An Introduction to Symplectic Geometry

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Contents

Preface

> Le caractère propre des méthodes de l'Analyse et de la Géométrie modernes consiste dans l'emploi d'un petit nombre de principes généraux, indépendants de la situation respective des différentes parties ou des valeurs relatives des différents symboles; et les conséquences sont d'autant plus étendues que les principes eux–mêmes ont plus de généralité.
>
> from G. DARBOUX: *Principes de Géométrie Analytique*

This text is written for the graduate student who has previous training in analysis and linear algebra, as for instance S. Lang's *Analysis I* and *Linear Algebra*. It is meant as an introduction to what is today an intensive area of research linking several disciplines of mathematics and physics in the sense of the Greek word $\sigma\upsilon\mu\pi\lambda\acute{\varepsilon}\kappa\varepsilon\iota\nu$ (which means *to interconnect*, or *to interrelate* in English).[1] The difficulty (but also the fascination) of the area is the wide variety of mathematical machinery required. In order to introduce this interrelation, this text includes extensive appendices which include definitions and developments not usually covered in the basic training of students but which lay the groundwork for the specific constructions

[1] I want to thank P. Slodowy for pointing out to me that the name *symplectic group*, which eventually gave rise to the term *symplectic geometry*, was proposed by H. WEYL, [**W**], 1938, in his book, *The Classical Groups* (see footnote on p. 165). The symplectic group was also called the *complex group* or an *Abelian linear group*, this last to honor ABEL, who was the first to study them.

needed in symplectic geometry. Furthermore, more advanced topics will continue to rely heavily on other disciplines, in particular on results from the study of differential equations.

Specifically, the text tries to reach the following two goals:

- To present the idea of the formalism of symplectic forms, to introduce the symplectic group, and especially to describe the symplectic manifolds. This will be accompanied by the presentation of many examples of how they come to arise; in particular the quotient manifolds of group actions will be described,

and

- To demonstrate the connections and interworking between mathematical objects and the formalism of theoretical mechanics; in particular, the Hamiltonian formalism, and that of the quantum formalism, namely the process of *quantization*.

The pursuit of these goals proceeds according to the following plan. We begin in Chapter 0 with a brief introduction of a few topics from theoretical mechanics needed later in the text. The material of this chapter will already be familiar to physics students; however, for the majority of mathematics students, who have not learned the connections of their subject to physics, this material will perhaps be new.

We are constrained, in the first chapter, to consider *symplectic* (and a little later *Kähler*) *vector spaces*. This is followed by the introduction of the associated notion of a symplectic group $Sp(V)$ along with its generation. We continue with the introduction of several specific and theoretically important subspaces, the isotropic, coisotropic and Lagrangian subspaces, as well as the hyperbolic planes and spaces and the radical of a symplectic space.

Our first result will be to show that the symplectic subspaces of a given dimension and rank are fixed up to symplectic isomorphism. A consequence is then that the Lagrangian subspaces form a homogeneous space $\mathcal{L}(V)$ for the action of the group $Sp(V)$. The greatest effort will be devoted to the description of the spaces of positive complex structures compatible with the given symplectic structure. The second major result will be that this space is a homogeneous space, and is, for dim $V = 2n$, isomorphic to the Siegel half space $\mathfrak{H}_n = Sp_n(\mathbb{R})/U(n)$.

The second chapter is dedicated to the central object of the book, namely *symplectic manifolds*. Here the consideration of differential forms is unavoidable. In Appendix A their calculus will be given. The first result of this chapter is then the derivation of a theorem by Darboux that says that the symplectic manifolds are all locally equivalent. This is in sharp contrast to the situation with *Riemannian manifolds*, whose definition is otherwise somewhat parallel to that of the symplectic manifolds. The chapter will then take a glance at new research by considering the assignment of invariants to symplectic manifolds; in particular, the symplectic capacities and the pseudoholomorphic curves will be given.

