Chapter 11

Topology

11.1. Multiply Connected Domains

Let $U \subseteq \mathbb{C}$ be a connected open set. In this chapter, such an open set will be called a *domain*. Fix a point $P \in U$. Consider the collection $\mathcal{C} = \mathcal{C}(U)$ of all curves $\gamma : [0,1] \to U$ such that $\gamma(0) = \gamma(1) = P$. We want to consider once again (as in Section 10.3) the relationship between two such curves γ_1, γ_2 given by the property of being homotopic (at P) to each other in the domain U.

Recall that this means the following: There is a continuous function $H : [0,1] \times [0,1] \to U$ such that

(1) $H(0,t) = \gamma_0(t)$ for all $t \in [0,1]$;

(2) $H(1,t) = \gamma_1(t)$ for all $t \in [0,1]$;

(3) $H(s,0) = H(s,1) = P$ for all $s \in [0,1]$.

The property of being homotopic is an equivalence relation on $\mathcal{C}$. We leave the details of this assertion to the reader. Let the collection of all equivalence classes be called $\mathcal{H}$.

Now we can define a binary operation on $\mathcal{H}$ which turns it into a group. Namely, suppose that γ, μ are curves in $\mathcal{C}$. Then we want to define $\gamma \cdot \mu$ to be the curve γ *followed* by the curve μ. Although the particular parametrization has no significance, for specificity we define $\gamma \cdot \mu : [0,1] \to \mathbb{C}$ by

$$(\gamma \cdot \mu)(t) = \begin{cases} \gamma(2t) & \text{if} \quad 0 \le t \le 1/2 \\ \mu(2t-1) & \text{if} \quad 1/2 < t \le 1. \end{cases}$$

We invite the reader to check that **(i)** this operation is well defined on equivalence classes and **(ii)** it is associative on equivalence classes—even though $(\gamma \cdot \mu) \cdot \delta$ is not literally equal to $\gamma \cdot (\mu \cdot \delta)$, the two are indeed homotopic.

Second, we declare the identity element to be the equivalence class $[e]$ containing the curve that is constantly equal to P. If $[\mu]$ is any equivalence class in $\mathcal{H}$, then $[e] \cdot [\mu] = [\mu] \cdot [e] = [\mu]$, as is easy to see.

Finally, let $\gamma \in \mathcal{C}$. The curve γ^{-1} is defined by $\gamma^{-1}(t) = \gamma(1-t)$. Both curves have the same image, and both curves begin and end at P; but γ^{-1} is "γ run backwards," so to speak. Check for yourself that the operation of taking inverses respects the equivalence relation. So the operation is well defined on equivalence classes. Therefore the inversion operation acts on $\mathcal{H}$. Also, one may check that $[\gamma] \cdot [\gamma]^{-1} = [\gamma]^{-1} \cdot [\gamma] = [e]$ (Exercise 20).

Thus $\mathcal{H}$, equipped with the operation $\cdot$, forms a group. This group is usually denoted by $\pi_1(U)$ and is called *the first homotopy group of U* or, more commonly, the *fundamental group* of U. [In conversation, topologists just call it π_1, or "pi-one."] The fundamental group is independent, up to group isomorphism, of the base point P with respect to which it is calculated (Exercise 22). Thus one can speak of *the* fundamental group of U.

Let $\Psi : U \to V$ be a continuous mapping that takes the point $P \in U$ to the point $Q \in V$. Then Ψ induces a mapping Ψ_* from $\pi_1(U)$ at P to $\pi_1(V)$ at Q by

$$\pi_1(U) \ni [\gamma] \mapsto [\Psi \circ \gamma] \in \pi_1(V).$$

Of course one needs to check that the mapping Ψ_* is well defined, and we leave the details of that argument to Exercise 26 (or see [MCC] or [GG]). In fact the mapping Ψ_* is a group homomorphism. It can be checked that if $\Psi : U \to V$ and $\Phi : V \to W$, then

$$(\Phi \circ \Psi)_* = \Phi_* \circ \Psi_*. \tag{$*$}$$

Now suppose that Ψ is a homeomorphism; that is, Ψ is continuous, one-to-one, and onto, and has a continuous inverse. Then Ψ^{-1} also induces a map of homotopy groups, and it follows from $(*)$ that Ψ_* is a group isomorphism of homotopy groups.

