic (anti-holomorphic) functions.

In higher-dimensional Euclidean spaces the question whether there exists a square root $\sqrt{\Delta}$ of the Laplacian was raised in the following discussion by P.A.M. Dirac (1928). Let T be a free classical particle in \mathbb{R}^3 with spin $\frac{1}{2}$ whose motion is to be studied in special relativity. Denoting its mass by m , its energy by E and its momentum by $p = \frac{vm}{\sqrt{1-v^2/c^2}}$, we have

$$E = \sqrt{c^2 p^2 + m^2 c^4}.$$

In quantum mechanics T is described by a state function $\psi(t, x)$ defined on $\mathbb{R}^1 \times \mathbb{R}^3$, and energy as well as momentum are to be replaced by the differential operators

$$E \mapsto ih \frac{\partial}{\partial t} \quad \text{and} \quad p \mapsto -ih \text{grad},$$

respectively. The state function ψ then becomes a solution of the equation

$$ih \frac{\partial \psi}{\partial t} = \sqrt{c^2 h^2 \Delta + m^2 c^4} \psi$$

involving the 3-dimensional Laplacian $\Delta = -\frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$. Mathematically speaking we now move to an n -dimensional Euclidean space and look for a square root $P = \sqrt{\Delta}$ of the Laplacian $\Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. The obvious assumption that P should be a first order differential operator with constant coefficients leads to the ansatz

$$P = \sum_{i=1}^n \gamma_i \frac{\partial}{\partial x_i}.$$

Now the equation $P^2 = \Delta = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ holds if and only if the coefficients γ_i of P satisfy the conditions

$$\gamma_i^2 = -E, \quad i = 1, \dots, n; \quad \gamma_i \gamma_j + \gamma_j \gamma_i = 0, \quad i \neq j.$$

For $n = 3$, there is an obvious solution to these equations. The vector space \mathbb{C}^2 can be identified with the set of quaternions via $\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = z_1 + jz_2$, and $\gamma_1, \gamma_2, \gamma_3 : \mathbb{C}^2 = \mathbb{H} \rightarrow \mathbb{H} = \mathbb{C}^2$ then correspond to multiplication by the quaternions $i, j, k \in \mathbb{H}$, respectively. Writing these as complex (2×2) -matrices, we obtain

$$\gamma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

The algebra multiplicatively generated by n elements $\gamma_1, \dots, \gamma_n$ satisfying the relations

$$\gamma_i^2 = -E, \quad \gamma_i \gamma_j + \gamma_j \gamma_i = 0 \quad (i \neq j),$$

is called the Clifford algebra \mathcal{C}_n (W.K. Clifford, 1845-1879) of the negative definite quadratic form $(\mathbb{R}^n, -x_1^2 - \dots - x_n^2)$. Thus, the question whether there is a square root $\sqrt{\Delta}$ of the Laplacian leads to the study of complex representations $\kappa : \mathcal{C}_n \rightarrow \text{End}(V)$ of the Clifford algebra. It turns out that \mathcal{C}_n has a smallest representation of dimension $\dim_{\mathbb{C}} V = 2^{\lfloor \frac{n}{2} \rfloor}$. The corresponding vector space is denoted by Δ_n and its elements are the Dirac spinors. Moreover, $\sqrt{\Delta}$ is a constant coefficient first order differential operator acting on the space $\mathbb{C}^\infty(\mathbb{R}^n; \Delta_n)$ of smooth Δ_n -valued functions on \mathbb{R}^n .

Spinors can be multiplied by vectors from Euclidean space. In order to define this product we represent a vector $x \in \mathbb{R}^n$ as a linear combination with respect to an orthonormal basis e_1, \dots, e_n ,

$$x = \sum_{i=1}^n x^i e_i,$$

and then define its product $x \cdot \psi$ by a spinor $\psi \in \Delta_n$ as

$$x \cdot \psi = \sum_{i=1}^n x^i \kappa(\gamma_i)(\psi).$$

From the defining relations of the Clifford algebra one immediately deduces the formula

$$x \cdot (x \cdot \psi) = -||x||^2 \psi.$$

In particular, the product $x \cdot \psi$ vanishes if and only if either the vector $x \in \mathbb{R}^n$ or the spinor $\psi \in \Delta_n$ is equal to zero. There is no non-trivial representation ε of the linear or the orthogonal group in the space Δ_n of spinors that is compatible with Clifford multiplication, i.e. which satisfies the relation

$$A(x) \cdot \varepsilon(A)(\psi) = \varepsilon(A)(x \cdot \psi)$$

for every $A \in SO(n; \mathbb{R})$, $x \in \mathbb{R}^n$ and $\psi \in \Delta_n$. Hence spinors on Riemannian manifolds cannot be defined as sections of a vector bundle that is associated with the frame bundle of the manifold. It is for this reason that in differential geometry the question to what extent the concept of spinors could be transferred from flat space to general Riemannian manifolds remained open for decades. In 1938 Elie Cartan expressed this difficulty in his book “Leçons sur la théorie des spineurs” with the following words:

“With the geometric sense we have given to the word ‘spinor’ it is impossible to introduce fields of spinors into the classical Riemannian technique.”

Only the development of the framework of principal fibre bundles and their associated bundles as well as the general theory of connections within differential geometry at the end of the forties made it possible to overcome this difficulty. The group $SO(n; \mathbb{R})$ is not simply connected. For $n \geq 3$ its universal covering, the group denoted by $Spin(n)$, is compact and covers $SO(n; \mathbb{R})$ twice. On the other hand, there exists a representation $\varepsilon : Spin(n) \rightarrow GL(\Delta_n)$ of the spin group which is compatible with Clifford multiplication. Considering now those special Riemannian manifolds M^n , today called spin manifolds, the frame bundle of which allows a reduction to the double cover $Spin(n)$ of the structure group $SO(n; \mathbb{R})$, we can define the vector bundle S associated with this reduction via the representation $\varepsilon : Spin(n) \rightarrow GL(\Delta_n)$, the so-called spinor bundle of M^n . Then spinor fields over M^n are sections of the bundle S and, as in the Euclidean case, the Dirac operator D can be introduced by the formula

$$D\psi = \sum_{i=1}^n e_i \cdot \nabla_{e_i} \psi.$$

Here ∇ denotes the covariant derivative corresponding to the Levi-Civita connection of the Riemannian manifold.

