Lecture Notes in Algebraic Topology

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Preface

To paraphrase a comment in the introduction to a classic point-set topology text, this book might have been titled *What Every Young Topologist Should Know.* It grew from lecture notes we wrote while teaching second–year algebraic topology at Indiana University.

The amount of algebraic topology a student of topology must learn can be intimidating. Moreover, by their second year of graduate studies students must make the transition from understanding simple proofs line-by-line to understanding the overall structure of proofs of difficult theorems.

To help our students make this transition, the material in these notes is presented in an increasingly sophisticated manner. Moreover, we found success with the approach of having the students meet an extra session per week during which they took turns presenting proofs of substantial theorems and writing lecture notes to accompany their explanations. The responsibility of preparing and giving these lectures forced them to grapple with "the big picture" and also gave them the opportunity to learn how to give mathematical lectures, preparing for their participation in research seminars. We have collated a number of topics for the students to explore in these sessions; they are listed as projects in the table of contents and are enumerated below.

Our perspective in writing this book was to provide the topology graduate students at Indiana University (who tend to write theses in geometric topology) with the tools of algebraic topology they will need in their work, to give them a sufficient background to be able to interact with and appreciate the work of their homotopy theory cousins, and also to make sure that they are exposed to the critical advances in mathematics which came about with the development of topology in the years 1950-1980. The topics discussed in varying detail include homological algebra, differential topology, algebraic K-theory, and homotopy theory. Familiarity with these topics is important not just for a topology student but any student of pure mathematics, including the student moving towards research in geometry, algebra, or analysis.

The prerequisites for a course based on this book include a working knowledge of basic point-set topology, the definition of CW-complexes, fundamental group/covering space theory, and the construction of singular homology including the Eilenberg-Steenrod axioms. In Chapter 8, familiarity with the basic results of differential topology is helpful. In addition, a command of basic algebra is required. The student should be familiar with the notions of R-modules for a commutative ring R (in particular the definition of tensor products of two R-modules) as well as the structure theorem for modules over a principal ideal domain. Furthermore, in studying non simply-connected spaces it is necessary to work with tensor products over (in general non-commutative) group rings, so the student should know the definition of a right or left module over such a ring and their tensor products. Basic terminology from category theory is used (sometimes casually), such as category, functor, and natural transformation. For example, if a theorem asserts that some map is natural, the student should express this statement in categorical language.

In a standard first-year course in topology, students might also learn some basic homological algebra, including the universal coefficient theorem, the cellular chain complex of a CW-complex, and perhaps the ring structure on cohomology. We have included some of this material in Chapters 1, 2, and 3 to make the book more self-contained and because we will often have to refer to the results. Depending on the pace of a first-year course, a course based on this book could start with the material of Chapter 2 (Homological Algebra), Chapter 3 (Products), or Chapter 4 (Fiber Bundles).

Chapter 6 (Fibrations, Cofibrations and Homotopy Groups) and Chapter 9 (Spectral Sequences) form the core of the material; any second-year course should cover this material. Geometric topologists must understand how to work with non simply-connected spaces, and so Chapter 5 (Homology with Local Coefficients) is fundamental in this regard. The material in Chapters 7 (Obstruction Theory and Eilenberg-MacLane Spaces) and 8 (Bordism, Spectra, and Generalized Homology) introduces the student to the modern perspective in algebraic topology. In Chapter 10 (Further Applications of Spectral Sequences) many of the fruits of the hard labor that preceded this chapter are harvested. Chapter 11 (Simple-Homotopy theory) introduces the ideas which lead to the subject of algebraic K-theory and to the s-cobordism theorem. This material has taken a prominent role in research in topology, and although we cover only a few of the topics in this area (K_1, the Whitehead group, and Reidemeister torsion), it serves as good preparation for more advanced courses.

These notes are meant to be used in the classroom, freeing the student from copying everything from the chalkboard and hopefully leaving more time to think about the material. There are a number of exercises in the text; these are usually routine and are meant to be worked out when the student studies. In many cases, the exercises fill in a detail of a proof or provide a useful generalization of some result. Of course, this subject, like any subject in mathematics, cannot be learned without thinking through some exercises. Working out these exercises as the course progresses is one way to keep up with the material. The student should keep in mind that, perhaps in contrast to some areas in mathematics, topology is an example driven subject, and so working through examples is the best way to appreciate the value of a theorem.

We will omit giving a diagram of the interdependence of various chapters, or suggestions on which topics could be skipped, on the grounds that teachers of topology will have their own opinion based on their experience and the interests of the students. (In any case, every topic covered in this book is related in some way to every other topic.) We have attempted (and possibly even succeeded) to organize the material in such a way as to avoid the use of technical facts from one chapter to another, and hence to minimize the need to shuffle pages back and forth when reading the book. This is to maximize its usefulness as a textbook, as well as to ensure that the student with a command of the concepts presented can learn new material smoothly and the teacher can present the material in a less technical manner. Moreover, we have not taken the view of trying to present the most elementary approach to any topic, but rather we feel that the student is best served by learning the high-tech approach, since this ultimately is faster and more useful in research. For example, we do not shrink from using spectral sequences to prove basic theorems in algebraic topology.

Some standard references on the material covered in this course include the books [**14**], [**36**], [**43**], [**9**], [**17**] [**31**], and [**7**]. A large part of the material in these notes was distilled from these books. Moreover, one can find some of the material covered in much greater generality and detail in these tomes. Our intention is not to try to replace these wonderful books, but rather to offer a textbook to accompany a course in which this material is taught.

We recommend that students look at the article "Fifty years of homotopy theory" by G. Whitehead [**44**] for an overview of algebraic topology, and look back over this article every few weeks as they are reading this book. The books a student should read after finishing this course (or in conjunction with this course) are Milnor and Stasheff, *Characteristic Classes* [**30**] (every mathematician should read this book), and Adams, *Algebraic Topology: A Student's Guide* [**1**].

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Projects

The following is a list of topics to be covered in the extra meetings and lectured on by the students. They do not always match the material of the corresponding chapter but are usually either related to the chapter material or preliminary to the next chapter. Sometimes they form interesting subjects which could reasonably be skipped. Some projects are quite involved (e.g. "state and prove the Hurewicz theorem"), and the students and instructor should confer to decide how deeply to cover each topic. In some cases (e.g. the Hopf degree theorem, the Hurewicz theorem, and the Freudenthal suspension theorem) proofs are given in later chapters using more advanced methods.

- **Chapter 1.**
  1. The cellular approximation theorem.
  2. Singular homology theory.
- **Chapter 2.**
  1. The acyclic models theorem and the Eilenberg-Zilber map.
- **Chapter 3.**
  1. Algebraic limits and the Poincaré duality theorem.
  2. Exercises on intersection forms.
- **Chapter 4.**
  1. Fiber bundles over paracompact bases are fibrations.
  2. Classifying spaces.
- **Chapter 5.**
  1. The Hopf degree theorem.
  2. Colimits and limits.

- **Chapter 6.**
  1. The Hurewicz theorem.
  2. The Freudenthal suspension theorem.
- **Chapter 7.**
  1. Postnikov systems.
- **Chapter 8.**
  1. Basic notions from differential topology.
  2. Definition of K-theory.
  3. Spanier-Whitehead duality.
- **Chapter 9.**
  1. Construction of the Leray-Serre-Atiyah-Hirzebruch spectral sequence.
- **Chapter 10.**
  1. Unstable homotopy theory.
- **Chapter 11.**
  1. Handlebody theory and torsion for manifolds.

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