If a domain has fundamental group consisting of just one element (the identity), then it is called *(topologically) simply connected*. [We have earlier studied the concept of a domain being analytically, or holomorphically, simply connected. It turns out that the analytic notion and the topological notion are equivalent when the topological space in question is a planar domain. This assertion will be proved in Section 11.3.] If a domain has a fundamental group consisting of more than one element, then it is called *multiply connected*.

It is immediate from our definitions that a simply connected domain and a multiply connected domain cannot be homeomorphic. In detail, suppose that Φ were a homeomorphism of a simply connected domain U with a multiply connected domain V. Then Φ_* would be a group isomorphism of the groups $\pi_1(U)$ and $\pi_1(V)$. But $\pi_1(U)$ consists of a single element and $\pi_1(V)$ contains more than one element. This is impossible, and the contradiction establishes our assertion.

Here is a more profound application of these ideas. Let U be the unit disc. Then U is simply connected. To see this, let $P = (0, 0)$ be the base point in U. If $\gamma(t) = (\gamma_1(t), \gamma_2(t))$ is any curve in $\mathcal{C}(U)$, then the mapping

$$H(s,t) = ((1-s)\gamma_1(t), (1-s)\gamma_2(t))$$

is a homotopy of γ with the constantly-equal-to-P mapping. This shows that $\mathcal{H}$ has just one equivalence class, that is, that $\pi_1(U)$ is a group containing just one element.

Now let V be the annulus $\{z : 1 < |z| < 3\}$. Let the base point be $Q = (2, 0)$. Then the important elements of $\mathcal{C}$ are the mappings $\gamma_j : [0, 1] \to V$ defined by

$$\gamma_j(t) = 2e^{2\pi i j t},\ j \in \mathbb{Z}.$$

Notice that γ_0 is the curve that is constantly equal to Q. Also γ_1 is the curve that circles the annulus once in the counterclockwise direction. Furthermore, γ_{-1} is the curve that circles the annulus once in the clockwise direction. In general, γ_j circles the annulus j times in the counterclockwise direction when j is positive; it circles the annulus $|j|$ times in the clockwise direction when j is negative.

It is intuitively plausible, but not straightforward to prove rigorously, that if $j \neq k$, then γ_j is not homotopic to γ_k (at the point 2); this assertion will follow from Corollary 11.2.5, to be proved in the next section. [If $[\gamma_j \cdot \gamma_k^{-1}]$ is $[e]$, then Corollary 11.2.5 implies that $\oint (1/z)\, dz$ over $\gamma_j \cdot \gamma_k^{-1}$ equals 0, while the integral is easily computed to be $2\pi i(j-k)$. Therefore $[\gamma_j \cdot \gamma_k^{-1}] = [e]$ if and only if $j = k$.] Thus there are (at least) countably many distinct homotopy classes. It is easy to see that $[\gamma_1] \cdot [\gamma_1] = [\gamma_2]$ and, in general, $[\gamma_1] \cdot [\gamma_j] = [\gamma_j] \cdot [\gamma_1] = [\gamma_{j+1}]$ when $j \geq 0$. Also $\gamma_{-1} = [\gamma_1]^{-1}$. It requires some additional work to prove that every closed curve at the base point Q is homotopic to one of the γ_j (and to only one, since the γ_j's are not homotopic to each other)—see Exercises 27, 28, and 29. Once this is shown, it follows that $[\gamma_1]$ generates $\pi_1(V)$ as a cyclic group of countably many elements. In summary, $\pi_1(V)$ is group-theoretically isomorphic to the additive group $\mathbb{Z}$.

It follows, in particular, that the annulus is not homeomorphic to the unit disc: This assertion follows simply from the fact that γ_1 is not homotopic to the constant curve γ_0, for instance (or by direct comparison of homotopy groups). One can think of the annulus as a domain with one hole in it. Similarly, there is an obvious, intuitive idea of a domain with k holes, $k \geq 0$ ($k = 0$ is just the case of a domain homeomorphic to the disc). It turns out that any planar domain with k holes is homeomorphic to any other planar domain with k holes; also, a domain with k holes is never homeomorphic to a domain with ℓ holes if $k \neq \ell$. This statement is difficult to make precise, and even more difficult to prove. So we shall leave this topic as a sort of invitation to algebraic topology, with some further ideas discussed in Section 11.5.