Therefore, spinor fields and Dirac operators cannot be introduced on every Riemannian space; but, nevertheless, they can be introduced for a large class. The existence of a $Spin(n)$ -reduction of the frame bundle of M^n translates into a topological condition on the manifold, i.e. the first two Stiefel-Whitney classes have to vanish:

$$w_1(M^n) = 0 = w_2(M^n).$$

In dimension $n = 4$, for a compact simply connected manifold M^4 , this topological condition is equivalent to the condition that the intersection form on $H^2(M^4; \mathbb{Z})$, considered as a quadratic form over the ring \mathbb{Z} , is even and unimodular. The algebraic theory of quadratic \mathbb{Z} -forms then implies that the signature σ is divisible by 8. Surprisingly, in 1952 Rokhlin proved

a further divisibility by 2: the signature $\sigma(M^4)$ of a smooth compact 4-dimensional spin manifold M^4 is divisible by 16:

$$\sigma(M^4)/16 \in \mathbb{Z}.$$

This additional divisibility of the signature of a 4-dimensional spin manifold, which does not result from purely algebraic considerations, was an essential aspect for the introduction of spinor fields and Dirac operators into mathematics. The consideration behind that may be outlined as follows. Could it be possible that there exists an elliptic operator P on every compact smooth 4-dimensional manifold with even intersection form on $H^2(M^4; \mathbb{Z})$, the index of which coincides with $\sigma/16$? Today we know the answer to that question: it is essentially given by the Dirac operator on a spin manifold, eventually introduced for Riemannian manifolds by M.F. Atiyah in 1962 in connection with his elaboration of the index theory for elliptic operators. Since then it has occurred in many branches of mathematics and has become one of the basic elliptic differential operators in analysis and geometry.

This book was written after a one-semester course held at Humboldt-University in Berlin during 1996/97. It contains an introduction into the theory of spinors and Dirac operators on Riemannian manifolds. The reader is assumed to have only basic knowledge of algebra and geometry, such as a two or three year study in mathematics or physics should provide. The presentation starts with an algebraic part comprising Clifford algebras, spin groups and the spin representation. The topological aspects concerning the existence and classification of spin reductions of principal $SO(n)$ -bundles are discussed in Chapter 2. Here the approach essentially requires only elementary covering theory of topological spaces. At the same time, each result will also be translated into the cohomological language of characteristic classes. The subsequent Chapter 3 deals with analysis in the spinor bundle, the twistor operator and the Dirac operator in detail. Here the general techniques of principal bundles and the theory of connections are applied systematically. To make the book more self-contained, these results of modern differential geometry are presented without proof in Appendix B. Chapter 4 contains special proofs for the analytic properties of Dirac operators (essential self-adjointness, Fredholm property) avoiding the general theory for elliptic pseudo-differential operators. Eigenvalue estimates and solution spaces of special spinorial field equations (Killing spinors, twistor spinors) are the topic of Chapter 5. We mainly discuss the general approach, referring to the literature for detailed investigations of these problems. The book is concluded in Appendix A with an extended version of a talk on

Seiberg-Witten theory given by the author in the seminar of the Sonderforschungsbereich 288 “Differentialgeometrie und Quantenphysik” in Berlin on February 9, 1995.

Since the eighties a group of younger mathematicians at Humboldt-University in Berlin has been working on spectral properties of Dirac operators and solution spaces of spinorial field equations. Many of the results from this period are collected in the references. On the other hand, the present book may serve as an introduction for a closer study. I would like to thank all those students and colleagues whose remarks and hints had an impact on the contents of this text in various ways.

I am particularly grateful to Dr. Ines Kath for her careful and detailed corrections of the text, and to Heike Pahlisch, whose typing of the manuscript took into account every single wish.

Thomas Friedrich
Berlin, March 1997

The English translation of this book has been prepared in the beginning of the year 2000. It does not differ essentially from the original text, although I made many changes in details which are not worth listing. During the last three years many new results have been published in this still dynamic area of mathematics. I included the corresponding references in the bibliography of the translation. During the academic year 1996/97 Dr. Andreas Nestke provided the exercises for students of my lectures at Humboldt University which furnished the starting point for this book. Two years later he had to leave the University. I thank him, as well as Dr. Ilka Agricola and Heike Pahlisch, for all the work and help related with the preparation of the English edition of this book.

Thomas Friedrich
Berlin, March 2000

Eigenvalue Estimates for the Dirac Operator and Twistor Spinors

5.1. Lower estimates for the eigenvalues of the Dirac operator

In this chapter we will consider a compact Riemannian manifold (M^n, g) with fixed spin structure and its Dirac operator D which, in this case, is exclusively determined by the Levi-Civita connection. By integration, from the Schrödinger-Lichnerowicz formula,

$$D^2 = \Delta + \frac{1}{4}R,$$

we immediately obtain the inequality $\lambda^2 \geq \frac{1}{4}R_0$ for every eigenvalue λ of the Dirac operator, where $R_0 = \min\{R(m) : m \in M^n\}$ is the minimum of the scalar curvature. However, this estimate is not optimal. We have (Th. Friedrich, 1980)

Proposition. *Let (M^n, g) be a compact Riemannian manifold with spin structure, and λ an eigenvalue of the Dirac operator D . Then,*

$$\lambda^2 \geq \frac{1}{4} \frac{n}{n-1} R_0.$$

Moreover, if $\lambda = \pm \frac{1}{2} \sqrt{\frac{n}{n-1} R_0}$ is an eigenvalue of the Dirac operator and ψ a corresponding eigenspinor, then ψ is a solution of the field equation

$$\nabla_X \psi = \mp \frac{1}{2} \sqrt{\frac{R_0}{n(n-1)}} X \cdot \psi$$

and the scalar curvature R is constant.

