Introduction

Why you want to read this book

Algebraic geometry is an old subject. Descartes' introduction in 1637 of coordinates in the plane and in space made it possible to relate the algebra of polynomials to the geometry of their zero loci. Gauss' proof of the fundamental theorem of algebra showed that the degree of a polynomial, an algebraic invariant, was equal to the number of roots counted with multiplicity, a geometric invariant. Mathematicians have been doing what is recognizably algebraic geometry for more than two centuries.

Within algebraic geometry, the study of algebraic curves is the oldest topic. Newton already classified all the possible types of real affine cubics. By the middle of the nineteenth century a rich theory of curves in the complex projective plane was a central topic, overturned by Riemann's work in mid-century — what became the theory of Riemann surfaces, which introduced new complex-analytic techniques to the field. This was taken up and made algebraic as a theory of plane curves by Alexander Brill, Max Noether, F. S. Macaulay and many others. By the end of the century Halphen and others were interested in the classification of space curves as well. The interested reader will find more about the early work on algebraic curves in the historical appendix to this book written by historian Jeremy Gray (page 353).

The development continues unabated today: in the second half of the twentieth century Grothendieck's foundations led to the solution of many classical problems and, in particular, to a firm foundation for the theory of moduli, allowing mathematicians to exploit the fact that algebraic varieties generally come in families parametrized by other algebraic varieties — something that we will return to often in this book. Algebraic geometry has merged more and more with number theory, and the moduli space of curves plays an important role in string theory, responding to the needs of physicists.

For these reasons, the subject of algebraic curves is one of the richest in algebraic geometry, if not in all of mathematics. If you want to know whether a conjecture is plausible, you can generally find well-understood special cases on which to test it. Some of the fundamental constructions of algebraic geometry, like the construction of moduli spaces and their description, can be carried out in the setting of algebraic curves with a degree of precision and detail far beyond what has been possible in higher dimensions.

Already the geometry of plane curves of degrees 3 and 4 shows some of the promise of the subject:

Every smooth plane cubic has exactly 9 flexes — points where the tangent line has contact of order 3 with the curves — and these points form a remarkable configuration; they're the only known example of a finite, noncollinear set of points in the plane such that the line joining any two contains a third. Generalizing the notion of the flexes of a plane cubic leads us to the subject of inflectionary behavior of linear series on curves in general, which arose separately in the theory of Weierstrass points on Riemann surfaces, and has become a powerful tool.

Every smooth quartic curve has exactly 28 bitangents, also forming a beautiful and mysterious configuration; see Figures 6.4 and A.5. Salmon [1852, p. 197] computed the number (315) of 4-tuples of bitangents whose eight points of tangency lie on a conic. The extension of these ideas to curves of higher genus leads us to the rich theory of theta characteristics, bringing together algebra (in the form of a bilinear pairing on the points of order 2 in the Jacobian of a curve), analysis (in the form of theta-functions) and of course projective geometry.

The richest subject in what is arguably the richest branch of mathematics[1] — of course you want to read this book!

Why we wrote this book

The wealth of beauty, both in theory and in examples, certainly makes the study of algebraic curves an attractive prospect. But it comes at a price: to absorb in detail all the things we've learned over the centuries about algebraic curves would take years, if not decades. This is, in essence, the conundrum facing anyone who undertakes to write a book on the subject: how to convey the wealth of information (and the many many ways in which our knowledge is incomplete) without writing an encyclopedia. We have chosen to try to be useful and broad but not necessarily complete.

[1] Number theorists may quibble…

When we introduce a technique or a construction without full proofs, we do so as a "cheerful fact:"[2]

I'm very well acquainted, too, with matters mathematical,
I understand equations, both the simple and quadratical,
About binomial theorem I am teeming with a lot o' news,
With many cheerful facts about the square of the hypotenuse.

— Gilbert and Sullivan, *Pirates of Penzance*, Major General's Song

Our intended audience is a graduate student considering working in the field of algebraic curves, or a researcher in a related field whose work has led them to questions about algebraic curves. Our goal is to equip the reader with the understanding of both the techniques and the state of our knowledge necessary to read the current literature and work on open problems.

What's with the title?

> Be simple by being concrete. Listeners are prepared to accept unstated (but hinted) generalizations much more than they are able, on the spur of the moment, to decode a precisely stated abstraction and to re-invent the special cases that motivated it in the first place.

–Paul Halmos, *How to Talk Mathematics*

This book aims to present those ideas and methods from the theory of algebraic curves that are used *in practice* by mathematicians working in a variety of fields of mathematical research.

Although mathematicians aspire to understand their subjects deeply, we feel that we learn in stages: in early stages we accept large and difficult results as black boxes and explore the rich examples that they yield. That is how we have tried to organize this book: We begin with two chapters that we hope will bridge the gap between first courses in algebraic geometry/commutative algebra at the level of Fulton's [1969] or Reid's [1988] well-known books, and the professional language of invertible sheaves, cohomology and linear series. An ideal background would be Hartshorne's textbook [1977] or Vakil's about to be published book [2023], but very much less will suffice if the reader is willing to accept some advanced ideas or look them up at leisure; we have tried to give precise references where this might be required. Subsequent chapters roughly alternate between expositions of basic techniques (partly without proofs) and families of examples, treated in detail.

[2] Hats off to the "Many Cheerful Facts" seminar run by the U. C. Berkeley graduate students, which gave us this idea!

What's in this book

In organizing this book we faced a common problem of mathematical exposition:

> We are dealing here with a fundamental and almost paradoxical difficulty. Stated briefly, it is that learning is sequential but knowledge is not. A branch of mathematics... consists of an intricate network of interrelated facts, each of which contributes to the understanding of those around it. When confronted with this network for the first time, we are forced to follow a particular path, which involves a somewhat arbitrary ordering of the facts.
> –Robert Osserman [2011]

In Chapters 1 and 2 we lay out the central objects of algebraic curve theory: invertible sheaves, linear series, canonical sheaves and the Riemann–Roch theorem, as well as Hurwitz's theorem, the adjunction formula and some elementary facts about the geometry of surfaces. We prove or sketch the proofs of many of these basic results. These chapters may serve as review for someone who has been exposed already to material roughly equivalent to Chapter IV of [Hartshorne 1977].

Thereafter, we alternate between chapters focused on special cases and chapters developing more of the theory. Chapter 3 describes the geometry of curves of genus 0 in projective space. We emphasize some of the things that make rational normal curves so special, and take the opportunity to introduce the conditions implying that a curve is *arithmetically Cohen–Macaulay*.

In Chapter 4 We begin by explaining the Riemann–Roch theorem and its consequences for smooth plane curves, in the style of the late nineteenth century, computing the canonical series and showing algorithmically how to compute the complete linear series of effective divisors linearly equivalent to a given (not necessarily effective) divisor, and we use this to describe the low-degree linear series on curves of genus 1, and demonstrate how counts of parameters suggest the presence of a moduli space.

Because most curves cannot be represented as smooth plane curves, a general treatment requires dealing with singular curves and their normalizations. This requires somewhat more algebraic technique, so we postpone the general case to Chapter 15.

In genus ≥ 2 there is a fundamental shift: not all invertible sheaves of a given degree on a curve C of genus $g \geq 2$ are congruent modulo the automorphism group of the curve. It is a salient feature of algebraic geometry that families of similar algebro-geometric objects are often naturally parametrized by algebraic varieties. This idea goes back to the beginning of algebraic geometry and the family of plane curves of degree d, but was only fully clarified in the work of Grothendieck, Deligne and Mumford.

Chapter 5 introduces our first examples of this phenomenon: the *fine moduli spaces* of divisors on a curve, as well as the Jacobian, for which we give the classic analytic construction, and the Picard varieties $\mathrm{Pic}_d(C)$ parametrizing isomorphism classes of invertible sheaves. We also introduce the subvarieties $W^r_d(C)$ parametrizing invertible sheaves with many sections.

Since the Jacobian is irreducible, we can speak meaningfully of a *general* invertible sheaf, and of the dimension of various families of special invertible sheaves. With this in hand we prove that every curve of genus $g > 1$ can be embedded in $\mathbb{P}^3$ as a curve of degree $g + 3$.

Equipped with information about the Jacobian, we proceed in Chapter 6 to study curves of genus 2 and 3, describing in particular how the geometry of a map of the curve to projective space, given by sections of an invertible sheaf, depends on the sheaf. We show how a hyperelliptic curve of genus g can be described by a set of $2g + 2$ points in $\mathbb{P}^1$, and we describe the canonical maps. Again, this suggests the existence of moduli spaces.

We spend Chapters 7 and 8 on other moduli spaces as they appear in the world of curves, though we do not give complete proofs of all the assertions made. We start in Chapter 7 with some central examples of fine moduli spaces, primarily the Hilbert scheme. Using properties of the Hilbert scheme we show that curves of genus ≥ 2 can have only finitely many automorphisms, and more generally that there can only be finitely many morphisms from one such curve to another.

The most interesting example of a moduli space, the moduli of smooth (or stable) curves of genus g, is not a fine moduli space, and we spend Chapter 8 describing what it is and isn't. In that chapter we also describe the Hurwitz space of coverings and the Severi variety of nodal plane curves.

After this, it's back to examples: in Chapter 9 we analyze aspects of the geometry of curves of genus 4 and 5.

In the following two chapters we take up the properties of the points of a general hyperplane sections of a curve in projective space. In Chapter 10 we give Rathmann's proof that the points of a general hyperplane section of a reduced irreducible curve are in linearly general position (independent of the characteristic). Consequences include Castelnuovo's bound on the maximal genus of curves of a given degree. We show that the curves that achieve the maximum genus, called Castelnuovo curves, are arithmetically Cohen–Macaulay. These include all plane curves, as well as canonical curves and linearly normal curves of high degree compared to their genera.

We also use the linear general position result to prove the strong forms of Clifford's and Martens' theorems on special linear series; to show that every curve has a nodal plane model; and to show that a general invertible sheaf of degree $g+2$ on a curve of genus g maps the curve to a nodal curve in the plane — unless the curve is hyperelliptic, in which case the image has a single multiple point of multiplicity g.

In characteristic 0 an even stronger statement than linearly general position is true: the monodromy of the hyperplane divisors is the full symmetric group. We take up this and some other monodromy questions in Chapter 11 and prove consequences for secant planes and for sums of linear series.

In Chapter 12 we return to the general question: "What linear series exist on curves of genus g?" We give three possible interpretations of this question, and the answers to each: Clifford's theorem, the Castelnuovo bound, and the Brill–Noether theorem, postponing the proof of the latter. We apply the Brill–Noether result to explore various special classes of curves of genus 6.

Chapter 13 prepares for the proof of the Brill–Noether theorem with a discussion of inflection points of linear series, generalizing the flexes of plane curves. The Plücker formula counts the number of inflection points, with appropriate weights. In this chapter we explain the remarkable connection between the study of inflections on a rational curve and the Schubert calculus of cycles in the Grassmannian. In Chapter 14 we use this material, together with a degeneration to cuspidal rational curves, to give a relatively short proof of the Brill–Noether theorem.

In Chapter 15 we take the classical point of view of Brill and Noether, and describe an explicit algorithm for finding the complete linear series on a smooth curve, using a singular plane model. This involves the result classically known as the completeness of the adjoint linear series. For the general case we use a simple case of the theory of dualizing sheaves, introduced in more generality in the next chapter.

Returning to curves in $\mathbb{P}^3$ in Chapter 16 we explain the theory of Hartshorne and Rao classifying curves up to linkage. This theory applies to all purely 1-dimensional subschemes of $\mathbb{P}^3$, and to explain this we detour to discuss dualizing sheaves and Grothendieck's $f^!$ operation.

Curves that lie on a quadric in $\mathbb{P}^3$ are easy to understand. In Chapter 17 we systematically describe a natural generalization: rational normal scrolls and the curves that lie on them. This includes all the Castelnuovo curves of high degree.

Chapter 18 presents some aspects of the theory of syzygies of the homogeneous ideals of curves and the famous — and, as of this writing, open — conjecture of Mark Green that connects syzygies of canonical curves to the Clifford index and the theory of rational normal scrolls. A table reproduced from work of Frank-Olaf Schreyer shows the sensitivity of the numerical information in the free resolution of a canonical curve to questions of the existence of special linear series on the curve.

Chapter 19 is in some ways the culmination of the book. It is concerned with *Hilbert schemes*, schemes $\mathcal{H}_{g,3,d}$ that parametrize smooth curves of genus g and degree d in $\mathbb{P}^3$. We work out examples up to degree 7, define the "principal component" — the only one dominating the moduli space of curves — and derive dimension estimates from deformation theory and from the Brill–Noether theorems using most of the ideas we have introduced.

The historical appendix written by Jeremy Gray (page 353) is a survey of some of the work on algebraic curves up to that of Brill and Noether, and we are grateful to him for allowing us to include it.

Exercises and hints. There are many exercises, clumped at the end of the chapters. Hints, and occasionally solutions, are given at the end of the book for exercises (or parts thereof) marked with ◆.

Relation of this book to other texts. Chapter IV of [Hartshorne 1977] has a similar flavor to that of this book, and contains details of most of the results in our Chapters 1 and 2. A beautiful (and brief) account of a number of topics in a style we particularly admire is found in [Mumford 1975].

A far more extensive treatment, partially overlapping that of the present book and containing many other topics, can be found in the important and encyclopaedic [Arbarello et al. 1985] and [Arbarello et al. 2011], and yet even these works do not cover all the major topics in the field. One of the topics we do not cover is the construction of the moduli space of curves, which can be found in [Arbarello et al. 2011]. Though not a complete account, [Harris and Morrison 1998] deals directly with this topic.

There are more elementary accounts of some of our material, in [Fulton 1969] and [Walker 1950] (who goes farther than we do into local resolution of singularities) as well as [Griffiths 1989]. The book [Kunz 2005] treats plane curves and their normalizations via the theory of valuations, and contains a detailed account of the ideas we treat in Chapter 15 from this more algebraic point of view. There is a comprehensive treatment of the local topological theory of plane curves and their singularities in [Brieskorn and Knörrer 1986]. The topological questions there are developed in different directions in [Milnor 1968] and [Eisenbud and Neumann 1985]. An idiosyncratic collection of interesting topics is presented in [Clemens 2003].

The Riemann surface point of view is well represented in the books [Forster 1981], [Gunning 1966], [Hulek 1995], [Kirwan 1992], and [Miranda 1995].

Prerequisites, notation and conventions

The reader should be familiar with the (Krull) dimension of rings and varieties, and their primary decomposition at the level of [Atiyah and Macdonald 1969]. Ideally the reader will already have some familiarity with the geometry of curves and surfaces at the level of Chapter IV and the beginning of Chapter V of [Hartshorne 1977], though our summary of the necessary material in the first two chapters of this book may suffice for the intrepid.

Unless otherwise mentioned, we assume that the ground field is the field of complex numbers $\mathbb{C}$, though much of what we do could be done over any algebraically closed field.

Commutative algebra. All the rings we consider are commutative with unit and Noetherian.

Since we are working over a field of characteristic 0, we use the terms smooth and nonsingular interchangeably.

