A Course in Differential Geometry

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Preface

This book provides an introduction to differential geometry, with principal emphasis on Riemannian geometry. It can be used as a course for second-year graduate students. The main theorems are presented in complete detail, but the student is expected to provide the details of certain arguments. We assume that the reader has a good working knowledge of multidimensional calculus and point-set topology.

Many readers have been exposed to the elementary theory of curves and surfaces in three-space, including tangent lines and tangent planes. But these techniques are not necessary prerequisites for this book.

In this book we work abstractly, so that the notion of tangent space does not necessarily have a concrete realization. Nevertheless we will eventually prove Whitney's theorem asserting that any abstract n-dimensional manifold may be imbedded in the Euclidean space $\mathbb{R}^p$ if p is sufficiently large.

In order to develop the abstract theory, one must work hard at the beginning, to develop the notion of local charts, change of charts, and atlases. Once these notions are understood, the subsequent proofs are much easier, allowing one to obtain great generality with maximum efficiency. For example, the proof of Stokes' theorem—which is difficult in a concrete context—becomes transparent in the abstract context, reducing to the computation of the integral of a derivative of a function on a closed interval of the real line.

In Chapter I we find the first definitions and two important theorems, those of Whitney and Sard.

Chapter II deals with vector fields and differential forms.

Chapter III concerns integration of vector fields, then extends to p-plane fields. We cite in particular the interesting proof of the Frobenius theorem, which proceeds by mathematical induction on the dimension.

Chapter IV deals with connections, the most difficult notion in differential geometry. In Euclidean space the notion of parallel transport is intuitive, but on a manifold it needs to be developed, since tangent vectors at distinct points are not obviously related. Loosely speaking, a connection defines an infinitesimal direction of motion in the tangent bundle, or, equivalently, a connection defines a sort of directional derivative of a vector field with respect to another vector. This concrete notion of connection is rarely taught in books on connections. In our work we devote ten pages to developing these ideas, together with the related notions of torsion, curvature and a working knowledge of the covariant derivative. All of these notions are essential to the study of real or complex manifolds.

In Chapter V we specialize to Riemannian manifolds. The viewpoint here is to deduce global properties of the manifold from local properties of curvature, the final goal being to determine the manifold completely.

In Chapter VI we explore some problems in partial differential equations which are suggested by the geometry of manifolds.

The last three chapters are devoted to global notation, specifically to using the covariant derivative instead of computing in local coordinates with partial derivatives. In some cases we are able to reduce a page of computation in local coordinates to just a few lines of global computation. We hope to further encourage the use of global notation among differential geometers.

The aim of this book is to facilitate the teaching of differential geometry. This material is useful in other fields of mathematics, such as partial differential equations, to name one. We feel that workers in PDE would be more comfortable with the covariant derivative if they had studied it in a course such as the present one. Given that this material is rarely taught, one may ask why? We feel that it requires a substantial amount of effort, and there is a shortage of good references. Of course there are reference books such as Kobayashi and Nomizu [5], which can be consulted for specific information, but that book is not written as a text for students of the subject.

The present book is made to be teachable on a chapter-by-chapter basis, including the solution of the exercises. The exercises are of varying difficulty, some being straightforward or solved in existing literature; others are more challenging and more directly related to our approach.

This book is an outgrowth of a course which I presented at the Université Paris VI. I have included many problems and a number of solutions. Some of these originated from examinations in the course. I am very grateful to my friend Mark Pinsky, who agreed to read the manuscript from beginning

to end. His comments allowed me to make many improvements, especially in the English. I would like to thank also one of my students, Sophie Bismuth, who helped me to prepare the final draft of this book.