In the course of the second chapter, we will present several examples of symplectic manifolds:

- First, the example which forms the origin of the theory and remains the primary application to physics is the *cotangent bundle* T^*Q of a given manifold Q.
- Second, the general *Kähler manifold*.
- Third, the *coadjoint orbits*. This description of symplectic manifolds with the operation of a Lie group G can be taken as the second major result of this chapter. We describe a theorem of Kostant and Souriau that says that for a given Lie group G with Lie algebra $\mathfrak{g}$ satisfying the condition that the first two cohomology groups vanish, that is $H^1(\mathfrak{g}) = H^2(\mathfrak{g}) = 0$, there is, up to covering, a one–to–one correspondence between the symplectic manifolds with transitive G–action and the G–orbits in the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$. Here we will need several facts from the theory of Lie algebras and systems of differential equations, and we will at least cover some of the rudiments we require. This will then offer yet another means for introducing one of the central concepts of the field, namely the *moment map*. This will, however, be somewhat postponed so that
- In the fourth and last example, *complex projective space* can be presented as a symplectic manifold; this will be seen as a specific example of the third example, as well as the second; that is, as a coadjoint orbit as well as as a Kähler manifold.

As preparation for the higher level construction of symplectic manifolds, Chapter 3 will introduce the standard concepts of a *Hamiltonian vector field* and a *Poisson bracket*. With the aid of these ideas, we can give the Hamiltonian formulation of classical mechanics and establish the following fundamental short exact sequence:

$$0 \to \mathbb{R} \to \mathcal{F}(M) \to \mathrm{Ham}\ M \to 0,$$

where $\mathcal{F}(M)$ is the space of smooth functions f defined on the symplectic manifold and given the structure of Lie algebra via the Poisson bracket, and Ham M is the Lie algebra of Hamiltonian vector fields on the manifold.

The third chapter continues with a brief introduction to *contact manifolds*. A theory for these manifolds in odd dimension can be developed which corresponds precisely to that of the symplectic manifolds. On the other hand, both may be viewed as pre-symplectic manifolds. Here the connection will be given through the example of a contact manifold as the surface of constant energy of a Hamiltonian system.

The fourth and fifth chapters will be a mix of further mathematical constructions and their physical interpretations. This will begin with the description of the *moment map* attached to the situation of a Lie group G acting symplectically on a symplectic manifold such that every Hamiltonian vector field is global Hamiltonian. This is a certain function

$$\Phi : M \to \mathfrak{g}^*, \ \ \mathfrak{g} = \text{Lie } G.$$

The most important examples of the moment maps are the Ad^*–equivariant ones, that is, those that satisfy a compatibility condition with respect to the coadjoint representation Ad^*. The first result of Chapter 4 is that for a symplectic form $\omega = -d\vartheta$ and a G invariant 1-form ϑ such an Ad^*–equivariant map can be constructed. This will then be applied to the cotangent bundle T^*Q, as well as to the tangent bundle TQ, where it will turn out that for a regular Lagrangian function $L \in \mathcal{F}(Q)$ the associated moment map is an integral for the Lagrangian equation associated to L. As examples, we will discuss the *linear* and *angular momenta* in the context of the formalism of the moment map, and so make clear the reason for this choice of terminology.

Next, we describe *symplectic reduction*. Here, we are given a symplectic G-operation on M and an Ad^*–equivariant moment map Φ; under some relatively easy–to–check conditions, for $\mu \in \mathfrak{g}^*$, the quotient

$$M_\mu = \Phi^{-1}(\mu)/G_\mu$$

is again a symplectic manifold. This central result of Chapter 4 has many applications, including the construction of further examples of symplectic manifolds (in particular, we obtain other proofs that the projective space $\mathbb{P}^n(\mathbb{C})$ as well as the coadjoint orbits are symplectic). Another application is the result of classical mechanics on the reduction of the number of variables by the application of symmetry, leading to the appearance of some integrals of the motion.

In the fifth and last chapter, we consider *quantization*; that is, the transition from classical mechanics to quantum mechanics, which leads to many interesting mathematical questions. The first case to be considered is the simplest: $M = \mathbb{R}^{2n} = T^*\mathbb{R}^n$. In this case the important tools are the groups $SL_2(\mathbb{R})$, $Sp_{2n}(\mathbb{R})$, the Heisenberg group $\text{Heis}_{2n}(\mathbb{R})$, the Jacobi group $G^J_{2n}(\mathbb{R})$ (as a semidirect product of the Heisenberg and symplectic groups) and their associated Lie algebras. It will follow that quantization assigns to the polynomials of degree less than or equal to 2 in the variables p and q of $\mathbb{R}^{2n}$ an operator on $L^2(\mathbb{R})$ with the help of the Schrödinger representation of the Heisenberg group and the Weil representation of the symplectic group (more precisely, its metaplectic covering). The theorem of Groenewold and van Hove then says that this quantization is *maximal*; that is, it cannot be extended to polynomials of higher degree.