Some results that we use:

Theorem 0.1 (Lasker's theorem). *If $f_1, \dots, f_c \subset \mathbb{C}[x_0, \dots, x_n]$ generates an ideal of codimension c, then the ideal $(f_1, \dots, f_c)$ is unmixed (all its primary components have codimension c).*

Theorem 0.2. *If R is a domain that is a finitely generated algebra over a field or a localization of such an algebra, then the normalization (= integral closure) of R is a finitely generated R-module. If R is 1-dimensional, then its normalization is nonsingular.*

Projective geometry. Schemes are assumed quasiprojective, and *varieties* (including curves) are reduced and irreducible schemes unless otherwise stated. "Points" will always be closed points unless we say otherwise. **Note to the patient reader: We usually use the term curve to refer to a smooth irreducible projective purely 1-dimensional scheme, but in some of the later chapters we explicitly allow more general 1-dimensional schemes.**

Though we occasionally use the classical topology, the term "open set" refers to the Zariski topology unless otherwise stated.

We adopt the Grothendieck convention that points of projective space $\mathbb{P}(V)$ are 1-dimensional quotients of V, or hyperplanes in V, so that $\mathrm{Sym}(V)$ is the homogeneous coordinate ring of $\mathbb{P}(V)$ and lines in V correspond to points of $\mathbb{P}(V^*)$.

However we write $G(k, V)$ for the variety of k-dimensional linear subspaces of V, so that in particular $G(1, V) = \mathbb{P}(V^*)$. We also write $\mathbb{G}(k, r)$ for k-dimensional projective subspaces of $\mathbb{P}^r$.

The results we assume are well represented by the following classical theorems:

Theorem 0.3 (Bézout's theorem)**.** *If $X, Y \subset \mathbb{P}^r$ are subvarieties satisfying the condition* $\operatorname{codim}(X \cap Y) = \operatorname{codim} X + \operatorname{codim} Y$*, then* $\deg(X \cap Y) = \deg X \deg Y$.

Theorem 0.4 (Bertini's theorem)**.** *If $X \subset \mathbb{P}^r$ is a nonsingular quasiprojective variety, and $\{H_\lambda \mid \lambda \in \Lambda\}$ is a linear family of hyperplanes of $\mathbb{P}^r$, then for an open subset of $\lambda \in \Lambda$ the scheme $H_\lambda \cap X$ is nonsingular away from the union of the* base locus $\bigcap_{\lambda\in\Lambda} H_\lambda$ *and the singular locus of X.*

Theorem 0.5 (main theorem of elimination theory)**.** *Any morphism $\phi : X \to Y$ of projective varieties* (*or schemes*) *is closed: if $X' \subset X$ is a Zariski closed subset, then $\phi(X') \subset Y$ is also closed.*

Corollary 0.6. *If $\phi : C \to D$ is a nonconstant morphism of* (*projective*) *curves, then ϕ is finite and surjective.*

If D is a smooth curve then the local ring of D at any point is a discrete valuation ring, so any torsion free module is flat. Thus:

Proposition 0.7. *If $\phi : C \to D$ is a nonconstant morphism of smooth curves, then ϕ is finite, surjective, and flat.*

If $X \subset \mathbb{P}^r$ is any scheme, we define the *homogeneous coordinate ring* of $X \subset \mathbb{P}^r$ to be $R_X = R_{X/\mathbb{P}^r} := S/I(X)$, where $S = \mathbb{C}[x_0, \dots, x_r]$ is the homogeneous coordinate ring of $\mathbb{P}^r$. We emphasize that, unlike the coordinate ring of an affine variety, this is not an intrinsic invariant of X, but depends on the embedding in $\mathbb{P}^r$.

When X is reduced and irreducible we write $\kappa(X)$ for the field of rational functions on X. In particular, if p is a closed point we write $\kappa(p) \cong \mathbb{C}$ for the residue class field at p.

Sheaves and cohomology. Some familiarity with coherent sheaves is recommended; a possible source is the first chapter of [Eisenbud and Harris 2000]. As for cohomology, it is probably enough if the reader can write down H^i and exact sequences without blushing. In any case we review some of the theory of coherent sheaves and their cohomology theory, and that of divisors on projective varieties, in the first two Chapters.

We occasionally use the bijection between algebraic and analytic sheaves on smooth projective curves, which preserves cohomology and exact sequences. This is a special case of the results in [Serre 1955/56].

If $\mathcal{F}$ is a sheaf on X and $X \subset Y$ then we will identify $\mathcal{F}$ with the sheaf usually written $\iota_*\mathcal{F}$, where $\iota : X \to Y$ is the inclusion map, and thus regard $\mathcal{F}$ as a sheaf on Y as well. The cohomology $H^i(X, \mathcal{F}) = H^i(Y, \iota_*\mathcal{F})$ canonically, so we will simply and unambiguously write $H^i(\mathcal{F})$ for either of these. We write $h^i(\mathcal{F})$ or (if D is a divisor) $h^i(D)$ for $\dim_{\mathbb{C}} H^i(\mathcal{F})$ or $\dim_{\mathbb{C}} H^i(\mathcal{O}_X(D))$.

If $\mathcal{F}$ is a sheaf on projective space (perhaps supported on a subvariety) we write $H^i_*(\mathcal{F})$ for $\bigoplus_{m\in\mathbb{Z}} H^i(\mathcal{F}(m))$.

Chapter 1

Linear series and morphisms to projective space

We start by laying out four major definitions: divisors, linear series, invertible sheaves, and maps to projective space. These ideas are used throughout this book. In the last section we explore some special cases, culminating with the condition for an invertible sheaf to provide an embedding of a curve in projective space.

We prove only some of our assertions; a reader who wants to see all the proofs should keep handy a copy of [Hartshorne 1977] or the equivalent. A more experienced reader, could instead skip ahead to Chapter 3.

As an analytic space, a complex projective smooth curve is a compact Riemann surface, a compact 1-dimensional complex manifold. In this sense its local structure is trivial, but its global structure can be hard to visualize. Any two Riemann surfaces with the same genus[1] are C^∞ isomorphic, but except for genus 0, where the sphere has a unique complex structure, there are continuous families of nonisomorphic global structures. The differences among Riemann surfaces or algebraic curves are revealed in the geometry of their

[1]The notion of genus was introduced by Riemann in his great paper [1857], though he never used the term. Instead, he defined the *connectivity* (Zusammenhang) of a Riemann surface X to be the number of disjoint closed curves (one-dimensional real submanifolds) that can be drawn on X such that the complement of their union is connected. The term genus (Geschlecht) was first used by Clebsch. Both uses were subtly different from their modern counterparts: whereas we speak of genus as a property of a given surface, Riemann and Clebsch used this idea as a way of partitioning the family of all Riemann surfaces into groups. For a discussion, see [Lê 2020].

maps to projective spaces. Throughout this book we will study curves in this way. It turns out that in characteristic 0 the algebraic and complex analytic theories are equivalent.

If U is a Zariski open set of a projective variety, then a *regular* function on U is a rational function whose denominator does not vanish on U. Because the only global regular functions on a projective variety are the constant functions, interesting maps must be described and studied using a different approach, and a large part of this chapter is devoted to the necessary machinery for doing this. The idea is simple: a point in projective space is the intersection of the hyperplanes containing it, so a map $\phi : C \to \mathbb{P}^r$ from a curve to projective space can be described set-theoretically by the set of preimages of these hyperplanes. In other words, a point $p \in C$ is sent to the point that is the intersection of those hyperplanes whose preimages contain p, as in Figure 1.1.

Some hyperplanes in $\mathbb{P}^r$ will be tangent to the image $\phi(C)$ at a point $\phi(p)$, and in that case the point p should be counted with higher multiplicity in the preimage of that hyperplane; in this way we arrive at a notion of divisor as a sum of points with multiplicities.

1.1. Divisors

On a smooth projective curve C, we define a *divisor* to be a finite formal sum of points of C with integer coefficients, compactly written as $\sum_{p\in C} m_p \cdot p$ with all but finitely many $m_p = 0$. The coefficient m_p is called the *multiplicity* of the point p in the divisor D; if all coefficients m_p are nonnegative we say that D is *effective*. Thus the group of divisors $\operatorname{Div} C$ is the free abelian group whose generators are the points of C.

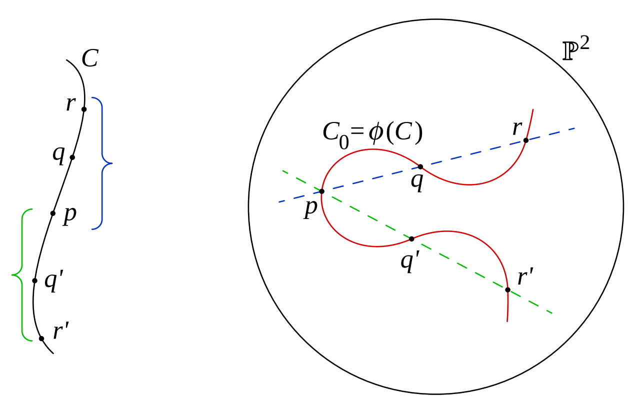

Figure 1.1. Divisors of the linear series corresponding to the map ϕ from C to $\mathbb{P}^2$ are preimages of the hyperplanes — in this case the dashed lines — in the plane. The image of p is determined by the intersection of the two lines.

The *degree* of $D = \sum_{p \in C} m_p \cdot p$ is by definition the sum $\sum m_p$ of its coefficients. Because the curve is smooth each local ring is a discrete valuation ring, so two elements of $\mathcal{O}_{C,p}$ with the same order of vanishing at p generate the same ideal in the local ring. Thus an effective divisor can be identified with a finite subscheme of C, and vice versa. If $D = \sum m_p \cdot p$ and $E = \sum n_p \cdot p$ are any two divisors, we write $E \geq D$ to mean $n_p \geq m_p$ for all $p \in C$.

1.2. Divisors and rational functions

Let C be a smooth projective curve, $U \subset C$ an open subset and $f \in \mathcal{O}_C(U)$ a regular function, not identically zero. At every point $p \in U$, we define the *order* of f at p, denoted $\mathrm{ord}_p(f)$, to be the highest power of the maximal ideal $m_p \subset \mathcal{O}_p$ containing f, and we define the divisor (f) associated to the function f by

$$(f) := \sum_{p \in U} \mathrm{ord}_p(f) \cdot p.$$

More generally, if h is any rational function on C, given locally as a quotient f/g of regular functions f and g, we define the divisor of h to be

$$(h) := (f) - (g);$$

this is independent of the choice of f and g. The divisor of a rational function h is called a *principal divisor*; principal divisors form a subgroup $\mathrm{Div}_0(C)$ of the group $\mathrm{Div}(C)$ of all divisors. We say that two divisors D and E on C are *linearly equivalent* if their difference $D - E$ is a principal divisor. The group $\mathrm{Div}(C)/\mathrm{Div}_0(C)$ of linear equivalence classes of divisors is called the *Picard group* of C, denoted $\mathrm{Pic}\, C$.

Linear equivalence is important in the description of a map $C \to \mathbb{P}^r$ by divisors because in $\mathbb{P}^r$ any two hyperplanes, defined by the vanishing of linear forms $\ell_1 = 0$ and $\ell_2 = 0$, differ by the globally defined rational function ℓ_1/ℓ_2. The preimages of these hyperplanes differ by the pullback of this rational function, and are thus linearly equivalent divisors.

Generalizations. We can generalize the notion of a divisor on a smooth curve in two respects.

First, we can extend the notion to higher-dimensional smooth varieties X, defining a divisor to be a finite formal linear combination of irreducible subvarieties of codimension 1 in X. In this setting, most of the notions introduced above have natural analogues; for example, a divisor is called *effective* if all the irreducible varieties in its expression as a sum appear with nonnegative coefficients. We can similarly define the divisor of a rational function on X, and the notion of linear equivalence of divisors. (One notion that does not have an obvious analogue is the degree of a divisor.)

We can also extend the notion to possibly singular varieties X, though this requires more care. We can mimic the definition above, defining a divisor to be a finite formal linear combination of irreducible subvarieties of codimension 1 in X; this is called a *Weil divisor* on X. But for most purposes we have to introduce a slightly different notion of divisor, called a *Cartier divisor*. We first define an *effective Cartier divisor* to be a subscheme locally defined on some open covering U_i by the vanishing of one nonzerodivisor in $\mathcal{O}_X(U_i)$; and a *Cartier divisor* to be a difference of effective Cartier divisors.

A simple example will serve to illustrate this. Suppose $C \subset \mathbb{A}^2$ is a plane curve with a node at a point $p = (0,0) \in C$. The point p itself (that is, the reduced subscheme) is *not* a Cartier divisor, since its ideal (x, y) is not generated by a single element. Likewise, the *fat point* — that is, the subscheme $\Gamma = V(x^2, xy, y^2) \subset C$ — is not Cartier. But a subscheme of degree 2 supported at p,

$$\Gamma_{\alpha,\beta} := V(\alpha x + \beta y, x^2, xy, y^2) \subset C,$$

— is in general Cartier, since it is defined locally by the equation $\alpha x + \beta y = 0$. For distinct lines L the schemes Γ_L are also distinct, since $\Gamma_L \subset L$. The two exceptions, when Γ_L is not Cartier, occur when the line L is tangent to either branch of C at the node. In these cases the scheme $\Gamma_{\alpha,\beta}$ is not a Cartier divisor on C: the intersection $L \cap C$ has multiplicity 3 at p, and is not equal to $\Gamma_{\alpha,\beta}$. For all these assertions see Exercise 1.7 at the end of the chapter; the reader not already familiar with the notion of Cartier divisor is encouraged to carry out the verifications.

Here is another way to characterize Cartier divisors on a scheme X: for each open affine set $U \subset X$ we define $K_X(U)$ to be the field of fractions of $\mathcal{O}_X(U)$, and write K_X for the associated sheaf. We write $\mathcal{O}_X^*$ and K_X^* for the sheaves of groups of units in $\mathcal{O}_X(U)$ and $K_X(U)$. A Cartier divisor on X is then by definition a global section of $K_X^*/\mathcal{O}_X^*$. Moreover, if X is a *normal* scheme (so that its local rings at the generic points of codimension-1 subvarieties are discrete valuation rings) then Cartier divisors are the same as locally principal Weil divisors. See [Hartshorne 1977, Section II.6] for more information.

In this book, we deal for the most part with smooth curves; even when a singular curve C_0 arises, it will be viewed as the image of its normalization C and its geometry analyzed in terms of that of C. Thus for the most part we will be dealing with divisors in the simplest setting of smooth curves. But there will be occasions when we want to extend our analysis to singular curves or to higher-dimensional varieties, and in those cases the notion of Cartier divisor is called for.