11.2. The Cauchy Integral Formula for Multiply Connected Domains

In previous chapters we treated the Cauchy integral formula for holomorphic functions defined on holomorphically simply connected (h.s.c.) domains. [In Chapter 4 we discussed the Cauchy formula on an annulus, in order to aid our study of Laurent expansions.] In many contexts we focused our attention on discs and squares. However it is often useful to have an extension of the Cauchy integral theory to multiply connected domains. While this matter is straightforward, it is not trivial. For instance, the holes in the domain may harbor poles or essential singularities of the function being integrated. Thus we need to treat this matter in detail.

The first stage of this process is to notice that the concept of integrating a holomorphic function along a piecewise C^1 curve can be extended to apply to continuous curves. The idea of how to do this is simple: We want to define $\oint_\gamma f$, where $\gamma : [a, b] \to \mathbb{C}$ is a continuous curve and f is holomorphic on a neighborhood of $\gamma([a, b])$. If $a = a_1 < a_2 < \cdots < a_{k+1} = b$ is a subdivision of $[a, b]$ and if we set γ_i to be γ restricted to $[a_i, a_{i+1}]$, then certainly we would require that

$$\oint_\gamma f = \sum_{i=1}^{k} \oint_{\gamma_i} f \,.$$

Suppose further that, for each $i = 1, \ldots, k$, the image of γ_i is contained in some open disc D_i on which f is defined and holomorphic. Then we would certainly want to set

$$\oint_{\gamma_i} f = F_i(\gamma_i(a_{i+1})) - F_i(\gamma_i(a_i)) \,,$$

where $F_i : D_i \to \mathbb{C}$ is a holomorphic antiderivative for f on D_i. Thus $\oint_\gamma f$ would be determined. It is a straightforward if somewhat lengthy matter (Exercise 36) to show that subdivisions of this sort always exist and that the resulting definition of $\oint_\gamma f$ is independent of the particular such subdivision used. This definition of integration along continuous curves enables us to use, for instance, the concept of index for continuous closed curves in the discussion that follows. [We take continuity to be part of the definition of the word "curve" from now on, so that "closed curve" means continuous closed curve, and so forth.] This extension to continuous, closed curves is important: A homotopy can be thought of as a continuous family of curves, but these need not be in general piecewise C^1 curves, and it would be awkward to restrict the homotopy concept to families of piecewise C^1 curves.

Definition 11.2.1. Let U be a connected open set. Let $\gamma : [0, 1] \to U$ be a continuous closed curve. We say that γ is *homologous to* 0 if $\text{Ind}_\gamma(P) = 0$ for all points $P \in \mathbb{C} \setminus U$.

The intuition here is just this: Suppose that γ is a simple closed curve in U. Intuitively, one expects the complement of γ to have two components, a bounded "interior" and an unbounded "exterior", as indeed it does (see the Jordan curve theorem in [WHY]). If the simple closed curve γ encircles a hole in U, then it is not homologous to zero. For if c is a point of $\mathbb{C} \setminus U$ that lies in that hole, then the index of γ with respect to c will not be zero. On the other hand, if γ does not encircle any hole, then any point in $\mathbb{C} \setminus U$ would lie *outside* the closed curve. Also, the index of γ with respect to that point would be 0—see Exercise 9. Thus, intuitively, a (simple) closed curve is expected to be homologous to 0 in U if and only if it does not encircle any hole in U.

Definition 11.2.2. A connected open set U is *homologically trivial* if every closed curve in U is homologous to 0.

It is natural to wonder whether a curve that is homotopic to a point is then necessarily homologous to zero, or vice versa. In fact it is easy to see that a "homotopically trivial curve" γ is indeed homologous to zero (cf. Exercise 23 and also Exercise 30).

Lemma 11.2.3. *If $U \subseteq \mathbb{C}$ is a connected open set and γ is a closed curve in U, based at $P \in U$, that is homotopic to the constant curve at P, then γ is homologous to 0. In particular, if U is simply connected, then it is homologically trivial.*

Proof. Let $H(s, t)$ be a homotopy of γ to a point $P \in U$. Set $H_s(t) = H(s, t)$. Let c be a point in the complement of U. Then

$$\text{Ind}_\gamma(c) = \frac{1}{2\pi i} \oint_\gamma \frac{1}{\zeta - c}\, d\zeta$$

by definition. Rewrite this as

$$I_0 = \frac{1}{2\pi i} \oint_{H_0} \frac{1}{\zeta - c}\, d\zeta.$$

It is easy to check that the value of

$$I_s = \frac{1}{2\pi i} \oint_{H_s} \frac{1}{\zeta - c}\, d\zeta$$

is a continuous function of s; since it is integer-valued, it must be constant. When s is sufficiently near to 1, then the curve $H(s, \,\cdot\,)$ will be contained in a small open disc contained in U and centered at P. But then it is immediate that $I_s = 0$. Since the expression is constant in s, it follows that $I_0 = 0$. □