Proof. The idea of the proof is based on not using the Levi-Civita connection but, instead, considering a suitably modified covariant derivative in the spinor bundle. To this end, fix a real-valued function $f : M^n \rightarrow \mathbb{R}^1$ and introduce the covariant derivative ∇^f in the spinor bundle S by the formula

$$\nabla_X^f \psi = \nabla_X \psi + f X \cdot \psi.$$

The algebraic properties of Clifford multiplication imply that ∇^f is a metric covariant derivative in the spinor bundle S :

$$X(\psi, \psi_1) = (\nabla_X^f \psi, \psi_1) + (\psi, \nabla_X^f \psi_1).$$

Let $\Delta^f = - \sum_{i=1}^n \nabla_{e_i}^f \nabla_{e_i}^f - \sum_{i=1}^n \operatorname{div}(e_i) \nabla_{e_i}^f$ be the corresponding Laplace operator, and denote by

$$|\nabla^f \psi|^2 = \sum_{i=1}^n |\nabla_{e_i}^f \psi|^2 = \sum_{i=1}^n |\nabla_{e_i} \psi + f e_i \cdot \psi|^2$$

the length of the 1-form $\nabla^f \psi$. We will compute the operator $(D - f)^2$. First,

$$(D - f)^2 = (D - f)(D - f) = D^2 - 2fD - \operatorname{grad}(f) + f^2,$$

and the Schrödinger-Lichnerowicz formula implies

$$(D - f)^2 = \Delta + \frac{1}{4}R - 2fD - \operatorname{grad}(f) + f^2.$$

On the other hand,

$$\begin{aligned} \Delta^f &= - \sum_{i=1}^n (\nabla_{e_i} + f e_i)(\nabla_{e_i} + f e_i) - \sum_{i=1}^n \operatorname{div}(e_i)(\nabla_{e_i} + f e_i) \\ &= \Delta - 2fD - \operatorname{grad}(f) + n f^2. \end{aligned}$$

Summing up, this yields

$$(D - f)^2 = \Delta^f + \frac{1}{4}R + (1 - n)f^2,$$

and, by integration over M^n , we obtain the formula

$$\int_{M^n} ((D - f)^2 \psi, \psi) = \int_{M^n} \left\{ |\nabla^f \psi|^2 + \frac{1}{4}R|\psi|^2 + (1 - n)f^2|\psi|^2 \right\}.$$

Suppose now that $D\psi = \lambda\psi$. Then we can insert the function $f = \frac{\lambda}{n}$ into the last formula and obtain

$$\lambda^2 \left(\frac{n-1}{n}\right)^2 \|\psi\|_{L^2}^2 = \|\nabla_n^\lambda \psi\|_{L^2}^2 + \lambda^2 \frac{1-n}{n^2} \|\psi\|_{L^2}^2 + \frac{1}{4} \int_{M^n} R|\psi|^2.$$

An algebraic transformation yields

$$\lambda^2 \frac{n-1}{n} \|\psi\|_{L^2}^2 = \|\nabla_n^\lambda \psi\|_{L^2}^2 + \frac{1}{4} \int_{M^n} R|\psi|^2 \geq \frac{1}{4} R_0 \|\psi\|_{L^2}^2,$$

i.e. $\lambda^2 \geq \frac{1}{4} \frac{n}{n-1} R_0$. Discussing the boundary case in this estimate, we immediately obtain the remaining assertions of the proposition. \square

The method of proof applied here may be refined in various ways. Consider, for example, for a fixed smooth real-valued function $f : M^n \rightarrow \mathbb{R}^1$ the (non-metric) covariant derivative

$$\tilde{\nabla}_X \psi = \nabla_X \psi + \frac{\lambda}{n} X \cdot \psi + \mu X \cdot \text{grad}(f) \cdot \psi + \nu df(X)\psi$$

with the “optimal” parameters $\mu = -\frac{1}{n-1}$, $\nu = -\frac{n}{n-1}$, and perform a calculation with the length $\|e^{\mu f} \tilde{\nabla} \psi\|_{L^2}^2$ similar to the one in the proof above. Then one obtains the inequality (O. Hijazi, 1986)

Proposition.

$$\lambda^2 \geq \frac{n}{n-1} \min \left\{ \frac{1}{4} R + \Delta(f) - \frac{n-2}{n-1} |\text{grad } f|^2 \right\}.$$

In particular, in dimension 2 the summand $|\text{grad } f|^2$ drops out. Then the formula simplifies to

$$\lambda^2 \geq \min \left\{ \frac{1}{2} R + 2\Delta(f) \right\}.$$

The Gauß curvature K of the Riemann surface (M^2, g) is equal to $K = \frac{1}{2}R$, and we can choose f as a solution to the differential equation

$$2\Delta(f) = -K + \frac{1}{\text{vol}(M^2, g)} \int_{M^2} K = -K + \frac{2\pi \mathcal{X}(M^2)}{\text{vol}(M^2)}.$$

Thus $\frac{1}{2}R + 2\Delta(f) = \frac{2\pi \mathcal{X}(M^2)}{\text{vol}(M^2)}$ is constant, and we obtain

$$\lambda^2 \geq \frac{2\pi \mathcal{X}(M^2)}{\text{vol}(M^2)}.$$

Of course, the last inequality is interesting only for 2-dimensional Riemannian manifolds which, topologically, are spheres. Summarizing, we obtain the following proposition, originally due to Lott, Hijazi, and Bär.