Divisors of functions. Returning to the case of smooth curves, we have:

Theorem 1.1. *Let C be a smooth projective curve. If $f \in K(C)$, the degree of the divisor (f) is 0. Thus any two linearly equivalent divisors on C have the same degree.*

Proof. The result is evident on $\mathbb{P}^1$, where a rational function is the ratio of two forms of the same degree. In general, a rational function ϕ on a smooth curve C defines a map $\pi : C \to \mathbb{P}^1$, such that

$$(\phi) = \pi^{-1}(0) - \pi^{-1}(\infty).$$

If $C \to D$ is any map of smooth projective curves, then restricting to an affine open subset D' of D and its preimage $C' \subset C$ the map is represented by a homomorphism of rings $\mathcal{O}_D(D') \to \mathcal{O}_C(C')$ in such a way that $\mathcal{O}_C(C')$ becomes a finitely generated module over $\mathcal{O}_D(D')$, which is torsion-free because C is reduced and irreducible. Since D is smooth, $\mathcal{O}_D(D')$ is a Dedekind domain, so $\mathcal{O}_C(C')$ is free of constant rank equal to the degree of the extension of the corresponding fields of rational functions $\kappa(D) \subset \kappa(C)$. This implies that the degree of the fibers is constant.

Returning to the case of the map $C \to \mathbb{P}^1$ defined by a rational function, we see that $\pi^{-1}(0)$ and $\pi^{-1}(\infty)$ have the same degree, so (ϕ) has degree 0 as required. □

In particular, we see that if $C \to \mathbb{P}^r$ is a morphism then the pullbacks of different hyperplanes have the same degree, and we define this to be the *degree of the morphism*.

It follows from Theorem 1.1 that we can write $\operatorname{Pic} C$ as a disjoint union

$$\operatorname{Pic} C = \bigsqcup_{d \in \mathbb{Z}} \operatorname{Pic}_d C,$$

where $\operatorname{Pic}_d C$ is the set of linear equivalence classes of divisors of degree d.

Invertible sheaves. To deal with families of linearly equivalent divisors we will use the language of invertible sheaves.

Recall first that a *coherent sheaf* $\mathcal{L}$ on X may be defined by specifying

- an open affine cover $\{U_i\}$ of X;
- for each i, a finitely generated $\mathcal{O}_X(U_i)$-module L_i;
- for each i, j, an isomorphism $\sigma_{i,j} : L_i|_{U_i \cap U_j} \to L_j|_{U_i \cap U_j}$ satisfying the compatibility conditions $\sigma_{j,k}\sigma_{i,j} = \sigma_{i,k}$.

A *global section* of $\mathcal{L}$ is a family of elements $t_i \in L_i$ such that $\sigma_{i,j} t_i = t_j$. Such a section may be realized as the image of the constant function 1 under a homomorphism of sheaves $\mathcal{O}_X \to \mathcal{L}$.

A coherent sheaf $\mathcal{L}$ is said to be *locally free* if the modules L_i are all free; when X is irreducible, the ranks of the free modules L_i are all the same, and this is called the *rank* of $\mathcal{L}$. An *invertible sheaf* is a locally free coherent sheaf of rank 1; that is, $L|_U \cong \mathcal{O}_U$, for every open set in some covering of X.

Invertible sheaves form a group: the tensor product $\mathcal{L}_1 \otimes_{\mathcal{O}_X} \mathcal{L}_2$ of two invertible sheaves is again an invertible sheaf. The inverse of an invertible sheaf $\mathcal{L}$ is the dual invertible sheaf $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ (Proof: Evaluation of functions defines a natural map $\mathcal{F} \otimes \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{O}_X)$, and when $\mathcal{F} = \mathcal{L}$ is locally free of rank 1 this is locally an isomorphism.) Invertible sheaves can be defined from divisors on a smooth scheme (or more generally from Cartier divisors on any scheme). In case C is a smooth projective curve, this is concrete: if $D = \sum m_p \cdot p$, then we define the sections of $\mathcal{O}_C(D)$ on any open set U by

$$\mathcal{O}_C(D)(U) := \left\{\text{rational functions } f \mid \operatorname{ord}_p(f) + m_p \geq 0 \text{ for all } p \in U\right\}.$$

Thus, at a point p with $m_p > 0$, we are allowing f to have a pole of order at most m_p; if $m_p \leq 0$ then f must be regular at p, with a zero of order at least $-m_p$.

More generally, we can associate an invertible sheaf to any Cartier divisor on a scheme X. First, if $D \subset X$ is an effective Cartier divisor (and thus locally defined by the vanishing of a nonzerodivisor) then D corresponds to a subscheme of X whose ideal sheaf $\mathcal{I}_{D/X}$ is invertible. We define $\mathcal{O}_X(-D)$ to be the ideal sheaf $\mathcal{I}_{D/X}$, and define $\mathcal{O}_X(D)$ to be its inverse. The dual of the inclusion of $\mathcal{O}_X(-D) \subset \mathcal{O}_X$ is a homomorphism $\sigma := \mathcal{O}_X \to \mathcal{O}_X(D)$, which we regard as a global section $\sigma \in H^0(\mathcal{O}_X(D))$, the image of $1 \in H^0(\mathcal{O}_X)$. It vanishes precisely along D.

For a general (Cartier) divisor $D = E - F$, we define

$$\mathcal{O}_X(D) := \mathcal{O}_X(E) \otimes \mathcal{O}_X(F)^{-1}.$$

If D and E are linearly equivalent divisors on C — that is, there is a rational function f on C with $(f) = D - E$ — then multiplication by f defines an isomorphism $\mathcal{O}_C(E) \to \mathcal{O}_C(D)$. Thus the isomorphism class of the invertible sheaf $\mathcal{O}_C(D)$ corresponds to the linear equivalence class of D. We will see in Corollary 2.2 that every invertible sheaf on a projective scheme has this form.

If $\sigma \in H^0(\mathcal{L})$ is a global section of an invertible sheaf on X and $p \in X$ is a point, then the *value* $\sigma(p)$ *of* σ *at* p is the image of σ under the natural map $H^0(\mathcal{L})$ to the fiber $\kappa(p) \otimes \mathcal{L}_p \cong \mathbb{C}$. Since the isomorphism is not canonical, σ does not define a function on X at p; but since any two isomorphisms differ by a unit in $\mathcal{O}_{X,p}$, the vanishing locus of σ, denoted $(\sigma)_0$, is a well-defined subscheme of X.[2] Moreover, if X is reduced and irreducible, the ratio σ/τ of two global sections of the same invertible sheaf is a well-defined rational function $\sigma(p)/\tau(p)$ at all the points where the denominator $\tau(p)$ is not 0, so the divisor class of $(\sigma)_0$ is independent of the choice of $\sigma \in H^0(\mathcal{L})$.

[2]When we say that σ vanishes at p, or $\sigma(p) = 0$, we mean that the image of σ in the stalk $\mathcal{L}_p = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_{X,p}$ is in the maximal ideal $\mathfrak{m}_{X,p}$ of $\mathcal{O}_{X,p}$ times $\mathcal{L}_p$, and *not* that σ is zero as an element of the stalk $\mathcal{L}_p$ itself. Similarly, we say that σ vanishes to order m at p if $\sigma(p)$ lies in $\mathfrak{m}_{X,p}^m \mathcal{L}_p$.

The tensor product of (rational) sections σ of $\mathcal{L}$ and σ' of $\mathcal{L}'$ is a rational section of $\mathcal{L} \otimes \mathcal{L}'$ whose divisor is the sum of the divisors of σ and σ'. Thus the group of divisor classes $\operatorname{Pic} X$ is naturally isomorphic to the group of invertible sheaves under $\otimes$, and we will identify the two.

Invertible sheaves and line bundles. If $\mathcal{L}$ is an invertible sheaf on a variety X and $p \in X$ is a point, the *stalk* $\mathcal{L}_p$ of $\mathcal{L}$ is isomorphic to the local ring $\mathcal{O}_{X,p}$. We write $\mathfrak{m}_{X,p}$ for the maximal ideal of $\mathcal{O}_{X,p}$. By definition the *fiber* of $\mathcal{L}$ is

$$\kappa(p) \otimes \mathcal{L} = \mathcal{L}_p / \mathfrak{m}_{X,p} \mathcal{L}_p \cong \mathbb{C},$$

where $\kappa(p) := \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}$ is the residue field of X at p. It is not hard to prove that the collection of these fibers forms a line bundle on X; that is, a morphism of schemes $L \to X$ whose fibers have the structure of 1-dimensional vector spaces such that $L|_{U_i} \cong U_i \times \mathbb{C}^1$ for some open covering $\{U_i\}$ of X.

Given a line bundle L on X, we can recover an invertible sheaf $\mathcal{L}$ associated to L by defining $\mathcal{L}(U)$ to be the set of sections of L defined over U. These two processes are inverse to one another, and allow us to think of invertible sheaves and line bundles interchangeably.

Though we generally use the invertible sheaf terminology, there are at least two points in which the line bundle approach is more natural. First, the vanishing of a section of an invertible sheaf at a point p is genuinely the vanishing of the section of the line bundle as a function. Second, and more serious, given a morphism $f : Y \to X$ of schemes, the pullback $f^*(\mathcal{L})$ of an invertible sheaf $\mathcal{L}$ on X is defined as the tensor product of $\mathcal{O}_Y$ with a sort of naive pullback, whereas the pullback of a line bundle is a straightforward set-theoretic operation.

Example 1.2 (Invertible sheaves on $\mathbb{P}^r$)**.** Since $\mathbb{C}[x_0, \dots, x_r]$ is a unique factorization domain, the ideal of any codimension 1 subvariety of $\mathbb{P}^r$ is generated by one nonzero element, which is thus a nonzerodivisor — it is a hypersurface. As we explained above, any two hypersurfaces of degree d differ by the divisor of a rational function, so the group of divisor classes on $\mathbb{P}^r$ is $\mathbb{Z}$. In other words, the class of a divisor is defined by its degree. Thus if $D = V(F) \subset \mathbb{P}^r$ is a hypersurface defined by the vanishing of a form F of degree d, it is natural to use the name $\mathcal{O}_{\mathbb{P}^r}(d)$ for $\mathcal{O}_{\mathbb{P}^r}(D)$.

If D is an effective divisor other than 0, then $H^0(\mathcal{O}_{\mathbb{P}^r}(-D)) = 0$, since there are no globally defined functions vanishing on D except 0.

To compute $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ directly, let $D = z_1 + z_2 + \cdots + z_d$ be a divisor of degree d and suppose that the coordinates are chosen so that none of the z_i are at infinity. The sections of $\mathcal{O}_{\mathbb{P}^1}(D)$ are the rational functions with poles in $\mathbb{P}^1$

only at the z_i. Identifying $\mathbb{P}^1 \setminus \{\infty\} = \mathbb{A}^1$ with $\mathbb{C}$ these can each be written as

$$\frac{g(z)}{(z - z_1)(z - z_2) \cdots (z - z_d)},$$

where g is a polynomial. The condition that the point at infinity is not a pole is the condition $\deg g \leq d$. With this condition, these rational functions form a vector space of dimension $d + 1$.

More generally, because every rational function on $\mathbb{P}^r$ has degree 0, and any two global sections differ by a rational function, every global section of $\mathcal{O}_{\mathbb{P}^r}(d)$ vanishes on a divisor of degree d. Thus we may identify $H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ with the $\binom{r+d}{r}$-dimensional vector space of forms of degree d on $\mathbb{P}^r$.

Putting this together for future reference we have:

Proposition 1.3. *Every invertible sheaf $\mathcal{L}$ on $\mathbb{P}^r$ has the form $\mathcal{L} \cong \mathcal{O}_{\mathbb{P}^r}(m)$ for a unique $m = \deg \mathcal{L} \in \mathbb{Z}$; and we have*

$$H^0(\mathcal{O}_{\mathbb{P}^r}(m)) = \mathbb{C}[x_0, \dots, x_r]_m,$$

the space of forms of degree m in $r + 1$ variables.

1.3. Linear series and maps to projective space

We will use invertible sheaves to describe maps of a given variety X to projective space. For this we add the notion of linear series (sometimes called linear system).

Definition 1.4. A *linear series* on a scheme X is a pair $\mathcal{V} = (\mathcal{L}, V)$ where $\mathcal{L}$ is an invertible sheaf on X and V is a nonzero vector space of global sections of $\mathcal{L}$. We defa *dimension* of the linear series to be

$$\dim \mathcal{V} := \dim_{\mathbb{C}} V - 1.$$

To every global section σ of an invertible sheaf $\mathcal{L}$ on a variety X we can associate an effective divisor $(\sigma) = (\sigma)_0$ defined by the vanishing of σ. If τ is a scalar multiple of σ, it has the same divisor; and if $H^0(\mathcal{O}_X) = \mathbb{C}$ (for example if X is reduced, connected, and projective) then the converse is true: two sections of $\mathcal{L}$ with the same divisor differ by multiplication by a scalar.

We sometimes write $|\mathcal{L}|$ for the complete linear series $(\mathcal{L}, H^0(\mathcal{L}))$. If $\mathcal{L} = \mathcal{O}(D)$ we often write this as $|D|$.

Thus a linear series $\mathcal{V} = (\mathcal{L}, V)$ gives rise to a family of effective divisors on X, all in the same linear equivalence class, parametrized by the projective space $\mathbb{P}V^*$ of nonzero $\sigma \in V$ mod scalars. This is indeed the way we think of a linear series: as a family of divisors parametrized by a projective space. The definition of the dimension of a linear series reflects this: it's not the dimension of V as a vector space, but the dimension of the corresponding projective space.

Similarly, we speak of “the divisors of a linear series $\mathcal{V}$” or as divisors “moving” in a linear series.

The intersection of the vanishing loci of all the sections in V is called the *base locus* of $\mathcal{V}$. It is in general a subscheme of C. The points in its support are called *basepoints* of $\mathcal{V}$. If the vector space V equals $H^0(\mathcal{L})$, the linear series is said to be *complete*; from the viewpoint of the *family of divisors*, this is the same as saying the linear series includes every effective divisor in the linear equivalence class.

Suppose that C is a smooth projective curve. Since the divisors in a linear series are linearly equivalent, they all have the same degree, called the degree of the linear series. If $\mathcal{V}$ is a linear series of degree d and dimension r we say that *$\mathcal{V}$ is a g^r_d*. This classical language means that $\mathcal{V}$ represents a group of d points moving within a linear equivalence class with r degrees of freedom.[3]

The next result is fundamental:

Theorem 1.5. *For any scheme X there is a natural bijection between the set of nondegenerate morphisms $\phi : X \to \mathbb{P}^r$ modulo PGL_{r+1} and base-point free linear series of dimension r on X up to isomorphism.*

Here *nondegenerate* means the image of the morphism ϕ is not contained in any hyperplane. The phrase “modulo PGL_{r+1}” is needed because the notation $\mathbb{P}^r$ supposes a choice of projective coordinates, and PGL_{r+1} is the group of linear coordinate transformations (actually all automorphisms, by Exercise 1.4). To get a correspondence without the dependence on a basis we could think of morphisms to $\mathbb{P}V$, the set of 1-quotients of V.

Suppose that $(\mathcal{L}, V)$ is a linear series on a smooth projective curve C that does have a base locus D_0. We can then subtract D_0 from all the divisors of the linear series, replacing $\mathcal{L}$ by $\mathcal{L}(-D_0)$ and dividing each section in V by the section σ of $\mathcal{O}_C(D_0)$ vanishing on D_0. This yields a new, base-point free linear series of the same dimension but lower degree.