As noted, a closed curve that is homotopic to a constant curve is homologous to 0. But the converse is not true in general. It is *not always true* for individual curves that a curve that is homologous to zero is homotopic to a point. What *is* true is that if every curve in the domain U is homologous to 0, then every curve in the domain is also homotopic to a point (i.e., homotopic to zero). We shall say something about this matter in Section 11.4. In particular, we shall prove that a connected open set $U \subseteq \mathbb{C}$ is (topologically) simply connected if and only if it is homologically trivial (see Theorem 11.4.1).

The main result of this section is the following theorem. The proof that we present uses an idea of A. F. Beardon, as implemented in [AHL2].

Theorem 11.2.4 (Cauchy integral theorem for multiply connected domains)**.** *Let $U \subseteq \mathbb{C}$ be a connected open set and suppose that f is holomorphic on U. Then*

$$\oint_\gamma f(z)\, dz = 0$$

for any curve γ in U that is homologous to zero.

Combining Lemma 11.2.3 with Theorem 11.2.4 yields the following corollary:

Corollary 11.2.5 (Homotopy version of the Cauchy integral theorem)**.** *If $f : U \to \mathbb{C}$ is holomorphic and if $\gamma : [0,1] \to \mathbb{C}$ is a closed curve at $P \in U$ that is homotopic to the constant curve at P, then $\oint_\gamma f(z)\, dz = 0$.*

This corollary can be proved directly, without reference to Theorem 11.2.4; a direct proof is given in Exercises 35–37.

Proof of Theorem 11.2.4. It is convenient to treat first the case of U bounded. Let γ be a curve that lies in U. Let $\mu > 0$ be the distance of the image of γ to $\mathbb{C} \setminus U$. Let $0 < \delta < \mu/2$.

Cover the plane by a mesh of closed squares, with sides parallel to the axes, disjoint interiors, and having side length δ. Let $\{Q_j\}_{j=1}^K$ be those closed squares from the mesh that lie entirely in U. Let U_δ denote the interior of

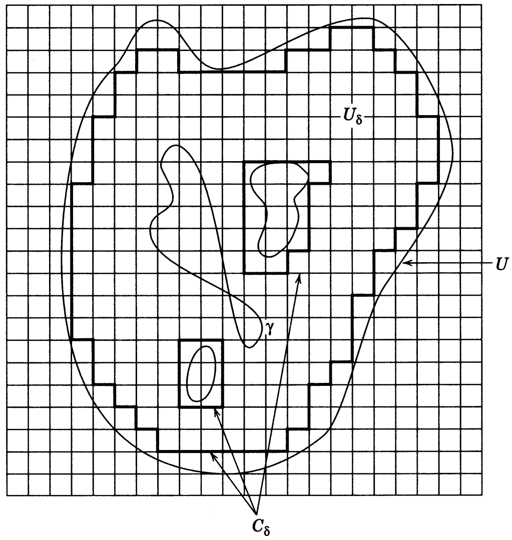

Figure 11.1

the union of the finitely many closed squares $Q_1, \ldots, Q_K$. Clearly, from the choice of δ, the image of γ lies in U_δ. Let C_δ denote the boundary of U_δ. Refer to Figure 11.1.

We orient C_δ as follows: Equip each Q_j with the counterclockwise orientation. When the sides of two of the Q_j meet, then integration along those two sides cancels. The edges that remain—that is, whose integrations do not cancel out—comprise the boundary C_δ of U_δ. Each of those edges is oriented, and their orientations are consistent. Take that as the orientation of C_δ, so that integration over C_δ is defined.

Let $c \in U \setminus U_\delta$. Then c lies in some square Q which is not one of the Q_j. Also, there must be a point $x \in Q$ that does not lie in U. The line segment joining c to x lies entirely in Q and therefore, in particular, does not intersect U_δ. Now $\mathrm{Ind}_\gamma(x) = 0$ by hypothesis. By the continuity of the integral, it also must be the case that $\mathrm{Ind}_\gamma(c) = 0$. Now c was an arbitrary element of $U \setminus U_\delta$. In particular c could be an arbitrary element of C_δ. We conclude that $\mathrm{Ind}_\gamma(c) = 0$ for every $c \in C_\delta$.