Proposition. *If (S^2, g) is a Riemannian metric on S^2 , then, for the first eigenvalue of the Dirac operator, we have*

$$\lambda^2 \geq \frac{4\pi}{\text{vol}(S^2, g)}.$$

The method we have outlined for estimating the eigenvalues of the Dirac operator may be refined even further when the Riemannian manifold carries additional geometric structures. Let us consider e.g. the case of a Kähler manifold (M^{2k}, J, g) with complex structure $J : T(M^{2k}) \rightarrow T(M^{2k})$. In this situation, consider the covariant derivative

$$\tilde{\nabla}_X \psi = \nabla_X \psi + fX \cdot \psi + hJ(X) \cdot \psi$$

depending on two parameters f and h which can be chosen freely. Elaborating on the Weitzenböck formulas for Riemannian manifolds with additional geometric structures, one will in general obtain better estimates than in the general case of a Riemannian manifold. For example, the following inequality, first proved by K.-D. Kirchberg, holds for Kähler manifolds:

Proposition. *Let (M^{2k}, J, g) be a compact Kähler spin manifold and λ an eigenvalue of the Dirac operator. Then,*

$$\lambda^2 \geq \begin{cases} \frac{1}{4} \frac{k+1}{k} R_0 & \text{if } k = \dim_{\mathbb{C}} M \text{ is odd,} \\ \frac{1}{4} \frac{k}{k-1} R_0 & \text{if } k = \dim_{\mathbb{C}} M \text{ is even.} \end{cases}$$

Remark. The quaternionic Kähler case has been investigated by Kramer, Semmelmann, and Weingart in 1997/98.

5.2. Riemannian manifolds with Killing spinors

By the proposition proved in Section 5.1, a spinor field ψ which is an eigen-spinor for the eigenvalue $\pm \frac{1}{2} \sqrt{\frac{n}{n-1}} R_0$ solves the stronger field equation

$$\nabla_X \psi = \mp \frac{1}{2} \sqrt{\frac{R_0}{n(n-1)}} X \cdot \psi.$$

This leads to the general notion of Killing spinors.

Definition. A spinor field ψ defined on a Riemannian spin manifold (M^n, g) is called a Killing spinor, if there exists a complex number μ such that

$$\nabla_X \psi = \mu X \cdot \psi$$

for all vectors $X \in T$. μ itself is called the Killing number of ψ .

We begin by listing a few elementary properties of Killing spinors.

Proposition. *Let (M^n, g) be a connected Riemannian manifold.*

- 1) *A not identically vanishing Killing spinor has no zeroes.*
- 2) *Every Killing spinor ψ belongs to the kernel of the twistor operator T . Moreover, ψ is an eigenspinor of the Dirac operator, $D(\psi) = -n\mu\psi$.*
- 3) *If ψ is a Killing spinor corresponding to a real Killing number $\mu \in \mathbb{R}^1$, then the vector field*

$$V^\psi = \sum_{i=1}^n (e_i \cdot \psi, \psi) e_i$$

is a Killing vector field of the Riemannian manifold (M^n, g) .

Proof. A Killing spinor restricted to the curve $\gamma(t)$, $\psi(t) = \psi(\gamma(t))$, satisfies the following first order ordinary differential equation along this curve:

$$\frac{d}{dt}\psi(t) = \mu \dot{\gamma}(t) \cdot \psi(t).$$

Now $\psi(0) = 0$ immediately implies $\psi(\gamma(t)) \equiv 0$, and this in turn yields property (1). Starting from $\nabla_X \psi = \mu X \cdot \psi$, we compute

$$D\psi = \sum_{i=1}^n e_i \nabla_{e_i} \psi = \mu \sum_{i=1}^n e_i \cdot e_i \cdot \psi = -n\mu\psi$$

and thus obtain

$$T(\psi) = \sum_{i=1}^n e_i \otimes (\nabla_{e_i} \psi + \frac{1}{n} e_i \cdot D\psi) = \sum_{i=1}^n e_i \otimes (\mu e_i \psi - \mu e_i \cdot \psi) = 0.$$

For a fixed point $m_0 \in M^n$ and a local orthonormal frame e_1, \dots, e_n with $\nabla_{e_i}(m_0) = 0$ we compute the covariant derivative $\nabla_X V^\psi$:

$$\begin{aligned} \nabla_X V^\psi &= \sum_{i=1}^n (e_i \cdot \nabla_X \psi, \psi) e_i + \sum_{i=1}^n (e_i \cdot \psi, \nabla_X \psi) e_i \\ &= \mu \sum_{i=1}^n (e_i \cdot X \cdot \psi, \psi) e_i + \sum_{i=1}^n \mu (e_i \psi, X \cdot \psi) e_i \\ &= \mu \sum_{i=1}^n ((e_i \cdot X - X e_i) \cdot \psi, \psi) e_i. \end{aligned}$$

This implies $g(\nabla_X V^\psi, Y) = \mu((YX - XY) \cdot \psi, \psi)$; hence $g(\nabla_X V^\psi, Y)$ is anti-symmetric in X, Y . But this property characterizes Killing vector fields on a Riemannian manifold. \square

Not every Riemannian manifold allows Killing spinors $\psi \neq 0$, and not every number $\mu \in \mathbb{C}$ occurs as a Killing number. We now derive a series of necessary conditions. To this end, recall the Weyl tensor of a Riemannian manifold. Let

$$R_{ijkl} = g(\nabla_{e_i}\nabla_{e_j}e_k - \nabla_{e_j}\nabla_{e_i}e_k - \nabla_{[e_i,e_j]}e_k, e_l)$$

be the components of the curvature tensor and

$$R_{ij} = \sum_{\alpha=1}^n R_{\alpha ij \alpha}$$

those of the Ricci tensor. Then define two new tensors K and W by

$$K_{ij} = \frac{1}{n-2} \left\{ \frac{R}{2(n-1)}g_{ij} - R_{ij} \right\},$$

$$W_{\alpha\beta\gamma\delta} = R_{\alpha\beta\gamma\delta} - g_{\beta\delta}K_{\alpha\gamma} - g_{\alpha\gamma}K_{\beta\delta} + g_{\beta\gamma}K_{\alpha\delta} + g_{\alpha\delta}K_{\beta\gamma}.$$