We prove Theorem 1.5 by describing the correspondence in both directions:

From morphisms to linear series. Let $f : X \to \mathbb{P}^r$ be any nondegenerate morphism. The associated linear series $\mathcal{V} = (\mathcal{L}, V)$ on X has $\mathcal{L} = f^*\mathcal{O}_{\mathbb{P}^r}(1)$, the pullback of the invertible sheaf $\mathcal{O}_{\mathbb{P}^r}(1)$, and

$$V = f^*H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \subset H^0(\mathcal{L}).$$

In geometric terms, if we think of a linear series as a family of effective divisors, this is the linear series on X consisting of preimages of hyperplanes in $\mathbb{P}^r$. Nondegeneracy assures us that the preimage of a hyperplane in $\mathbb{P}^r$ is indeed a divisor on X.

[3]This “g” is unrelated to the genus; it’s short for *gruppi di punti*, Italian for “divisors.”

From linear series to morphisms. Suppose that X is any scheme, and $\mathcal{V} = (\mathcal{L}, V)$ is a base-point free linear series of dimension r on X; we want to describe a corresponding morphism $f : X \to \mathbb{P}^r$. Choose a basis $\sigma_0, \dots, \sigma_r$ for V. If we let $D_i = (\sigma_i) \subset X$ be the divisor of zeroes of σ_i and set $U_i \coloneqq X \setminus D_i$, the ratio σ_j/σ_i is a regular function on U_i, and we can define a map $f_i : U_i \to \mathbb{P}^r$ by

$$f_i : p \mapsto \Big(\frac{\sigma_0}{\sigma_i}(p), \dots, \frac{\sigma_r}{\sigma_i}(p)\Big),$$

where the component $\sigma_i/\sigma_i = 1$ ensures that not all the components are 0, so that the image point is well-defined. The maps f_i and f_j agree on the overlap $U_i \cap U_j$, and by the hypothesis that $\mathcal{V}$ is base-point free the U_i cover X; so together they define a regular map f from X to $\mathbb{P}^r$.

We can describe this map set-theoretically without having to choose a basis: since $\mathcal{V}$ is assumed base-point free, for any point $p \in X$ the subspace $H_p \coloneqq \{\sigma \in V \mid \sigma(p) = 0\}$ is a hyperplane in V; thus we get a map $f : X \to \mathbb{P}V$.

Example 1.6. The morphism from $\mathbb{P}^r$ defined by the complete linear series $(\mathcal{O}_{\mathbb{P}^r}(d), H^0(\mathcal{O}_{\mathbb{P}^r}(d)))$ has target $\mathbb{P}^{\binom{r+d}{r}-1}$, and takes a point $(a_0, \dots a_r)$ to the point whose coordinates are all the monomials of degree d in $x_0, \dots x_r$. It is called the *d-th Veronese morphism* from $\mathbb{P}^r$. For example, on $\mathbb{P}^1$ this has the form

$$(x_0, x_1) \mapsto (x_0^d,\ x_0^{d-1}x_1,\ \dots, x_1^d).$$

The image of $\mathbb{P}^1$ under this morphism is called the *rational normal curve* of degree d; in the case $d = 2$ is the *plane conic*, and in the case $d = 3$ it is called the *twisted cubic*. Veronese himself studied the image of $\mathbb{P}^2$ by the Veronese morphism of degree 2 now simply called the *Veronese surface*.

1.4. The geometry of linear series

An upper bound on $h^0(\mathcal{L})$. We will develop sophisticated ways of estimating the dimensions of linear series. We begin with an elementary bound:

Theorem 1.7. *Let C be a smooth projective curve. If $\mathcal{L}$ is an invertible sheaf of degree $d \geq 0$ on C, then $h^0(\mathcal{L}) \leq d + 1$; and equality holds if and only if $C \cong \mathbb{P}^1$.*

Proof. Let $p_1, \dots, p_{d+1}$ be points of C. If $h^0(\mathcal{L})$ exceeded $d+1$, there would be a nonzero section $\sigma \in H^0(\mathcal{L})$ vanishing at $p_1, \dots, p_{d+1}$; the divisor (σ) would then have degree $\geq d+1$, contradicting the hypothesis that $\deg \mathcal{L} = d$.

For the second part, suppose that $h^0(\mathcal{L}) = d + 1$. If $p_1, \dots, p_d$ are points of C, there is a nonzero section $\sigma \in H^0(\mathcal{L})$ vanishing at $p_1, \dots, p_d$; by degree considerations, it cannot vanish anywhere else. It follows that any two divisors of degree d on C are linearly equivalent, hence that any two points $p, q \in C$ are linearly equivalent. Thus there is a rational function f on C with exactly one zero and one pole, giving an isomorphism $C \cong \mathbb{P}^1$. □

From the correspondence between invertible sheaves and maps to projective space, we now get:

Corollary 1.8. *If $C \subset \mathbb{P}^d$ is a nondegenerate curve, then the degree of C is at least d, with equality only in the case that C is a rational normal curve.* ☐

Rational normal curves arise often in the literature because they have many extremal properties, such as those of Corollaries 1.8 and 1.9. For a related result see Corollary 13.3. Since there is only one invertible sheaf of degree d on $\mathbb{P}^1$, any two rational normal curves of degree d differ by a transformation in PGL_{d+1}, and we will therefore often speak of *the* rational normal curve.

Corollary 1.9. *Let $C \subset \mathbb{P}^d$ be the rational normal curve of degree d. If E is an effective divisor on C of degree $e \leq d+1$, then the* span *of E (that is, the smallest linear space containing the subscheme E) has dimension $e-1$.*

Less formally: any finite set or subscheme of C is as linearly independent as possible.

Proof. Since E imposes at most e conditions on hyperplanes, the span of E has dimension at most $e-1$.

On the other hand, the hyperplanes containing E meet C in a divisor of the form $E+E'$, where $\deg E' = d-e$. Thus the projection of C from E is a nondegenerate curve of degree $d-e$ in $\mathbb{P}^{d-(\dim \operatorname{span} E)-1}$, so from Corollary 1.8 we get $d-e \geq d - \dim \operatorname{span} E - 1$, as required. ☐

In the case of distinct points on a rational normal curve it is easy to make a direct argument why they are as independent as possible: Choose coordinates so that none of the points are at infinity. We can identify the points $\lambda_1, \ldots, \lambda_{d+1} \in C \cong \mathbb{P}^1$ with distinct complex numbers, and the independence (for $\ell = d+1$) is equivalent to the the nonvanishing of the Vandermonde determinant

$$\begin{vmatrix} 1 & \lambda_1 & \lambda_1^2 & \cdots & \lambda_1^d \\ 1 & \lambda_2 & \lambda_2^2 & \cdots & \lambda_2^d \\ \vdots & & & & \vdots \\ 1 & \lambda_{d+1} & \lambda_{d+1}^2 & \cdots & \lambda_{d+1}^d \end{vmatrix} = \prod_{1 \leq i < j \leq d+1} (\lambda_j - \lambda_i).$$

Incomplete linear series. In classical algebraic geometry a linear series of dimension 1 is called a *pencil*,[4] a linear series of dimension 2 is called a *net* and, less commonly, a three-dimensional linear series is called a *web*. We will use only the first of these terms.

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[4]This usage harks back to the early meaning of "pencil" in English, an artist's fine brush, borrowed before 1400 from (old) French "pincel" and (late) Latin "pincellus". In the geometric sense the term initially referred to a set of lines (light rays, in a 1665 example attested in the OED) meeting in a point.

The morphism associated to an incomplete linear series $V \subset H^0(\mathcal{L})$ is the composition of the morphism associated to the complete linear series $|\mathcal{L}|$ with a linear projection. In general, if $V \subset W \subset H^0(\mathcal{L})$ are nested linear series, then a 1-dimensional quotient of W restricts to a 1-dimensional quotient of V unless it vanishes on V. Thus we have a partially defined linear morphism $\pi : \mathbb{P}W \to \mathbb{P}V$. The *indeterminacy locus* of the map consists of the set of 1-quotients vanishing on V, that is, $\mathbb{P}(W/V) \subset \mathbb{P}W$; we will call it the *center of the projection* π. (It is sometimes useful to think of the dual picture: lines in W^* map to lines in V^* except when they lie in the subspace $(W/V)^* = \operatorname{Ann} V \subset W^*$.) Thus there is a commutative diagram

$$\begin{array}{ccc} & & \mathbb{P}W^* \\ & \overset{\phi_W}{\nearrow} & \Big\downarrow{\scriptstyle \pi} \\ C & \xrightarrow{\phi_V} & \mathbb{P}V^*. \end{array}$$

If W is base-point free, then V is base-point free if and only if the center of the projection π is disjoint from $\phi_W(C)$. We then say that π is *regular* on C. An example is illustrated in Figure 1.2.

By way of language, we will say that a curve $C \subset \mathbb{P}^r$ embedded by a complete linear series $|\mathcal{L}|$ is *linearly normal*; this is equivalent to saying that the pullback map

$$H^0(\mathcal{O}_{\mathbb{P}^r}(1)) \to H^0(\mathcal{L})$$

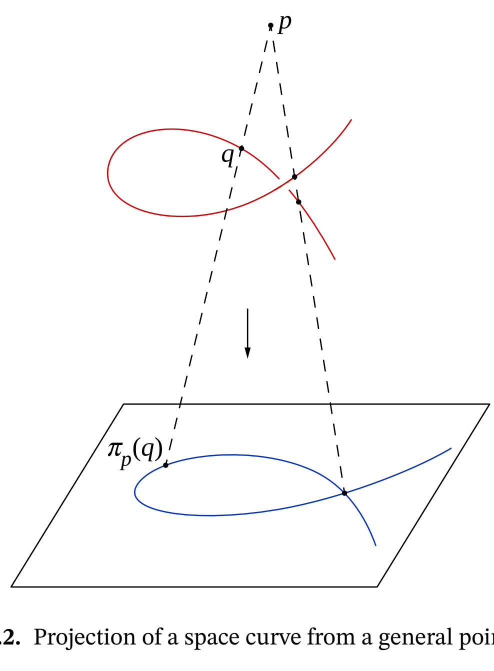

Figure 1.2. Projection of a space curve from a general point p to $\mathbb{P}^2$.

is surjective. Since regular projections of a curve correspond to subseries, this is equivalent to saying that C is *not* the regular projection of a nondegenerate curve $\tilde{C} \subset \mathbb{P}^{r+1}$.

Sums of linear series. If $\mathcal{D} = (\mathcal{L}, V)$ and $\mathcal{E} = (\mathcal{M}, W)$ be two linear series on a curve C. By the *sum* $\mathcal{D} + \mathcal{E}$ of $\mathcal{D}$ and $\mathcal{E}$ we will mean the pair

$$\mathcal{D} + \mathcal{E} = (\mathcal{L} \otimes \mathcal{M}, U)$$

where $U \subset H^0(\mathcal{L} \otimes \mathcal{M})$ is the subspace generated by the image of $V \otimes W$, under the multiplication/cup product map $H^0(\mathcal{L}) \otimes H^0(\mathcal{M}) \to H^0(\mathcal{L} \otimes \mathcal{M})$. In other words, it's the subspace of the complete linear series $|\mathcal{L} \otimes \mathcal{M}|$ spanned by divisors of the form $D + E$, with effective divisors $D \in \mathcal{D}$ and $E \in \mathcal{E}$.

The sum of two base-point free linear series is clearly base-point free; if the two series correspond to maps $\phi : C \to \mathbb{P}^r$ and $\psi : C \to \mathbb{P}^s$ then the sum corresponds to the composition of the map $(\phi, \psi) : C \to \mathbb{P}^r \times \mathbb{P}^s$ with the *Segre embedding* $\sigma : \mathbb{P}^r \times \mathbb{P}^s \to \mathbb{P}^{(r+1)(s+1)-1}$ given in coordinates by

$$((x_0, \dots, x_r), (y_0, \dots, y_s)) \mapsto (x_0 y_0, \dots, x_i y_j, \dots, x_r y_s).$$

Note that this says nothing about the dimension of $\mathcal{D} + \mathcal{E}$, since the composite map $\sigma \circ (\phi, \psi)$ will typically not be nondegenerate. In fact, the best we can do in general is the following proposition.

Proposition 1.10. *If $\mathcal{D}$ and $\mathcal{E}$ are two linear series that contain effective divisors on a curve C, then*

$$\dim(\mathcal{D} + \mathcal{E}) \geq \dim \mathcal{D} + \dim \mathcal{E}.$$

Proof. Saying that $\dim \mathcal{D} \geq m$ is equivalent to saying that we can find a divisor $D \in \mathcal{D}$ containing any given m points of C; since $\mathcal{D} + \mathcal{E}$ contains all pairwise sums $D + E$ with $D \in \mathcal{D}$ and $E \in \mathcal{E}$, we can certainly find a divisor $F \in \mathcal{D} + \mathcal{E}$ containing any given $\dim \mathcal{D} + \dim \mathcal{E}$ points of C. □

Which linear series define embeddings? A linear series $\mathcal{V} = (\mathcal{L}, V)$ on a projective variety is called *very ample* if it is base-point free and defines an embedding (what Hartshorne calls a "closed immersion"). If D is a Cartier divisor on X, then we say that D is *very ample* if the complete linear series $|D|$ is very ample, and we say that D is *ample* if mD is very ample for some integer $m > 0$.

Similarly, $\mathcal{V}$ or D are called *birationally very ample* if V or D are base-point free and define a map that is generically one-to-one or, equivalently, an embedding when restricted to an open set. This arises often when speaking of a map from a smooth curve C onto a curve C_0 in the plane, since the latter frequently has singularities, as in Figure 1.2.

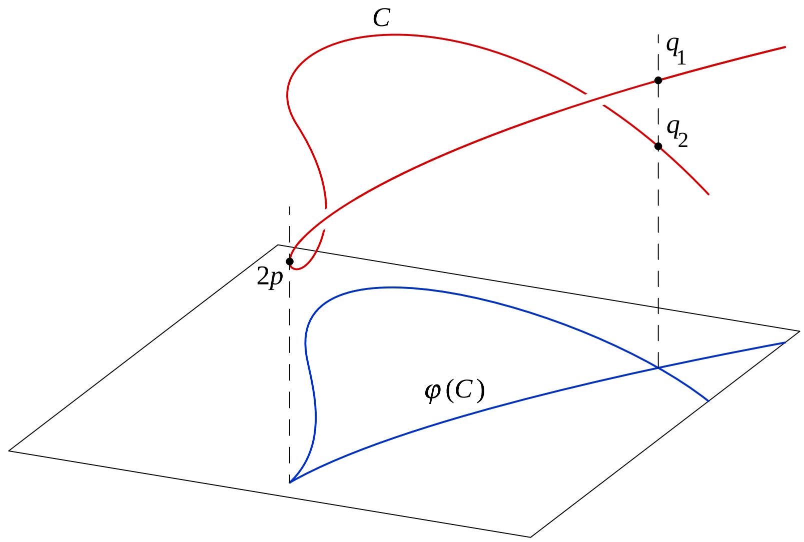

Figure 1.3. Two distinct points or one double point might impose just one condition on the linear series defined by projection from a point a (here illustrated by vertical projection).