The remainder of the fifth chapter consists in laying the groundwork for the general situation, which essentially follows KIRILLOV [**Ki**]. Here a subalgebra $\mathfrak{p}$, the *primary quantities*, comes into play, which for the case of $M = T^*Q$ turns out to be the arbitrary functions in q and the linear functions in p. Here yet more functional analysis and topology are required in order to demonstrate the result of Kirillov that for a symplectic manifold, with an algebra $\mathfrak{p}$ in $\mathcal{F}(M)$ of primary quantities relative to the Poisson bracket, a quantization is possible. That is, there is a map which assigns to each $f \in \mathfrak{p}$ a self–adjoint operator $\tilde{f}$ on Hilbert space $\mathcal{H}$ satisfying the conditions

(1) the function 1 corresponds to the identity $\mathrm{id}_{\mathcal{H}}$,

(2) the Poisson bracket of the two functions corresponds to the Lie bracket of operators, and

(3) the algebra of operators operates irreducibly.

There is a one–to–one correspondence between the set of equivalence classes of such representations of $\mathfrak{p}$ and the cohomology group $H^1(M, \mathbb{C}^*)$.

In the first two appendices, manifolds, vector bundles, Lie groups and algebras, vector fields, tensors, differential forms and their basic handling are covered. In particular, the various derivation processes are covered so that one may follow the proofs in the cited literature. A quick reading of this synopsis is perhaps recommended as an entrance to the second chapter. In Chapter 2 some material about cohomology groups will also be required. The third appendix presents some of the rudiments of cohomology theory. In the final appendix, the central concept of coadjoint orbits is prepared by a consideration of the fundamental concepts and constructions of representation theory.

As already mentioned, somewhat more from the theory of differential equations than is usually presented in a beginner's course on the topic, in particular Frobenius' theorem, is required to fully follow the treatment of symplectic geometry given here. Since in these cases the difficulty is not in grasping the statements, this material is left out of the appendices and simply used in the text as needed, though again without proof.

It is not the intention of this text to compete with the treatment of the classical and current literature over the research in the various subtopics of symplectic geometry as can be found, for example, in the books by

Abraham–Marsden [**AM**], Aebischer *et al.* [**Ae**], Guillemin–Sternberg [**GS**], Hofer–Zehnder [**HZ**], McDuff–Salamon [**MS**], Siegel [**Si1**], Souriau [**So**], Vaisman [**V**], Wallach [**W**] and Woodhouse [**Wo**]. Instead we have tried to introduce the reader to the material in these sources and, moreover, to follow the work contained in, for example, Gromov [**Gr**] and Kirillov [**Ki**]. In the hope that this will provide each reader with a starting point into this fascinating area a few parts of chapters 1, 2, and 4 may be skipped by those whose interests lie in physics, and one may begin directly with the sections on Hamiltonian vectorfields, moment maps and quantization.

This text is, with minor changes, a translation of the book "Einführung in die Symplektische Geometrie" (Vieweg, 1998). The production of this text has only been possible through the help of many. U. Schmickler–Hirzebruch and G. Fischer, on the staff of Vieweg–Verlag, have made many valuable suggestions, as has E. Dunne from the American Mathematical Society. My colleagues J. Michaliček, O. Riemenschneider and P. Slodowy, from the *Mathematische Seminar* of the Universität Hamburg, were always, as ever, willing to discuss these topics. A. Günther prepared one draft of this text, and I. Köwing did a newer draft and also showed great patience for my eternal desire to have something or other changed. I also had very successful technical consultation with F. Berndt, D. Nitschke and R. Schmidt. The last of these went through the German text with great attention and smoothed out at least some of what was rough in the text. I would also thank T. Wurzbacher, W. Foerg-Rob and P. Wagner for carefully reading (parts of) the German text and finding some misprints, wrong signs and other mistakes. The translation was done by M. Klucznik, who had an enormous task in producing very fluent English (at least in my opinion) and a fine layout of my often rather involved German style. It is a great joy for me to thank each of these.

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