W is called the Weyl tensor of the Riemannian manifold. Because of its symmetry properties the Weyl tensor can be considered as a bundle morphism defined on the 2-forms of (M^n, g) :

$$W : \Lambda^2(M^n) \rightarrow \Lambda^2(M^n), \quad W(e_i \wedge e_j) = \sum_{k<l} W_{ijkl}e_k \wedge e_l.$$

With these notations we have the following

Proposition. *Let (M^n, g) be a connected Riemannian spin manifold with a non-trivial Killing spinor ψ for the Killing number μ . Then:*

- 1) $\mu^2 = \frac{1}{4} \frac{1}{n(n-1)}R$ at each point. In particular, the scalar curvature of (M^n, g) is constant and μ is either real or purely imaginary.
- 2) (M^n, g) is an Einstein space.
- 3) $W(w^2) \cdot \psi = 0$ for every 2-form $w^2 \in \Lambda^2(M^n)$.

Proof. $\nabla_X\psi = \mu X \cdot \psi$ implies $\nabla_X\nabla_Y\psi = \mu(\nabla_XY) \cdot \psi + \mu^2Y \cdot X \cdot \psi$ and hence

$$(\nabla_X\nabla_Y - \nabla_Y\nabla_X - \nabla_{[X,Y]})\psi = \mu^2(Y \cdot X - X \cdot Y)\psi.$$

Computing $\sum_{\alpha=1}^n e_\alpha \cdot R(X, e_\alpha) \cdot \psi$ now yields

$$\sum_{\alpha=1}^n e_\alpha \cdot R(X, e_\alpha)\psi = \mu^2 \sum_{\alpha=1}^n e_\alpha(e_\alpha X - X e_\alpha) \cdot \psi = 2(1-n)\mu^2\psi.$$

On the other hand, in Section 3.1 we proved the formula

$$\sum_{\alpha=1}^n e_\alpha \cdot R(X, e_\alpha)\psi = -\frac{1}{2}\text{Ric}(X) \cdot \psi.$$

Hence, $\text{Ric}(X) \cdot \psi = 4(n-1)\mu^2 X \cdot \psi$, and, since ψ does not vanish at any point, this implies

$$\text{Ric}(X) = 4(n-1)\mu^2 X.$$

Thus (M^n, g) is an Einstein space of scalar curvature $R = 4n(n-1)\mu^2$. The curvature tensor $R(X, Y)$ in the spinor bundle S is related to the curvature tensor $R(X, Y)Z$ of the Riemannian manifold (M^n, g) via the formula

$$R(X, Y)\psi = \frac{1}{4} \sum_{\alpha=1}^n e_\alpha \cdot R(X, Y)e_\alpha \cdot \psi.$$

Hence, because $4\mu^2 = \frac{R}{n(n-1)}$, the equation

$$\nabla_X \nabla_Y \psi - \nabla_Y \nabla_X \psi - \nabla_{[X, Y]} \psi = \mu^2(YX - XY) \cdot \psi$$

can also be written as

$$\left\{ \sum_{\alpha=1}^n e_\alpha \cdot R(X, Y)e_\alpha + \frac{R}{n(n-1)}(X \cdot Y - Y \cdot X) \right\} \psi = 0$$

and, for an Einstein space,

$$\sum_{\alpha=1}^n e_\alpha \wedge R(X, Y)e_\alpha + \frac{R}{n(n-1)}(X \wedge Y - Y \wedge X)$$

coincides with $W(X \wedge Y)$. This, eventually, implies

$$W(w^2) \cdot \psi = 0.$$

□

From the proof of the preceding proposition we can also deduce the following geometrical property of manifolds with Killing spinors:

Proposition. *A Riemannian spin manifold admitting a Killing spinor $\psi \neq 0$ with Killing number $\mu \neq 0$ is locally irreducible.*

Proof. If M^n is locally the Riemannian product $M^n = M_1^k \times M_2^{n-k}$, then we may consider vectors X, Y tangent to M_1^k and M_2^{n-k} , respectively. This implies $R(X, Y)Z = 0$, and from

$$\left\{ \sum_{\alpha=1}^n e_\alpha R(X, Y)e_\alpha + \frac{R}{n(n-1)}(XY - YX) \right\} \psi = 0$$

we obtain

$$R \cdot X \cdot Y \cdot \psi = 0.$$

Since $\mu \neq 0$, the scalar curvature is different from zero. Moreover, X and Y are orthogonal vectors. But this implies $\psi = 0$, hence a contradiction. □

The next-to-last proposition shows that Killing spinors are divided into two types, depending on whether the Killing number μ is real or imaginary ($\mu \neq 0$):

real Killing spinors	$\mu \in \mathbb{R}$	M^n is an Einstein space of pos. scalar curvature $R > 0$
imaginary Killing spinors	$\mu \in i \cdot \mathbb{R}$	M^n is an Einstein space of neg. scalar curvature $R < 0$

Since $R = 4n(n - 1)\mu^2$, real Killing spinors precisely correspond to eigen-spinors of the Dirac operator for the eigenvalue $\pm \frac{1}{2} \sqrt{\frac{n}{n-1}R}$. The field equation $\nabla_X \psi = \mu X \cdot \psi$ could be generalized by allowing $\mu : M^n \rightarrow \mathbb{C}$ to be a complex-valued function. However, due to a result by A. Lichnerowicz, this does not lead to an actual generalization:

Proposition. *Let (M^n, g) be a connected spin manifold, $\mu : M^n \rightarrow \mathbb{C}$ a smooth function and ψ a non-trivial solution of the equation*

$$\nabla_X \psi = \mu X \cdot \psi.$$

If the real part $\text{Re}(\mu) \neq 0$ is not identically zero, then μ is constant and real. Hence ψ is a real Killing spinor.