Given a linear series $\mathcal{V} = (\mathcal{L}, V)$ and an effective divisor D on C, we set $\mathcal{V}(-D) = (\mathcal{L}(-D), V(-D))$ where

$$\mathcal{L}(-D) := \mathcal{L} \otimes \mathcal{O}(-D) \text{ and } V(-D) := \{\sigma \in V \mid \sigma(D) = 0\}.$$

The difference $\dim \mathcal{V} - \dim \mathcal{V}(-D)$ is called the *number of conditions imposed by D on the linear series $\mathcal{V}$*; we say that *D imposes independent conditions* on $\mathcal{V}$ if $\dim \mathcal{V} - \dim \mathcal{V}(-D) = \deg D$.

Via the correspondence of Theorem 1.5, statements about the geometry of a morphism $\phi : C \to \mathbb{P}^r$ can be formulated as statements about the relevant linear series. In the case of complete series, these are statements about the vector space $H^0(\mathcal{L})$ of global sections of $\mathcal{L}$. We write $h^0(\mathcal{L})$ for the vector space dimension of $H^0(\mathcal{L})$ (and similarly for other cohomology groups). It is useful to have criteria in these terms for when a linear series defines an embedding, or even to be base-point free so that it defines a morphism:

Proposition 1.11. [Hartshorne 1977, Theorem IV.3.1] *Let $\mathcal{L}$ be an invertible sheaf on a smooth curve C. The complete linear series $|\mathcal{L}|$ is base-point free if and only if*

$$h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1 \quad \forall p \in C;$$

and $\mathcal{L}$ is very ample if and only if

$$h^0(\mathcal{L}(-p - q)) = h^0(\mathcal{L}) - 2 \quad \forall p, q \in C.$$

Proof. First, if $\mathcal{L}$ is base-point free, then vanishing at a point imposes one linear condition on sections of $\mathcal{L}$, so $h^0(\mathcal{L}(-D)) \geq h^0(\mathcal{L}) - \deg D$ for any effective divisor D.

To say that $|\mathcal{L}|$ is base-point free means that for every point $p \in C$ there is a section of $\mathcal{L}$ that does not vanish at p; thus vanishing at p is a nontrivial linear condition on $H^0(\mathcal{L})$. Conversely, if $h^0(\mathcal{L}(-p)) = h^0(\mathcal{L}) - 1$ then p imposes a nontrivial condition, so some section of $\mathcal{L}$ does not vanish at p.

Since a divisor of degree d cannot impose more than d conditions on a linear series, the statement $h^0(\mathcal{L}(-p-q)) = h^0(\mathcal{L}) - 2$ for all p, q implies the condition for base-point freeness; and saying that $\phi_{\mathcal{L}}(p) \neq \phi_{\mathcal{L}}(q)$ implies that the linear series defines a set-theoretic injection.

Let $\phi : C \to \mathbb{P}^r$ be the map defined by $\mathcal{L}$. To say that ϕ is an embedding locally at a point $p \in C$, we need to know in addition that the map of local rings

$$\phi^* : \mathcal{O}_{\mathbb{P}^r,\phi(p)} \to \mathcal{O}_{C,p}$$

is surjective.

Since C is projective, the map $C \to \phi(C)$ is finite, so ϕ^* makes $\mathcal{O}_{C,p}$ into a finitely generated $\mathcal{O}_{\mathbb{P}^r,\phi(p)}$-module. By Nakayama's lemma it suffices to show that $\mathcal{O}_{C,p}/\phi^*(\mathfrak{m}_{\mathbb{P}^r,\phi(p)})$ is generated by the image of $\mathcal{O}_{\mathbb{P}^r,\phi(p)}/\mathfrak{m}_{\mathbb{P}^r,\phi(p)} = \mathbb{C}$.

Since the constants $\mathbb{C} = \mathcal{O}_{\mathbb{P}^r,\phi(p)}/\mathfrak{m}_{\mathbb{P}^r,\phi(p)}$ pull back to the constants in $\mathcal{O}_{C,p}/\mathfrak{m}_{C,p}$, surjectivity will follow if

$$\frac{\mathcal{O}_{C,p}}{\phi^*(\mathfrak{m}_{\mathbb{P}^r,\phi(p)})\mathcal{O}_{C,p}}$$

is 1-dimensional; that is, if the linear series contains a section that vanishes to order exactly 1 at p, which is equivalent to the condition that $h^0(\mathcal{L}(-2p)) \neq h^0(\mathcal{L}(-p))$. □

A more geometric version of the last part of the proof would be to say that the condition of the existence of a section vanishing to order exactly 1 implies that ϕ is an injection on the tangent space to C at p. This implies that the map is analytically an isomorphism onto its image, locally on the source; and together with the finiteness and set-theoretic injectivity of the map, this suffices. Figure 1.3 illustrates the two ways in which a linear series on a smooth curve can fail to be very ample.

If $\phi : X \to \mathbb{P}^r$ is a generically finite morphism, then the *degree* of ϕ is the number of points in the preimage of a general point of $\phi(X)$. It follows that if $D := \sum_{p\in C} n_p p$ is a divisor on a smooth curve, and the linear series $|D|$ is base-point free, then the degree of the morphism associated to $|D|$ is $\deg D := \sum_{p\in C} n_p$.

Exercises

Exercise 1.1. Show that there is no nonconstant morphism $\mathbb{P}^r \to \mathbb{P}^s$ when $s < r$ by showing that any nontrivial linear series of dimension $< r$ on $\mathbb{P}^r$ has a nonempty base locus. ◆

Exercise 1.2. Let

$$V = \langle s^{a_0}t^{d-a_0}, \ldots, s^{a_r}t^{d-a_r}\rangle \subset \mathbb{C}[s,t]_d,$$

and let $\mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), V)$. Determine the conditions for each of the following properties to hold:

(1) $\mathcal{V}$ is base-point free.

(2) $\mathcal{V}$ is ample.

(3) $\mathcal{V}$ is very ample.

(4) $\mathcal{V}$ is complete.

Exercise 1.3. Extend the statement of Proposition 1.11 to incomplete linear series; that is, prove that the morphism associated to a linear series $(\mathcal{L}, V)$ on a smooth curve is an embedding if and only if

$$\dim\big(V \cap H^0(\mathcal{L}(-p-q))\big) = \dim V - 2 \quad \forall p, q \in C.$$

Exercise 1.4. Show that an automorphism of $\mathbb{P}^r$ takes hyperplanes to hyperplanes. Deduce that it is given by the linear series $\mathcal{V} = (\mathcal{O}_{\mathbb{P}^r}(1), H^0(\mathcal{O}_{\mathbb{P}^r}(1)))$, and use this to show that $\operatorname{Aut}\mathbb{P}^r = \operatorname{PGL}(r+1)$. ◆

Exercise 1.5. Let $C \subset \mathbb{P}^r$ be any linearly normal curve and $\phi : C \to C$ an automorphism. Show that ϕ is induced by an automorphism of $\mathbb{P}^r$ if and only if ϕ carries the invertible sheaf $\mathcal{O}_C(1)$ to itself; that is, $\phi^*(\mathcal{O}_C(1)) \cong \mathcal{O}_C(1)$. In this case we say that the automorphism is projective. Show that every automorphism of a rational normal curve $C \subset \mathbb{P}^d$ extends to $\mathbb{P}^d$. Since the automorphism group PGL_2 of $\mathbb{P}^1$ acts transitively on $\mathbb{P}^1$, we say that C is *projectively homogeneous*.

Exercise 1.6. Show that the ring $\mathbb{C}[s^d, s^{d-1}t, \ldots, t^d]$ is normal (= integrally closed) by noting that its integral closure must be contained in $\mathbb{C}[s,t]$ and then showing that if f is any polynomial in the integral closure then the homogeneous components of f are also in the integral closure.

Exercise 1.7. Suppose $C = V(y^2 - x^2 - x^3) \subset \mathbb{A}^2$ (so that C is a plane curve with a node at the point $p = (0,0) \in C$).

(1) Show that the point p itself (that is, the reduced subscheme) is not a Cartier divisor, since its ideal (x, y) is not generated by a single element.

(2) Show that the fat point, that is, the subscheme $\Gamma = V(x^2, xy, y^2)$, is contained in C but is not a Cartier divisor on C.

(3) Let $\Gamma_{\alpha,\beta} \coloneqq V(\alpha x + \beta y, x^2, xy, y^2) \subset C$. Show that if $\beta \neq \pm\alpha$ then $\Gamma_{\alpha,\beta}$ is a Cartier divisor on C.

(4) With $\Gamma_{\alpha,\beta}$ as above, show that if $\beta = \pm\alpha$ then $\Gamma_{\alpha,\beta}$ is not a Cartier divisor on C. ◆

Chapter 3

Curves of genus 0

We begin our project of describing curves in projective space with the simplest case, that of genus 0. Even here, there are interesting statements to make about the geometry of their embeddings in $\mathbb{P}^r$, and there are many open problems. Though we won't treat such questions, curves of genus 0 over other fields pose their own interesting problems: the study of such curves over $\mathbb{Q}$ was a major part of Gauss's work.

Using Theorem 1.7 and the Riemann–Roch theorem, we can show that (over an algebraically closed field) $\mathbb{P}^1$ is the only curve of arithmetic genus 0:

Corollary 3.1. *Every reduced irreducible projective curve C of arithmetic genus 0 over an algebraically closed field is isomorphic to $\mathbb{P}^1$.*

Proof. The curve C is smooth since otherwise its normalization would have negative genus. By the Riemann–Roch theorem, any linear series $\mathcal{L}$ of degree d on C has $h^0(\mathcal{L}) \geq d+1$, so we may use Theorem 1.7 to conclude that $C \cong \mathbb{P}^1$. □

Even images of genus 0 curves must have genus 0. The following proof works as well if we replace $\mathbb{C}$ by any algebraically closed field.

Theorem 3.2 [Lüroth 1875]. (1) *If $C \to D$ is a nonconstant map of reduced, irreducible projective curves, then the geometric genus of C must be at least that of D. In particular, if C has geometric genus 0, then so does D.*

(2) *If K is a field with $\mathbb{C} \subsetneq K \subset \mathbb{C}(x)$ then $K = \mathbb{C}(y)$ for some $y \in \mathbb{C}(x)$.*

Proof. (1) Normalizing, we get a map $\tilde{C} \to \tilde{D}$, and the first statement follows from Hurwitz's theorem.

(2) Since $\mathbb{C}$ is algebraically closed, K is a transcendental extension. Since $\mathbb{C}(x)$

has transcendence degree 1, if $z \in K \setminus \mathbb{C}$ then x is algebraic over $\mathbb{C}(z)$. Thus $\mathbb{C}(x)$ is finite over $\mathbb{C}(z)$, so K is finite over $\mathbb{C}(z)$ as well. In particular, K is finitely generated over $\mathbb{C}$. It follows that K is the field of rational functions on a curve D, and the inclusion $K \subset \mathbb{C}(x)$ corresponds to a finite map $\mathbb{P}^1 \to D$. Applying part (1), we see that $D \cong \mathbb{P}^1$ so $K \cong \mathbb{C}(y)$ for some y. □

Cheerful Fact 3.3 (rational curves over other fields)**.** *Lüroth's theorem* refers to statement (2) in Theorem 3.2. For an elementary proof of it, valid over any field, see [Jacobson 1989, Section 8.13].

Over a non-algebraically closed field, a curve C of genus 0 need not have any points, or any invertible sheaves of odd degree. Since the canonical sheaf K_C has degree -2, there necessarily exist invertible sheaves of every even degree; thus an arbitrary curve of genus 0 is isomorphic to a plane conic.

A projective curve of genus 0 over a field k is called a *form* of $\mathbb{P}^1$ if it becomes isomorphic to $\mathbb{P}^1$ after extension of scalars to the algebraic closure of k. The unique example with $k = \mathbb{R}$ that is not isomorphic over $\mathbb{R}$ to $\mathbb{P}^1$ is the conic with no $\mathbb{R}$-rational points, $x^2 + y^2 + z^2 = 0$. The classification of curves of genus 0 over $\mathbb{Q}$ is a subject that goes back to Gauss.

Noncommutative algebras enter the subject of forms of $\mathbb{P}^1$ (and $\mathbb{P}^n$ more generally) in a surprising way: The curve $\mathbb{P}^1_k$ itself may be described as the scheme of left ideals of k-vector-space dimension 2 in the ring of 2×2 matrices over k: such an ideal can be embedded in the matrix ring as a linear combination of the 2 columns in an appropriate sense. More generally, any scheme that is a form of $\mathbb{P}^1$ over k may be described as the scheme of 2-dimensional left ideals in a 4-dimensional central simple (= Azumaya) algebra over k. For example, the conic $x^2 + y^2 + z^2 = 0$ with no points over $\mathbb{R}$ is the scheme of left ideals in the algebra of quaternions. See [Serre 1979, Section X.6].

There is an analogue of the last statement of part (1) of Theorem 3.2 for rational surfaces, proved by Castelnuovo: every complex surface admitting a dominant rational map from $\mathbb{P}^2$ is rational (see [Beauville 1996, Corollary V.5], for example). However, the analogue in higher dimensions is false; for example, [Clemens and Griffiths 1972] shows that a smooth cubic threefold X admits a dominant rational map $\mathbb{P}^3 \to X$ but is not rational.

3.1. Rational normal curves

The homogeneous coordinate ring of a rational normal curve. If $\mathcal{V} = (V, \mathcal{L})$ is a linear series on a scheme X, then the inclusion $V \subset H^0(\mathcal{L})$ induces by multiplication a map

$$\rho_{\mathcal{V}} : \mathrm{Sym}(V) \to \bigoplus_{n\in\mathbb{Z}} H^0(\mathcal{L}^n)$$

from the symmetric algebra of V. When $\mathcal{V}$ embeds X in $\mathbb{P}^r = \mathbb{P}V$, the image R_X of this map is called the *homogeneous coordinate ring* of X.

The rational normal curve $C \subset \mathbb{P}^d$ of degree d is embedded by the complete linear series $\mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), \mathbb{C}[s,t]_d)$. Since any form of degree nd in $\mathbb{C}[s,t]$ is a sum of products of n forms of degree d, the corresponding homogeneous coordinate ring of the rational normal curve is

$$\mathbb{C}[s,t]_{(d)} := \bigoplus_n (\mathbb{C}[s,t]_{nd}),$$

and the map $\rho_{\mathcal{V}}$ is surjective. This is expressed by saying that C is *arithmetically Cohen–Macaulay*; more generally, if $C \subset \mathbb{P}^r$ is a one-dimensional scheme, we say that C is linearly (respectively, quadratically, …, n-ically) normal if the natural maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$$

are surjective for $m = 1$ (respectively $m = 2, \ldots, m = n$). We say that C is arithmetically Cohen–Macaulay (usually abbreviated ACM) when ρ_m is surjective for all m. We'll discuss the significance of this condition in Section 3.3, and we will prove that it is equivalent to the condition that R_C is a Cohen–Macaulay ring, justifying the name.

The equations defining a rational normal curve. Choosing a basis s, t for the linear forms on $\mathbb{P}^1$, we can write the d-th Veronese map $\mathbb{P}^1 \to \mathbb{P}^d$ as

$$\phi_d : (s,t) \mapsto (s^d, s^{d-1}t, \ldots, t^d),$$

from which we see that the image C of ϕ_d lies in the zero locus of the homogeneous quadratic polynomial $x_i x_j - x_{i+1}x_{j-1}$ for every i, j. We can realize these quadratic forms as the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & \ldots & x_{d-1} \\ x_1 & x_2 & \ldots & x_d \end{pmatrix}.$$

Note that if we substitute $s^i t^{(d-i)}$ for x_i and identify $H^0(\mathcal{O}_{\mathbb{P}^1}(i))$ with $\mathbb{C}[s,t]_i$, this becomes the multiplication table

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \times H^0(\mathcal{O}_{\mathbb{P}^1}(d-1)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(d)).$$

It is easy to see that C is set-theoretically defined by the 2×2 minors of M: the affine set $s = 1$ in $\mathbb{P}^1$ maps to the affine set $x_0 = 1$ in $\mathbb{P}^d$, and the affine form of the map is $t \mapsto (t, t^2, \ldots, t^d)$. But if $x_1 = t$ then from the equations $x_0 x_i = x_1 x_{i-1}$ we see successively that $x_i = t^i$; that is, the vanishing of the minors of M at a point p implies that p lies on C.

A much stronger statement is true:

Proposition 3.4. *The homogeneous ideal of the rational normal curve*

$$\mathbb{P}^1 \twoheadrightarrow C \subset \mathbb{P}^d, \quad (s,t) \mapsto (s^d, s^{d-1}t, \ldots, t^d),$$

is generated by the 2×2 *minors of* M.

Proof. Let $I \subset \mathbb{C}[x_0, \dots, x_d]$ be the ideal generated by the 2×2 minors of M. As we have seen, the map

$$\phi : \mathbb{C}[x_0, \dots, x_d]/I \twoheadrightarrow \mathbb{C}[s,t]_{(d)}, \quad x_i \mapsto s^{d-i}t^i,$$

is surjective, and we must show that this is a monomorphism, or equivalently that the source of ϕ in degree n has (at most) the same dimension $nd + 1$ as the target of ϕ in degree nd.

If $0 < i \leq j < d$ then

$$x_i x_j \equiv x_{i-1}x_{j+1} \pmod{I}.$$

Thus every monomial in the x_i of degree t is equivalent, modulo I, to a monomial of the form

$$x_0^a x_1^{\epsilon_1} \cdots x_{d-1}^{\epsilon_{d-1}} x_d^b$$

where at most one ϵ_i is 1, and the rest are 0. There are $n + 1$ such elements of degree n with all the $\epsilon_i = 0$ and $n(d-1)$ elements with one of the $\epsilon_i = 1$. Thus there are $nd + 1$ such elements in all, proving that ϕ is an isomorphism. □

Cheerful Fact 3.5. The description of the equations above can be considerably extended; see Proposition 17.6. For example, the equations of the Veronese surface, which is the image of $\mathbb{P}^2$ under the complete linear series of quadrics, is defined by the 2×2 minors of the generic symmetric matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_2 & x_4 & x_5 \end{pmatrix},$$

which arises from the multiplication table of

$$H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^2}(1)) \to H^0(\mathcal{O}_{\mathbb{P}^2}(2)).$$

Corollary 3.6. *The dimension of the degree n part of the homogeneous ideal of the rational normal curve of degree d is*

$$H^0(\mathcal{I}_{C/\mathbb{P}^d}(n)) = \binom{n+d}{n} - (nd+1).$$

In particular the $\binom{d}{2}$ *minors of the matrix M are linearly independent.*

Proof. The homogeneous coordinate ring of the rational normal curve is

$$\mathbb{C}[s,t]_{(d)} \subset \mathbb{C}[s,t].$$

Comparing the dimension of $\mathbb{C}[x_0, \dots, x_d]_n$ with the dimension of $\mathbb{C}[s,t]_{nd}$ gives the result. □

The number of quadrics containing the rational normal curve is extremal:

Proposition 3.7. *If $C \subset \mathbb{P}^d$ is any irreducible, nondegenerate curve, then*

$$h^0(\mathcal{I}_{C/\mathbb{P}^d}(2)) \le \binom{d}{2}.$$

Equality holds only if C is a rational normal curve.

In the proof we will use a general fact about hyperplane sections:

Proposition 3.8. *If $X \subset \mathbb{P}^r$ is a nondegenerate, reduced, irreducible variety, then $H^0(\mathcal{I}_X(1)) = H^1(\mathcal{I}_X) = 0$. Moreover if H is any hyperplane of $\mathbb{P}^r$, defined by a linear form h, then the natural sequence*

$$0 \to \mathcal{I}_{X/\mathbb{P}^r} \xrightarrow{h} \mathcal{I}_{X/\mathbb{P}^r}(1) \to \mathcal{I}_{(H\cap X)/H}(1) \to 0$$

arising from the restriction of the ideal sheaf $\mathcal{I}_{X/\mathbb{P}^r}$ to H is exact, and the subscheme $H \cap X$ spans H; that is, $H^0(\mathcal{I}_{(H\cap X)/H}(1)) = 0$.

See Exercise 3.5 for the necessity of the hypotheses and Exercise 3.6 for a generalization.

Corollary 3.9. *If $X \subset \mathbb{P}^r$ is a nondegenerate, reduced, irreducible variety of codimension c, then* $\deg X \ge c + 1$.

Proof. The intersection of X with a general plane Λ of dimension c is a finite scheme spanning Λ. □

Proof of Proposition 3.8. Since X is reduced, irreducible and nondegenerate, h is a nonzerodivisor modulo $\mathcal{I}_{X/\mathbb{P}^r}$, so $h\mathcal{I}_{X/\mathbb{P}^r} = (h) \cap \mathcal{I}_{X/\mathbb{P}^r}$ where $(h) = h\mathcal{O}_X$. Setting $\Gamma = H \cap X$ we have

$$\begin{aligned}\mathcal{I}_{X/\mathbb{P}^r}(1)/h\mathcal{I}_{X/\mathbb{P}^r} &= \mathcal{I}_{X/\mathbb{P}^r}(1)/((h) \cap \mathcal{I}_{X/\mathbb{P}^r})\\ &= (\mathcal{I}_{X/\mathbb{P}^r}(1) + (h))/(h)\\ &= \mathcal{I}_{\Gamma/H}(1),\end{aligned}$$

from which the exactness assertion follows.

Again because X is reduced and irreducible, $H^0(\mathcal{O}_X)$ contains only the constant function, so the map $H^0(\mathcal{O}_{\mathbb{P}^r}) \to H^0(\mathcal{O}_X)$ is surjective, from which it follows that $H^1(\mathcal{I}_{X/\mathbb{P}^r}) = 0$. From the long exact sequence in cohomology it follows that the restriction map $H^0(\mathcal{I}_{X/\mathbb{P}^r}(1)) \to H^0(\mathcal{I}_{\Gamma/H}(1))$ is surjective. Since X is nondegenerate, $H^0(\mathcal{I}_{X/\mathbb{P}^r}(1)) = 0$, and the desired result follows. □

Proof of Proposition 3.7. Consider the restriction of the quadrics containing C to a general hyperplane $H \cong \mathbb{P}^{d-1} \subset \mathbb{P}^d$, and let $\Gamma = H \cap C$. There is an exact sequence:

$$0 \to \mathcal{I}_{C/\mathbb{P}^d}(1) \to \mathcal{I}_{C/\mathbb{P}^d}(2) \to \mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2) \to 0.$$

Proposition 3.8 shows that $h^0(\mathcal{I}_{C/\mathbb{P}^d}(1)) = h^1(\mathcal{I}_{C/\mathbb{P}^d}(1)) = 0$, so Γ imposes the same number of conditions on quadrics as C does.

Proposition 3.8 also shows that H is the linear span of Γ. Therefore the hyperplane section Γ of C must contain at least d linearly independent points (this gives another proof that such a curve must have degree $\geq d$). Any set of linearly independent points imposes independent conditions (cf. Definition 10.5) on linear forms, and thus also on quadrics (see Proposition 10.6 for a stronger result). Thus

$$h^0(\mathcal{I}_{\Gamma/\mathbb{P}^{d-1}}(2)) \leq h^0(\mathcal{O}_{\mathbb{P}^{d-1}}(2)) - d = \binom{d+1}{2} - d = \binom{d}{2},$$

establishing the desired inequality.

To prove that equality can be achieved only with the rational normal curve, suppose that $C \subset \mathbb{P}^d$ is not a rational normal curve, so that $\deg C > d$. Let

$$\Gamma' \subset H \cong \mathbb{P}^{d-1}$$

be a subset of d linearly independent points of a general hyperplane section $H \cap C$ and let $\Gamma = \Gamma' \cup \{p\} \subset H$ be a subset containing one more point.

We will show that p imposes a nontrivial vanishing condition on quadrics containing Γ', since then C will lie on fewer quadrics than the rational normal curve. Since the points of Γ' are linearly independent, each subset of $d-1$ of them spans a $(d-2)$-plane inside H, and the intersection of these spans is empty. Thus one of the spans Λ does not contain p. The union of a general hyperplane H containing Λ and a general hyperplane containing the point $\Gamma' \setminus \Gamma' \cap \Lambda$ is a quadric containing Γ' but not p, as required. □

For each m, rational normal curves lie on more hypersurfaces of degree m than any other irreducible, nondegenerate curve in $\mathbb{P}^d$; see Exercise 10.4.

Rational normal curves are projectively homogeneous. An important property of rational normal curves $C \subset \mathbb{P}^d$ is that they are *projectively homogeneous*, in the sense that the subgroup $G \subset \mathrm{PGL}_{d+1}$ of automorphisms of $\mathbb{P}^d$ that carry C to itself acts transitively on C. More generally:

Proposition 3.10. *If $X \subset \mathbb{P}^n$ is the image of $\mathbb{P}^r$ by the Veronese map associated to $|\mathcal{O}_{\mathbb{P}^r}(d)|$, then X is projectively homogeneous.*

Proof. $\mathbb{P}^r$ itself is a homogeneous variety in that $\operatorname{Aut}\mathbb{P}^r$ acts transitively. For any automorphism σ of $\mathbb{P}^r$, we have $\sigma^*\mathcal{O}_{\mathbb{P}^r}(d) = \mathcal{O}_{\mathbb{P}^r}(d)$ because $\mathcal{O}_{\mathbb{P}^r}(d)$ is the unique invertible sheaf of degree d on $\mathbb{P}^r$; thus σ induces an automorphism ϕ on $H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ and an automorphism $\bar{\phi}$ on the ambient space $\mathbb{P}^N := \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ of the target of the d-th Veronese map. If $\ell \in H^0(\mathcal{O}_{\mathbb{P}^r}(d))$ with divisor $D \subset X$, then $\phi(\ell) = \ell \circ \sigma$ has divisor $\sigma^{-1}(D)$, so $\bar{\phi}^{-1}$ induces σ on $\mathbb{P}^r$. □

The rational normal curve $C \subset \mathbb{P}^d$ can be characterized among irreducible, nondegenerate curves as the unique projectively homogeneous curve in $\mathbb{P}^d$ (Corollary 13.3).

Interpolation for rational normal curves. Another striking property of rational normal curves is expressed in the following proposition. Recall that a collection of points (or a finite subscheme) of $\mathbb{P}^d$ is *in linearly general position* if no $k+1$ of them lie in a $(k-1)$-plane with $k \leq n$.

Proposition 3.11. *Any $d + 3$ points $p_1, \dots, p_{d+3} \in \mathbb{P}^d$ in linearly general position lie on a unique rational normal curve $C \subset \mathbb{P}^d$.*

Proof. There is an automorphism $\Phi : \mathbb{P}^d \to \mathbb{P}^d$ carrying $p_1, \dots, p_{d+1}$ to the coordinate points $(0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{P}^d$ and the point p_{d+2} to the point $(1, 1, \dots, 1)$. Denote the image of the remaining point p_{d+3} by $[\alpha_0, \dots, \alpha_n]$. We consider maps $\mathbb{P}^1 \to \mathbb{P}^d$ given in terms of an inhomogeneous coordinate z on $\mathbb{P}^1$ by

$$z \mapsto \left(\frac{1}{z - \nu_0}, \frac{1}{z - \nu_1}, \dots, \frac{1}{z - \nu_d} \right)$$

with $\nu_0, \dots, \nu_d$ any distinct scalars. Clearing denominators, we see that the image of such a map is a rational normal curve, and it passes through the $d+1$ coordinate points of $\mathbb{P}^d$, which are the images of the points $z = \nu_0, \dots, \nu_d \in \mathbb{P}^1$. Moreover, the image of the point at infinity is $(1, 1, \dots, 1)$. Because the points are assumed linearly independent, the α_i must all be nonzero, and thus if we take $\nu_i = -1/\alpha_i$, the image of the point $z = 0$ is $(\alpha_0, \dots, \alpha_d)$. This proves existence; we leave uniqueness to Exercise 3.11. □

Cheerful Fact 3.12. A generalization of Proposition 3.11 is given, with a proof that is different even in our case, in [Harris 1982a, Proposition 3.19]. There is also a version replacing the set of $d + 3$ distinct points by a finite scheme of degree $d + 3$ satisfying necessary conditions, in [Eisenbud and Harris 1992].

Given some "natural" family of curves in projective space $\mathbb{P}^r$ (such as an open set of smooth curves in a component of the Hilbert scheme described in Chapters 7 and 19) and an integer m, we can ask whether there exists a curve in the family passing through m given general points of $\mathbb{P}^r$. Proposition 3.11 gives the only example we know of curves other than complete intersections for which there is a *unique* such curve.

3.2. Other rational curves

What about other rational curves in projective space?

Any linear series $\mathcal{D}$ of degree d on $\mathbb{P}^1$ is a subseries of the complete series $|\mathcal{O}_{\mathbb{P}^1}(d)|$, so any map $\phi : \mathbb{P}^1 \to \mathbb{P}^r$ of degree d may be given as the composition of the embedding $\phi_d : \mathbb{P}^1 \to \mathbb{P}^d$ of $\mathbb{P}^1$ as a rational normal curve with a linear projection $\pi : \mathbb{P}^d \to \mathbb{P}^r$. Since the natural degree d Veronese embedding of

$\mathbb{P}^1 = \mathbb{P}(V)$ as the rational normal curve is the map $\mathbb{P}V \to \mathbb{P}(\mathrm{Sym}^d(V))$, the projection is a map into $\mathbb{P}W$, where $W \subset \mathrm{Sym}^d(V)$.

We can make this more explicit in several ways: first, choosing a basis s, t for V and a basis of forms F_i of degree d on $\mathbb{P}^1$ for W, we may write the map as

$$(s,t) \mapsto (F_0(s,t), \dots, F_r(s,t)).$$

If the F_i have a greatest common factor $G(s,t)$ then the set $G = 0$ will be the base locus of the linear series, and the map

$$(s,t) \mapsto \left(\frac{F_0(s,t)}{G(s,t)}, \dots, \frac{F_r(s,t)}{G(s,t)}\right).$$

gives the same map, represented as a linear series of lower degree $d - \deg G$. Thus we will now assume that the forms in W have no common factor.