In low dimensions $n = \dim(M^n)$, the geometrical conditions for the existence of real or imaginary Killing spinors, respectively, are rather restrictive, and for $n \leq 4$ only Riemannian spaces of constant sectional curvature admit this kind of spinor fields. Consider e.g. the case $n = 3$. Then (M^3, g) is necessarily a 3-dimensional Einstein space, hence a space form. We meet the same situation in dimension $n = 4$.

Proposition. *Let (M^4, g) be a connected Riemannian spin manifold with a non-trivial Killing spinor ψ for the Killing number $\mu \neq 0$. Then (M^4, g) is a space of constant sectional curvature.*

Proof. Decompose the Killing spinor $\psi = \psi^+ + \psi^-$ according to the splitting of the spinor bundle, $S = S^+ \oplus S^-$. The equation for the Killing spinor then takes the form

$$\nabla_X \psi^+ = \mu X \psi^-, \quad \nabla_X \psi^- = \mu X \psi^+.$$

Define the set

$$N = \{m \in M^4 : \psi^+(m) = 0 \text{ or } \psi^-(m) = 0\}.$$

$N \subset M^4$ is a closed subset without inner points. Indeed, if N has inner points, then there is an open subset $U \subset N \subset M^n$ on which, e.g. ψ^+ vanishes, $\psi^+|_U = 0$. This implies $\nabla\psi^+|_U \equiv 0$ and, since $\mu \neq 0$, from the Killing equation we obtain $\psi^-|_U \equiv 0$. Hence $\psi = \psi^+ + \psi^-$ vanishes identically on the subset U , a contradiction to the fact proved above that non-trivial Killing spinors have no zeroes. Thus $U := M^4 \setminus N$ is a dense open subset of M^4 . The condition on the Weyl tensor W of M^4 now takes the form

$$W(w^2)\psi^+ = 0 \text{ and } W(w^2)\psi^- = 0.$$

However, a simple algebraic computation using the realization of the \mathcal{C}_4 -module $\Delta_4 = \Delta_4^+ \oplus \Delta_4^-$ explicitly given in Sections 1.3 and 1.5 proves the following fact.

If $\eta^2 \in \Lambda^2(\mathbb{R}^4)$ is a 2-form and $\psi^+ \in \Delta_4^+, \psi^- \in \Delta_4^-$ are two non-trivial spinors, then $\eta^2 \cdot \psi^+ = 0 = \eta^2 \cdot \psi^-$ implies that the 2-form η^2 is trivial, $\eta^2 = 0$.

Applying this, we immediately conclude that the Weyl tensor W vanishes on the set $M^4 \setminus N$. However, this set is dense. Hence (M^4, g) is a 4-dimensional Einstein space with vanishing Weyl tensor, i.e. a space of constant sectional curvature. \square

In view of this proposition, the search for necessary and sufficient conditions on a Riemannian space to have (real or imaginary) Killing spinors is interesting only in dimensions $n \geq 5$. There are extensive series of examples and studies, for which we refer to the book [BFGK] and the supplementary article [Bär] using the holonomy theory. In the case of real Killing spinors (e.g. in the important dimension $n = 7$) this classification problem has not been finally settled yet (compare the paper [FKMS]).

Seiberg-Witten Invariants

A.1. On the topology of 4-dimensional manifolds

The topology of manifolds looks back at a long history which began in the last century (Riemann). In the works of many mathematicians (Poincaré, Brouwer, Hopf, Morse etc.) during a first period lasting until the middle of the thirties of this century homological properties of manifolds were studied, the calculus of variations was developed and, in particular, complete proofs were given for the classification of compact 2-dimensional manifolds. The characteristic features in the topology of manifolds between 1935 and 1960 were the theory of characteristic classes (Whitney, Pontryagin), the calculation of the bordism ring (Thom) and the discovery of exotic differential structures (Milnor). In particular, it became obvious that in dimensions $n \geq 4$ the category $\text{Top}(n)$ of n -dimensional topological manifolds does not coincide with the category $\text{Diff}(n)$ of smooth manifolds, i.e. the natural mapping

$$\text{Diff}(n) \rightarrow \text{Top}(n)$$

forgetting the differential structure is neither injective nor surjective. In connection with the solution of the Poincaré conjecture in dimensions $n \geq 5$ and the proof of the h -cobordism theorem (Smale), the so-called surgery techniques were developed in the sixties (Wall, Browder, Novikov). They led to a far-reaching classification theory for certain classes of smooth compact manifolds in dimensions $n \geq 5$.

The situation in low dimensions, $n = 3, 4$, is rather special. On the one hand, the possible variety of forms is considerably larger already in dimension $n = 3$ than for surfaces, and the classification question is much more difficult. On the other hand, even in a 4-dimensional manifold there is not enough room to apply the surgery techniques which were so successful in higher dimensions. Dimension $n = 3$ turned out to be the last one in which Top and Diff coincide: every compact 3-dimensional manifold is triangulizable, every two triangulations are combinatorially equivalent and, moreover, every such manifold admits exactly one differential structure. The Poincaré conjecture in this dimension is still unsettled. Apart from the fundamental group $\pi_1(M^4)$, the integral intersection form $H^2(M^4; \mathbb{Z})$ is the most important invariant of an orientable compact 4-dimensional manifold. In 1982, M. Freedman proved that for simply connected compact topological 4-manifolds this intersection form almost determines the manifold itself in Top(4). In particular, each unimodular quadratic form over the ring \mathbb{Z} of integers can be realized as the intersection form of a compact simply connected topological manifold M^4 . On the other hand, it was already known for a long time (Rokhlin 1952) that if a unimodular form of even type can be realized by a closed smooth 4-manifold, then its signature is divisible by 16. Take, e.g., the positive definite unimodular \mathbb{Z} -form E_8 of dimension 8,

$$E_8 = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}.$$

E_8 is of even type and has signature 8, $\sigma(E_8) = \dim E_8 = 8$. Hence there exists a simply connected topological manifold M_0^4 with intersection form E_8 , and M_0^4 is by no means smooth, i.e. the mapping

$$\text{Diff}(4) \rightarrow \text{Top}(4)$$

is not surjective (and not injective as well). Thus the smooth topology in dimension 4 is already a completely different topic than the continuous.