Perhaps more simply, we can pass to an open affine $\mathbb{A}^1 \subset \mathbb{P}^1$ given by $t = 1$, with coordinate function $z = s/t$, and dehomogenize the F_i to get a vector space of polynomials $f_i = F(s,1)$ of degree $\leq d$. Then we may write the map as

$$z \mapsto (f_0(z), \dots, f_r(z)).$$

Since the F_i have no common factor, and thus are not all divisible by t, at least one of the f_i will have degree exactly d. For example, the twisted cubic itself can be represented by the map $z \mapsto (1, z, z^2, z^3)$.

Here, we think of a polynomial $f(z) = f(s/t)$ of degree $\leq d$ as a rational function on $\mathbb{P}^1$ having a pole of order at most d at the point at infinity $(1,0) \in \mathbb{P}^1$; but we could also take rational functions $F_i(s,t)/G_i(s,t)$ of total degree $\deg F_i - \deg G_i = d$ or, dehomogenizing, $\phi_i(z) = f_i(z)/g_i(z)$.

Smooth rational quartics. Given how easy it is to describe rational curves in projective space in this way, it is surprising how many open questions there are. We begin with one of the simplest cases: a smooth nondegenerate rational curve C of degree 4 in $\mathbb{P}^3$. As described above, such a curve is the image of $\mathbb{P}^1$ under a map given by 4 quartic forms, or, in the most coordinate-free formulation, a codimension 1 subspace of $\mathrm{Sym}^4(V)$, where $V \cong \mathbb{C}^2$.

Example 3.13. In Exercise 3.10 the reader is asked to prove that there is a 1-parameter family of PGL_4 orbits of smooth rational quartic curves in $\mathbb{P}^3$. Perhaps the simplest is given by the parametrization

$$\mathbb{P}^1 \ni (s,t) \mapsto (s^4, s^3t, st^3, t^4) \in \mathbb{P}^3,$$

or, more simply, $t \mapsto (t, t^3, t^4)$.

Proposition 3.14 shows that the homogeneous ideal of this curve requires 4 generators, and it is not hard to show that the ideal is generated scheme theoretically (that is, up to saturation) by 3 elements. Nevertheless it is one of the most famous open problems in the theory of curves to determine whether or not the ideal is generated up to radical by just 2 elements — that is, whether it can be written *set-theoretically* as the intersection of two surfaces. For a sample of the recent work on this, see [Hartshorne and Polini 2015].

Proposition 3.14. *If $C \subset \mathbb{P}^3$ is a smooth rational quartic curve, then C lies on a smooth quadric surface S in the divisor class* $(1,3)$*, and the homogeneous ideal of C is minimally generated by the quadric defining S and three cubic forms.*

Proof. We first consider the restriction maps

$$\rho_e : H^0(\mathcal{O}_{\mathbb{P}^3}(e)) \to H^0(\mathcal{O}_C(e)).$$

Since $C \cong \mathbb{P}^1$, we may identify $H^0(\mathcal{O}_C(e))$ with $H^0(\mathcal{O}_{\mathbb{P}^1}(4e))$. Since we assume that C is nondegenerate, it does not lie on a hyperplane, so ρ_1 is a monomorphism.

The source $H^0(\mathcal{O}_{\mathbb{P}^3}(2))$ of ρ_2 is 10-dimensional, and the target $H^0(\mathcal{O}_{\mathbb{P}^1}(8))$ is 9-dimensional, so ρ_2 has a kernel of dimension at least 1; that is, C lies on a quadric surface, and since C is irreducible and nondegenerate, any quadric surface containing C is irreducible. This quadric is smooth; see Example 2.41.

Given that C lies on a smooth quadric surface Q, we consider its divisor class (a,b) in the Picard group $\mathrm{Pic}(Q) = \mathbb{Z} \oplus \mathbb{Z}$. As we saw in Example 2.42, a smooth curve in the class (a,b) has degree $a+b$ and genus $(a-1)(b-1)$; solving the equations $a+b = 4$ and $(a-1)(b-1) = 0$ we see that $C \sim (1,3)$ or $C \sim (3,1)$. Since the cases are symmetric we assume $C \sim (1,3)$.

The source of ρ_3 is 20-dimensional and the target is 13-dimensional so there are at least 7 cubics in the ideal of $C \subset \mathbb{P}^3$. Four of these come from multiplying the quadric by the 4 variables on $\mathbb{P}^3$, so there are at least 3 more cubic generators in $I_{C/\mathbb{P}^3}$, the homogeneous ideal of C.

From the exact sequence

$$0 \to I_{Q/\mathbb{P}^3} \to I_{C/\mathbb{P}^3} \to I_{C/Q} \to 0$$

we see that $I_{C/\mathbb{P}^3}$ is generated by the form defining the quadric together with generators for $I_{C/Q}$. Moreover, the natural inclusion $I_{C/Q} \subset H^0_*(I_{C/Q})$ is an equality; this follows from the exact sequence of sheaves

$$0 \to \mathcal{I}_{Q/\mathbb{P}^3} \to \mathcal{I}_{C/\mathbb{P}^3} \to \mathcal{I}_{C/Q} \to 0$$

because $H^1_*(\mathcal{I}_{Q/\mathbb{P}^3}) = H^1_*(\mathcal{O}_{\mathbb{P}^3}(-2)) = 0$. Thus we need only compute generators for $H^0_*(I_{C/Q})$.

Since C lies in class $(1,3)$ we see that

$$\mathcal{I}_{C/Q}(d) = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(d-1, d-3).$$

By the Künneth formula,

$$h^0(\mathcal{I}_{C/Q}(3)) = h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0)) = h^0(\mathcal{O}_{\mathbb{P}^1}(2)) \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(0)) = 3,$$

and we see that $I(C)$ contains just 3 cubic generators modulo the quadric defining Q.

We claim that the three generators of $h^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0))$ generate all of

$$H^0_*(\mathcal{I}_{C/Q}(3)) = H^0_*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(2,0)) = \bigoplus_{m\geq 0} H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(m+2,m)).$$

For this we must show that the maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^3}(1)) \otimes H^0(\mathcal{I}_{C/Q}(m+3)) \to H^0(\mathcal{I}_{C/Q}(m+1+3))$$

are surjective for $m \geq 0$.

Since the restriction of $H^0(\mathcal{O}_{\mathbb{P}^3}(1))$ to the quadric is

$$H^0(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)) = H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(1)),$$

the map ρ_m is the tensor product of the two maps

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m+2)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(m+1+2))$$

and

$$H^0(\mathcal{O}_{\mathbb{P}^1}(1)) \otimes H^0(\mathcal{O}_{\mathbb{P}^1}(m)) \to H^0(\mathcal{O}_{\mathbb{P}^1}(m+1)),$$

both of which are surjective for $m \geq 0$. This completes the proof. □

Some open problems about rational curves. We can say far less about rational curves of higher degree, even in $\mathbb{P}^3$. For example, when d is large, we don't know the possible Hilbert functions for curves of degree d in $\mathbb{P}^3$, and the situation in $\mathbb{P}^r$ for $r > 3$ is even worse. However, we do know the Hilbert function of a *general* rational curve $C \subset \mathbb{P}^r$ of degree d.

Since such statements will come up often, we pause to explain exactly what it means to say "A general X has property Y." This statement presupposes a choice of a parameter space for objects of type X, that is to say, an algebraic variety Z whose points each correspond to an object of type X. Thus, to be precise, the statement should be, "An object of type X that is general with respect to parameter space Z has property Y." The statement then means: inside Z there is a dense open set whose elements correspond to objects of type X that have property Y. Often Z is irreducible, and then it is enough to have a nonempty open set, since every such set is Zariski dense.

In the case of nondegenerate rational curves of degree d in $\mathbb{P}^r$ we could take for Z the space of $(r+1)$-tuples of independent elements of $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$, or equivalently, taking the quotient by $\mathrm{PGL}(r+1)$, the Grassmannian of $(r+1)$-dimensional planes in $H^0(\mathcal{O}_{\mathbb{P}^1}(d))$. We will later make statements about general curves of genus g, referring either to a component of a Hilbert scheme (discussed in Chapter 7) or to the moduli space of curves that is discussed at length in Chapter 8.

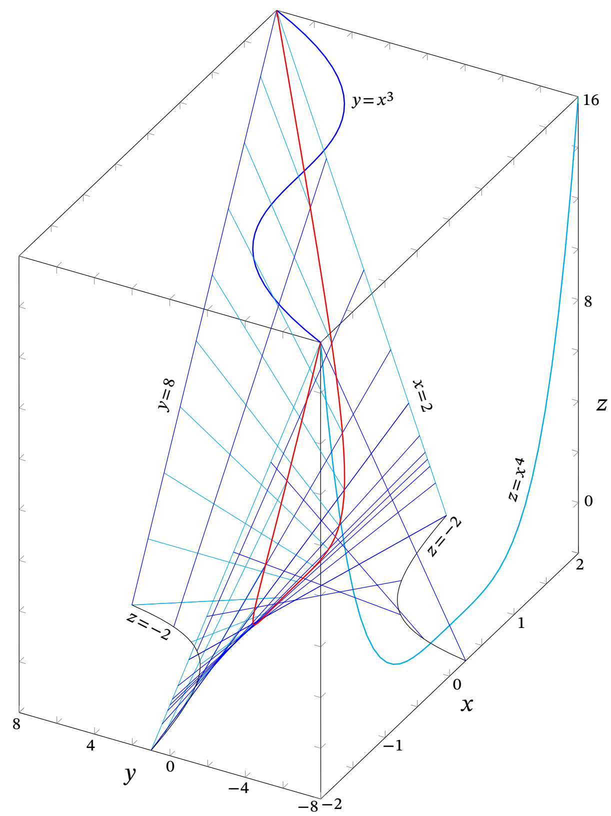

Figure 3.1. A rational quartic space curve of type $(1, 3)$ on a quadric surface. The quartic, in red, is given by $t \mapsto (t, t^3, t^4)$, and so projects to the cubic and quartic curves in the xy- and xz-planes (shown on the top and right surfaces of the box). The quartic, $z = xy$, is illustrated by means of its rulings and its intersections with the bounding planes. (Inspired by code of Enrique Acosta Jaramillo for the twisted cubic.)

As in Proposition 3.14, knowing the Hilbert function of a curve $C \subset \mathbb{P}^r$ is tantamount to knowing the ranks of the restriction maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m)) = H^0(\mathcal{O}_{\mathbb{P}^1}(md)).$$

Equivalently we ask: if V is a general $(r+1)$-dimensional vector space of homogeneous polynomials of degree d, what is the dimension of the space of polynomials spanned by m-fold products of polynomials in V?

We might guess that the answer is, “as large as possible,” meaning that the rank of ρ_m is $\binom{m+r}{r}$ or $md+1$, whichever is less — in other words, the map ρ_m is either injective or surjective for each m. This was proved in [Ballico and Ellia 1983], and is a special case of Larson’s maximal rank theorem [Larson 2017].

As we will see in subsequent chapters it is possible to speak of a general curve of genus g and a general invertible sheaf of degree d on such a curve; and the analogous statement about Hilbert functions was proven in [Larson 2017]; see Chapter 12.

Nevertheless, even the degrees of the generators of the homogeneous ideal of a general rational curve of degree d in $\mathbb{P}^3$ is unknown for larger d.

The secant plane conjecture. If $C \subset \mathbb{P}^r$, then an e-secant s-plane to C is an s-plane $\Lambda \cong \mathbb{P}^s \subset \mathbb{P}^r$ such that the intersection $\Lambda \cap C$ has degree $\geq e$.

Should you expect a curve $C \subset \mathbb{P}^r$ to have any e-secant s-planes? The set of s-planes in $\mathbb{P}^r$ is parametrized by the Grassmannian $\mathbb{G} = \mathbb{G}(s, r)$, which has dimension $(s+1)(r-s)$. Inside $\mathbb{G}$, the locus of planes that meet C has codimension $r-s-1$ (reason: the locus of planes containing a given point p is isomorphic by projection from p to $\mathbb{G}(s-1, r-1)$, and thus has codimension $r-s$). Thus one might conjecture that a curve $C \subset \mathbb{P}^r$ will have e-secant s-planes when

$$e \leq (s+1)\frac{r-s}{r-s-1},$$

perhaps with a few low-degree exceptions. Is this true of a general rational curve? For most e, r and s, we don't know.

3.3. The Cohen–Macaulay property

In Section 3.1 we defined a curve $C \subset \mathbb{P}^r$ to be arithmetically Cohen–Macaulay (ACM) if the natural maps

$$\rho_m : H^0(\mathcal{O}_{\mathbb{P}^r}(m)) \to H^0(\mathcal{O}_C(m))$$

are surjective for all m. When this is the case we can accurately predict the number of independent forms of each degree vanishing on the curve, and the condition has other important consequences that we will explore here and in Chapters 16 and 18. For general treatments of Cohen–Macaulay rings see [Eisenbud 1995, Chapter 18] or the book [Bruns and Herzog 1993]. One of the characterizations of the condition is in terms of regular sequences:

Definition 3.15. A sequence of elements $f_1, \dots, f_c$ in a ring R is *regular* if f_{i+1} is a nonzerodivisor modulo $(f_1, \dots, f_i)$ for $i = 0, \dots, c-1$ and the ideal $(f_1, \dots, f_c)$ is not the unit ideal.

The *grade* of an ideal I in a ring R (sometimes called the depth of I on R) is the maximal length of a regular sequence contained in I, or ∞ if $I = R$.

A local ring $(R, \mathfrak{m})$ is *Cohen–Macaulay* if grade $\mathfrak{m} = \dim R$. A Noetherian ring R is called Cohen–Macaulay if every localization is Cohen–Macaulay. Likewise, a scheme is Cohen–Macaulay if all its local rings are Cohen–Macaulay.

Example 3.16.
- The sequence of elements $x_0, \dots, x_r \subset S := \mathbb{C}[x_0 \dots, x_r]$ is regular, proving from the definition that the localization of S at the homogeneous maximal ideal is Cohen–Macaulay.

- Regular local rings are Cohen–Macaulay. Thus every smooth scheme is Cohen–Macaulay, and in particular $S := \mathbb{C}[x_0 \dots, x_r]$ is Cohen–Macaulay.
- It follows from the definition that any complete intersection in a Cohen–Macaulay scheme is again Cohen–Macaulay. Thus every plane curve, and more generally every complete intersection curve, is ACM; and we shall show in Section 10.2 that every canonically embedded curve and every curve embedded by a complete linear series of sufficiently high degree is also ACM.
- Zero-dimensional rings are all Cohen–Macaulay. The set of zerodivisors in a Noetherian ring is the union of the associated primes of 0, so a 1-dimensional ring is Cohen–Macaulay if and only if it is pure-dimensional (sometimes called "unmixed") — that is, every associated prime ideal is 1-dimensional. Thus any purely 1-dimensional scheme is Cohen–Macaulay.

Cheerful Facts 3.17. • We defined the grade as the maximal length of a regular sequence in I; but in fact all maximal regular sequences have the same length, equal to the smallest integer k such that $\mathrm{Ext}^k(R/I, R) \neq 0$ [Eisenbud 1995, Theorem 17.4 and Proposition 18.4].