Likewise, at the beginning of the eighties, S.K. Donaldson introduced a method for the study of Diff(4) based on associating with every smooth 4-dimensional manifold the moduli space of solutions of the self-dual Yang-Mills equation in a non-abelian gauge field theory as an invariant, and on deriving new invariants from it. In this way, he was able to exclude further unimodular quadratic forms over the ring \mathbb{Z} as intersection forms of smooth

simply connected and closed 4-manifolds M^4 . For example, if $H^2(M^4; \mathbb{Z})$ is positive definite, then this intersection form has to be trivial. In particular, $E_8 \oplus E_8$ cannot occur as the intersection form of a smooth manifold even though the Rokhlin condition $\sigma/16 \in \mathbb{Z}$ is satisfied. This obstruction eventually led to the proof that there exist exotic differential structures in \mathbb{R}^4 . On the other hand, using known algebraic surfaces, many unimodular \mathbb{Z} -forms can be realized as intersection forms. Computations for connected sums of $K3$ -surfaces with $(S^2 \times S^2)$ resulted in the conjecture that the form

$$2k(-E_8) \oplus m \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

does occur as the intersection form of a smooth 4-manifold if and only if $m \geq 3k$ (the so-called 11/8 conjecture; this inequality is equivalent to $b_2(M^4)/|\sigma(M^4)| \geq \frac{11}{8}$). Within the framework of Donaldson theory, this inequality could only be proved for small k . Another application of the invariants constructed by means of non-abelian gauge field theory concerns splitting questions. A compact simply connected complex surface S with $b_2^+(S) \geq 3$ cannot be smoothly represented as a connected sum $S = X_1 \# X_2$ with $b_2^+(X_i) > 0$. This led to the construction of different differential structures on compact simply connected 4-manifolds.

In autumn 1994, E. Witten suggested that all these results, and others reaching even further, could be obtained by considering the moduli space of a system of equations for a pair consisting of a spinor and an abelian connection. The spinor has to be harmonic with respect to the abelian gauge field and, on the other hand, algebraically related to the curvature form of the abelian connection (Seiberg-Witten equation). This system of equations is the 4-dimensional analogue to the 2-dimensional Ginzburg-Landau model (1950) of superconductivity. Witten's claim was elaborated by many mathematicians in the following months and turned out to be right. In contrast to non-abelian gauge field theories with their non-linear equations, one could now return to an abelian theory, and the analytical theory of smooth 4-dimensional topology gets drastically simplified !

The first observation forming the base of Seiberg-Witten theory is that each orientable compact 4-dimensional manifold M^4 has a $\text{spin}^{\mathbb{C}}$ structure (though, possibly, no spin structure!). Hence spinors may be defined on it. We briefly sketch a proof: The universal coefficient theorem implies the formula

$$H^3(M^4; \mathbb{Z}) = \{H_3(M^4; \mathbb{Z})/\text{Tor}(H_3(M^4; \mathbb{Z}))\} \oplus \text{Tor}(H_2(M^4; \mathbb{Z})),$$

and, from Poincaré duality, $H_2(M^4; \mathbb{Z}) = H^2(M^4; \mathbb{Z})$, we conclude the relations

$$\mathrm{Tor}(H^3(M^4; \mathbb{Z})) = \mathrm{Tor}(H_2(M^4; \mathbb{Z})) = \mathrm{Tor}(H^2(M^4; \mathbb{Z})).$$

Let $T \subset H^2(M^4; \mathbb{Z})$ be the torsion subgroup. Consider the exact sequence

$$\begin{array}{ccccccc} H^2(M^4, \mathbb{Z}) & \xrightarrow{2} & H^2(M^4, \mathbb{Z}) & \xrightarrow{r} & H^2(M^4, \mathbb{Z}_2) & & \\ & & \xrightarrow{\beta_*} & & H^3(M^4, \mathbb{Z}) & \xrightarrow{2} & H^3(M^4, \mathbb{Z}) \rightarrow \dots \end{array}$$

Then,

$$\mathrm{im}(\beta_*) = \{\alpha^3 \in \mathrm{Tor}(H^3(M^4; \mathbb{Z})) : 2\alpha^3 = 0\} \approx \{\gamma^2 \in T : 2\gamma^2 = 0\}.$$

The sequence $\{\gamma^2 \in T : 2\gamma^2 = 0\} \rightarrow T \xrightarrow{2} T \rightarrow T/2T$ is an exact sequence of \mathbb{Z}_2 -vector spaces. Thus, $\dim_{\mathbb{Z}_2}(T/2T) = \dim_{\mathbb{Z}_2}\{\gamma^2 \in T : 2\gamma^2 = 0\}$ and, since $r(T) = T/2T$, we obtain

$$\begin{aligned} \dim_{\mathbb{Z}_2}(H^2(M^4; \mathbb{Z}_2)) &= \dim_{\mathbb{Z}_2}(\mathrm{im}(r)) + \dim_{\mathbb{Z}_2}(\mathrm{im}(\beta_*)) \\ &= \dim_{\mathbb{Z}_2}(\mathrm{im}(r)) + \dim_{\mathbb{Z}_2}(r(T)). \end{aligned}$$

The following inclusion is obvious:

$$r(T) \subset \mathrm{im}(r) \subset H^2(M^4; \mathbb{Z}_2).$$

Consider $x \in \mathrm{im}(r)$ with $x = r(\alpha)$ and $\alpha \in H^2(M^4; \mathbb{Z})$. If $y \in r(T)$, then there exists an element $\beta \in T \subset H^2(M^4; \mathbb{Z})$ with $r(\beta) = y$. β is torsion and hence $\alpha \cup \beta$ in $H^4(M^4; \mathbb{Z}) \approx \mathbb{Z}$ vanishes. This in turn implies

$$x \cup y = 0 \quad \text{for } x \in \mathrm{im}(r), y \in r(T).$$

The set $\Gamma = \{\gamma \in H^2(M^4; \mathbb{Z}) : \forall y \in r(T) \gamma \cup y = 0\}$ thus contains $\mathrm{im}(r)$. On the other hand,

$$\dim_{\mathbb{Z}_2}(\Gamma) = \dim_{\mathbb{Z}_2}(H^2(M^4; \mathbb{Z}_2)) - \dim_{\mathbb{Z}_2}(r(T)) = \dim_{\mathbb{Z}_2}(\mathrm{im}(r))$$

by \mathbb{Z}_2 -Poincaré duality. Hence, $\mathrm{im}(r) = \Gamma$, i.e.