• For any proper ideal $I \subsetneq R$ we have $\operatorname{grade} I \leq \operatorname{codim} I$. On the other hand, if R is Cohen–Macaulay, then $\operatorname{grade} I = \operatorname{codim} I$ for every ideal I of R. In this sense the grade is an arithmetic approximation to the codimension.

• Every localization of a Cohen–Macaulay ring is Cohen–Macaulay. If $X \subset \mathbb{P}^r$ has homogeneous coordinate ring R_X, then we say that $X \subset \mathbb{P}^r$ is arithmetically Cohen–Macaulay if R_X is Cohen–Macaulay, a property that depends on the embedding, and implies the intrinsic property that X is Cohen–Macaulay (since the local rings of X are essentially localizations of R_X). Previously we defined a curve $C \subset \mathbb{P}^r$ to be arithmetically Cohen–Macaulay if the natural map $R_C \to H^0_*(\mathcal{O}_C)$ is surjective. We will prove that this is equivalent to R_C being Cohen–Macaulay in Proposition 3.18. The same definitions and theorem apply to a positively graded ring, and homogeneous ideals.

• By Serre's criterion [Eisenbud 1995, Section 11.2] the homogeneous coordinate ring R_C of a curve C is normal (that is, integrally closed) if and only if C is both nonsingular and ACM, and sometimes C is said to be *projectively normal* in this case. (This is also the excuse for the terminology "linearly normal" for a curve embedded by a complete linear series, quadratically normal if ρ_2 is surjective, and so on.)

Proposition 3.18. *Suppose that $C \subset \mathbb{P}^r$ is a 1-dimensional subscheme. Let $S = H^0_*(\mathcal{O}_{\mathbb{P}^r})$ be the homogeneous coordinate ring of $\mathbb{P}^r$, and let $R_C = S/I_C$ be the homogeneous coordinate ring of C. The following conditions are equivalent:*

(1) *The natural injective map $R_C \to H^0_*(\mathcal{O}_C(1))$ is surjective.*

(2) $H^1_*(\mathcal{I}_{C/\mathbb{P}^r}) = 0$.

(3) *The homogeneous coordinate ring R_C of C is a Cohen–Macaulay ring; that is, there are linear forms h, h' on $\mathbb{P}^r$ whose images in R_C form a regular sequence. In particular, R_C has no 0-dimensional primary components, so C is purely 1-dimensional and thus Cohen–Macaulay as a scheme.*

(4) *For every hyperplane $H \subset \mathbb{P}^r$ that does not contain any component of C, the homogeneous ideal of $H \cap C$ is equal to $I_C + (h)$, where h is a linear form defining H.*

Proof. (1) $\iff$ (2): We may assume that $r \geq 2$, so $H^1_*(\mathcal{O}_{\mathbb{P}^r}(n)) = 0$ for all n. Using this and the exact sequence $0 \to \mathcal{I}_{C/\mathbb{P}^r} \to \mathcal{O}_{\mathbb{P}^r} \to \mathcal{O}_C \to 0$ we see that $H^0(\mathcal{O}_{\mathbb{P}^r}(n)) \to H^0(\mathcal{O}_C(n))$ is surjective for all n if and only if $H^1(\mathcal{I}_{C/\mathbb{P}^r}(n)) = 0$ for all n, proving the equivalence of (1) and (2).

(1) $\iff$ (3): First let $C \subset \mathbb{P}^r$ be an arbitrary 1-dimensional subscheme, and let $R = H^0_*(\mathcal{O}_C) := \bigoplus_{n\in\mathbb{Z}} H^0(\mathcal{O}_C(n))$. If H is a general hyperplane, with equation $h = 0$, then h does not vanish on any primary component of C, and thus the sequence

$$0 \to \mathcal{O}_C(-1) \xrightarrow{h} \mathcal{O}_C \to \mathcal{O}_{C\cap H} \to 0$$

is exact. Applying H^0_* and using (2), we see that h is a nonzerodivisor on R, and that R/hR is a subring of $H^0_*(\mathcal{O}_{C\cap H})$. A general linear form h' doesn't vanish on any point of $C \cap H$, so h' is a unit on $H^0_*(\mathcal{O}_{C\cap H})$ and thus a nonzerodivisor on R/hR.

The ring R_C is the image of the natural map $S \to R$, and by definition C is ACM if and only if this map is surjective, so that $R_C = R$. This shows that if C is arithmetically Cohen–Macaulay then R_C is a Cohen–Macaulay ring, and is in particular unmixed; that is, C has no 0-dimensional primary components (see for example [Eisenbud 1995, Chapter 18] for general information about Cohen–Macaulay rings). This proves the equivalence of conditions (1) and (3).

(3) $\iff$ (4): If h does not vanish on any component of C then h is a nonzerodivisor on R_C. The ideal $I_{C\cap H}$ is in any case the saturation of $I_C + (h)$. If $I_{C\cap H} = I_C + (h)$, then any linear form h' not containing a point of $C \cap H$ is a nonzerodivisor on $I_C + (h)$, so C satisfies condition (2). Conversely, in a 2-dimensional positively graded Cohen–Macaulay ring, any series of parameters is a regular sequence [Eisenbud 1995, Section 18.2], so $I_C + (h)$ is unmixed, and in particular, saturated. □

Cheerful Fact 3.19. The Cohen–Macaulay property is hard to interpret geometrically; the definition is justified by its usefulness. Here are two results that help our intuition:

(1) A scheme X is Cohen–Macaulay if some (equivalently every) finite map $f : X \to P$ to a smooth scheme P of the same dimension is flat, or equivalently the pushforward $f_*(\mathcal{O}_X)$ is locally free. (This follows from the Auslander–Buchsbaum formula [Eisenbud 1995, Section 19.3].)

(2) (Hartshorne) If a scheme X is Cohen–Macaulay then X is connected in codimension 1 (that is, X remains connected after removing any closed subset of codimension ≥ 2). See [Eisenbud 1995, Theorem 18.12] for a proof.

3.4. Exercises

Exercise 3.1. Let $\mathcal{V} = (\mathcal{O}_{\mathbb{P}^1}(d), V)$ be the linear series of degree d on $\mathbb{P}^1$ defined by the vector space $V = \langle s^d, s^{d-1}t, st^{d-1}, t^d\rangle$, where s, t are coordinates on $\mathbb{P}^1$. Show that $\mathcal{V}$ defines an isomorphism from $\mathbb{P}^1$ onto a smooth curve of type $(1, d-1)$ on a quadric surface. ◆

Exercise 3.2. With notation as in Section 3.1, show that the sheaf associated to the graded module coker M, that is, the cokernel of the map $\mathcal{O}^d_{\mathbb{P}^d}(-1) \to \mathcal{O}^2_{\mathbb{P}^d}$ defined by M, is the unique invertible sheaf of degree 1 on the rational normal curve C, and that thus the associated complete linear series defines the isomorphism $C \to \mathbb{P}^1$ inverse to the Veronese map.

Exercise 3.3. Considering $\mathbb{P}^n$ as Proj $\mathbb{C}[x_0, \dots, x_n]$, we may index the variables of $\mathbb{P}^{\binom{n+d}{d}-1}$ by monomials p of degree d in the x_i. Let $M_{n,d}$ be an $(n+1)\times\binom{n+d-1}{n}$ matrix whose rows are indexed by the variables x_i, whose columns are indexed by the monomials m of degree $d-1$ in the x_i and whose (i, m) entry is the variable corresponding to the monomial $x_i m$. (The matrix M of Section 3.1 is $M_{1,d}$.) Show that the 2×2 minors of $M_{n,d}$ generate the ideal of the image $V_{n,d}$ of the Veronese map $\mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$, and that the cokernel of $M_{n,d}$ is the unique invertible sheaf of degree 1 supported on $V_{n,d}$.

Exercise 3.4. Let $\nu_d : \mathbb{P}^r \to \mathbb{P}^{\binom{r+d}{r}-1}$ be the d-Veronese map, and let $C \subset \mathbb{P}^r$ be the rational normal curve of degree r. Is $\nu_d(C)$ nondegenerate? If not, what is the dimension of its linear span (that is, of the smallest linear space that contains it)? ◆

Exercise 3.5. Let C_1 be the union of two skew lines in $\mathbb{P}^3$, and let C_2 be the double line on a smooth quadric in $\mathbb{P}^3$. Show that a general hyperplane section of C_1 and of C_2 violates the conclusion of Proposition 3.8.

Exercise 3.6. Suppose that $X \subset \mathbb{P}^r$ is a subscheme and Z is the hypersurface in $\mathbb{P}^r$ defined by a polynomial $F \in H^0(\mathcal{O}_{\mathbb{P}^r}(m))$. If the restriction of F is a nonzerodivisor in $\mathcal{O}_X(m)$ then there is a short exact sequence

$$\mathcal{I}_X(d-m) \xrightarrow{F} \mathcal{I}_X(m) \to \mathcal{I}_{X\cap Z}(m) \to 0.$$

Exercise 3.7. Let $C \subset \mathbb{P}^3$ be the smooth rational quartic (or any smooth curve embedded by an incomplete linear series), and let h be a linear form defining a hyperplane H. Show that the irrelevant ideal is associated to the homogeneous ideal $I_C + (h)$, and thus $I_C(1)/hI_c$ is not the saturated homogeneous ideal of the finite set $C \cap H$.

Exercise 3.8. Show that the twisted cubic is the unique irreducible, nondegenerate space curve lying on three quadrics by considering the possible intersections of two of the quadrics. ◆

Exercise 3.9. As a consequence of our description of rational quartic rational quartic curves on a smooth quadric in Proposition 3.14, show that a general g^3_4 on $\mathbb{P}^1$ is uniquely expressible as a sum of the g^1_1 and a g^1_3 (in other words, a general 4-dimensional vector space of quartic polynomials on $\mathbb{P}^1$ is uniquely expressible as the product of a 2-dimensional vector space of cubics and the 2-dimensional space of linear forms. ◆

Exercise 3.10. Show that, up to projective equivalence, there is a 1-parameter family of embeddings of $\mathbb{P}^1$ as a smooth quartic curve in $\mathbb{P}^3$ by constructing an invariant that distinguishes them. ◆

Exercise 3.11. Complete the proof of Proposition 3.11 by showing that if C, C' are two rational normal curves in $\subset \mathbb{P}^n$ meeting in at least $n+3$ distinct points, then $C = C'$. ◆

Exercise 3.12. Let $V = \mathbb{C} \cdot e_1 \oplus \mathbb{C} \cdot e_2$ be a 2-dimensional vector space.

The group $\mathrm{SL}_2 = \mathrm{SL}(V)$ acts on the rational normal curve of degree d through automorphisms induced from its action on the ambient space $\mathbb{P}^d$ of the rational normal curve, which may be identified with $\mathbb{P}(\mathrm{Sym}^d(V))$.

In [Fulton and Harris 1991, pp. 146–150] it is shown that every finite-dimensional rational representation of $\mathrm{SL}(V)$ is a direct sum of representations of the form $\mathrm{Sym}^e(V)$ for various $e \geq 0$. There it is explained that to understand how a given representation decomposes one should look at the action of the torus generator

$$\alpha := \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \in \mathrm{SL}(V).$$

The eigenvectors of α are called the *weight vectors* of the representation. Note that $\mathrm{Sym}^e(V)$ is spanned by the weight vectors $w_s := e_1^{e-s}e_2^s$ that satisfy $\alpha w_s = t^{e-2s}$ for $s = 0, \dots e$. To decompose an arbitrary representation W, knowing that W is a direct sum of $\mathrm{Sym}^{e_i} V$, it is enough to know the eigenvalues for the action of α: We begin by finding an element $w \in W$ that is an eigenvector of α and transforms by α as $\alpha w = t^m w$ with the highest possible m (this is called a "highest weight vector"). Such a w must be contained in a summand $\mathrm{Sym}^m(V)$, and after removing the eigenvalues of the action of SL_2 on $\mathrm{Sym}^m(V)$, we continue.

(1) Use this method to show that

$$\mathrm{Sym}^d(V)\otimes \mathrm{Sym}^d(V) = \mathrm{Sym}^{2d}(V)\oplus \mathrm{Sym}^{2d-2}(V)\oplus \mathrm{Sym}^{2d-4}(V)\otimes\cdots,$$
$$\mathrm{Sym}^2(\mathrm{Sym}^d(V)) = \mathrm{Sym}^{2d}(V)\oplus \mathrm{Sym}^{2d-4}(V)\oplus \mathrm{Sym}^{2d-8}(V)\otimes\cdots,$$
$$\bigwedge^2(\mathrm{Sym}^d(V)) = \mathrm{Sym}^{2d-2}(V)\oplus \mathrm{Sym}^{2d-6}(V)\oplus \mathrm{Sym}^{2d-10}(V)\otimes\cdots,$$

where we take $\mathrm{Sym}^m(V)=0$ when $m<0$.

(2) Show that the space of quadrics containing the rational normal curve is a representation of SL_2 of the form

$$\mathrm{Sym}^{2d-4}(V)\oplus \mathrm{Sym}^{2d-8}(V)\cdots$$

(3) Show there is a distinguished nonsingular skew-symmetric form (up to scalars) on the ambient space of the twisted cubic. Thus a twisted cubic in $\mathbb{P}^3$ determines, for each point of $\mathbb{P}^3$, a distinguished plane containing that point.

(4) Show that if d is divisible by 4 there is a distinguished quadric in the ideal of the rational normal curve.

Exercise 3.13. Let $\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^3$ be a twisted cubic. Show that the normal bundle $\mathcal{N}_{C/\mathbb{P}^3}$ (defined to be the quotient of the restriction $T_{\mathbb{P}^3}|_C$ to C of the tangent bundle of $\mathbb{P}^3$ by the tangent bundle T_C) is $\mathcal{N}_{C/\mathbb{P}^3}\cong \mathcal{O}_{\mathbb{P}^1}(5)\oplus\mathcal{O}_{\mathbb{P}^1}(5)$. ◆

Exercise 3.14. Let $\mathbb{P}^1 \hookrightarrow C \subset \mathbb{P}^d$ be a rational normal curve. Show that the normal bundle $\mathcal{N}_{C/\mathbb{P}^d}$ is

$$\mathcal{N}_{C/\mathbb{P}^d}\cong \bigoplus_{i=1}^{d-1}\mathcal{O}_{\mathbb{P}^1}(d+2). \quad ◆$$

Exercise 3.15. In the situation of Exercise 3.14, the set of direct summands of $\mathcal{N}_{C/\mathbb{P}^d}$ is a projective space $\mathbb{P}^{d-2}$. How does the group of automorphisms of $\mathbb{P}^d$ carrying C to itself act on this $\mathbb{P}^{d-2}$? ◆

See [Coskun and Riedl 2018], for example, for more on normal bundles of rational curves.

Exercise 3.16. If $C=\bigcap_{i=1}^{r-1}X_i\subset\mathbb{P}^r$ is a complete intersection of hypersurfaces, then C is arithmetically Cohen–Macaulay. ◆

Exercise 3.17. Give a proof of Corollary 3.9 without using Bertini's theorem, by projecting X from a general point and using induction on $\operatorname{codim} X$. ◆