$$\mathrm{im}(r) = \{\gamma \in H^2(M^4; \mathbb{Z}) : \forall y \in r(T) \gamma \cup y = 0\}.$$

So we arrive at a more precise description of the image of the \mathbb{Z}_2 -reduction $r : H^2(M^4; \mathbb{Z}) \rightarrow H^2(M^4; \mathbb{Z}_2)$. We are going to use this to prove that M^4 has a spin^c structure. The necessary and sufficient condition to be considered is that the second Stiefel-Whitney class w_2 belongs to the image $\mathrm{Im}(r)$. We thus have to check whether $w_2 \cup y = 0$ for all $y \in r(T)$. By the Wu formulas for an orientable 4-dimensional manifold, w_2 is the only cohomology class $w_2 \in H^2(M^4; \mathbb{Z}_2)$ satisfying the condition $x^2 = w_2 \cup x$ for all $x \in H^2(M^4; \mathbb{Z}_2)$. Now for $y \in r(T)$ we have $y^2 = 0$, since y is the \mathbb{Z}_2 -reduction of a torsion element from $H^2(M^4; \mathbb{Z})$. This implies

$$w_2 \cup y = y^2 = 0$$

for all $y \in r(T)$, i.e. the second Stiefel-Whitney class $w_2(M^4)$ is the \mathbb{Z}_2 -reduction of an integral cohomology class. Altogether we obtain the

Proposition (Wu 1950; Hirzebruch and Hopf 1958). *Every compact orientable 4-dimensional manifold M^4 has a $\text{spin}^{\mathbb{C}}(4)$ structure.* \square

Next we will collect a few special formulas from 4-dimensional spin algebra. The Hodge operator $*$: $\Lambda^2(\mathbb{R}^4) \rightarrow \Lambda^2(\mathbb{R}^4)$ acting on 2-forms splits Λ^2 into the self-dual and the anti-self-dual 2-forms:

$$\Lambda^2(\mathbb{R}^4) = \Lambda_+^2(\mathbb{R}^4) \oplus \Lambda_-^2(\mathbb{R}^4).$$

The 4-dimensional spin representation Δ_4 also decomposes into $\Delta_4 = \Delta_4^+ \oplus \Delta_4^-$ with $\Delta_4^+ \cong \mathbb{C}^2$. The endomorphisms $e_i \cdot e_j : \Delta_4^+ \rightarrow \Delta_4^+$ induced by Clifford multiplication have the following matrices:

$$\begin{aligned} e_1 e_2 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, & e_1 e_3 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & e_1 e_4 &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \\ e_2 e_3 &= \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, & e_2 e_4 &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & e_3 e_4 &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}. \end{aligned}$$

In particular, as endomorphisms in Δ_4^+

$$e_1 e_2 - e_3 e_4 = 0, \quad e_1 e_3 + e_2 e_4 = 0, \quad e_1 e_4 - e_2 e_3 = 0.$$

For a spinor $\Phi \in \Delta_4^+$ we define a 2-form w^Φ by the formula

$$w^\Phi(X, Y) = \langle X \cdot Y \cdot \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2,$$

where $X, Y \in \mathbb{R}^4$. w^Φ is a 2-form with imaginary values. This results from the following calculations:

$$\begin{aligned} w^\Phi(X, Y) &= \langle X \cdot Y \cdot \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2 \\ &= \langle (-YX - 2\langle X, Y \rangle) \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2 \\ &= -\langle YX \Phi, \Phi \rangle - \langle X, Y \rangle |\Phi|^2 = -w^\Phi(Y, X), \end{aligned}$$

$$\begin{aligned} \overline{w^\Phi(X, Y)} &= \overline{\langle X \cdot Y \cdot \Phi, \Phi \rangle} + \langle X, Y \rangle |\Phi|^2 = \langle \Psi, X \cdot Y \cdot \Phi \rangle + \langle X, Y \rangle |\Phi|^2 \\ &= \langle Y \cdot X \Phi, \Phi \rangle + \langle X, Y \rangle |\Phi|^2 = w^\Phi(Y, X) = -w^\Phi(X, Y). \end{aligned}$$

Using the explicitly described spin representation, we easily obtain the proof of the next

Proposition.

- 1) Let $\Phi \in \Delta_4^+$ and $w^2 \in \Lambda_-^2$. Then, $w^2 \cdot \Phi = 0$.
- 2) $\langle w^\Phi \cdot \Phi, \Phi \rangle = -2|\Phi|^4$ and $|w^\Phi|^2 = 2|\Phi|^4$.

Proof. Setting $\Phi = \begin{pmatrix} \Phi_1 \\ \Phi_2 \end{pmatrix} \in \mathbb{C}^2 = \Delta_4^+$ yields

$$w^\Phi(e_1, e_2) = i\{|\Phi_1|^2 - |\Phi_2|^2\} = w^\Phi(e_3, e_4),$$

$$w^\Phi(e_1, e_3) = -\Phi_2\bar{\Phi}_1 + \Phi_1\bar{\Phi}_2 = -w^\Phi(e_2, e_4),$$

$$w^\Phi(e_1, e_4) = -i\Phi_2\bar{\Phi}_1 - i\Phi_1\bar{\Phi}_2 = w^\Phi(e_2, e_3),$$

and the formulas simply result from inserting these entities. □