Suppose that $z \in U_\delta$ is any point. Suppose also at the outset that z lies in the interior of some square Q_{j_0}. [The case that z lies on the edge of some square will follow afterward by the continuity of the integral.] Then, by the Cauchy integral formula for squares, and for Q_j any of the chosen squares that make up U_δ,

$$\frac{1}{2\pi i} \oint_{\partial Q_j} \frac{f(\zeta)}{\zeta - z} \, d\zeta = \begin{cases} f(z) & \text{if } \ j = j_0 \\ 0 & \text{if } \ j \neq j_0. \end{cases}$$

Summing over j, and noting the cancellation of the integrals over edges of the squares that lie in the interior of U_δ, we see that

$$\frac{1}{2\pi i}\oint_{C_\delta}\frac{f(\zeta)}{\zeta - z}\,d\zeta = f(z).$$

Now integrate both sides in a complex line integral over $z \in \gamma$. We obtain

$$\oint_\gamma f(z)\,dz = \oint_\gamma\left(\frac{1}{2\pi i}\oint_{C_\delta}\frac{f(\zeta)}{\zeta - z}\,d\zeta\right)dz.$$

We may apply the version of Fubini's theorem that appears in Appendix A, since the integrand is a jointly continuous function of both its arguments (see Exercise 39). Therefore we find that

$$\oint_\gamma f(z)\,dz = \oint_{C_\delta} f(\zeta)\left(\frac{1}{2\pi i}\oint_\gamma\frac{1}{\zeta - z}\,dz\right)d\zeta.$$

But of course the integral inside the parentheses is just the (negative of the) index of the curve γ about the point $\zeta \in C_\delta$. This has already been established to be 0. As a result,

$$\oint_\gamma f(z)\,dz = 0$$

as desired.

In case U is not bounded, we may take the intersection of U with a large disc that contains γ. Then the proof proceeds essentially as before, with the additional observation that $\mathrm{Ind}_\gamma(z) = 0$ for z outside this disc and hence $\mathrm{Ind}_\gamma(z) = 0$ for all z in the complement of $U \cap$ (the disc). □

As discussed earlier, we are looking for a Cauchy integral formula for multiply connected regions; in such a formula, the index ought to be used to take into account the possibility that the curve of integration goes around one or more holes. Such a formula in fact follows directly, as it did in the case of the disc and the square, from the Cauchy integral theorem:

Theorem 11.2.6 (Cauchy integral formula for multiply connected domains)**.** *Let $U \subseteq \mathbb{C}$ be a connected open set. Let γ be a closed curve, with image in U, that is homologous to zero. If z is a point of U, and if f is holomorphic on U, then*

$$\mathrm{Ind}_\gamma(z)\cdot f(z) = \frac{1}{2\pi i}\oint_\gamma\frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

In particular, the formula holds for any closed curve γ that is homotopic to a constant curve.

Proof. The proof of the first statement is obtained by applying Theorem 11.2.4 to the function $\zeta \mapsto [f(\zeta) - f(z)]/(\zeta - z)$. The details are left as an exercise. The second statement follows from the first by Lemma 11.2.3, or by applying Corollary 11.2.5 to the function $\zeta \mapsto [f(\zeta) - f(z)]/(\zeta - z)$. □

11.3. Holomorphic Simple Connectivity and Topological Simple Connectivity

In Section 6.7 we proved that if a proper subset U of the complex plane is holomorphically (analytically) simply connected, then it is conformally equivalent to the unit disc. At that time we promised to relate holomorphic simple connectivity to some more geometric notion. This is the purpose of the present section.

It is plain that if a domain is holomorphically simply connected, then it is (topologically) simply connected. For the hypothesis implies (by the analytic form of the Riemann mapping theorem, Theorem 6.6.3) that the domain is either conformally equivalent to the disc or to $\mathbb{C}$. Hence it is certainly topologically equivalent to the disc. Therefore it is simply connected in the topological sense.

For the converse direction, we notice that if a domain U is (topologically) simply connected, then simple modifications of ideas that we have already studied show that any holomorphic function on U has an antiderivative: Suppose that $f : U \to \mathbb{C}$ is holomorphic. Choose a point $P \in U$. If γ_1, γ_2 are two piecewise C^1 curves from P to another point Q in U, then $\oint_{\gamma_1} f(z)\,dz = \oint_{\gamma_2} f(z)\,dz$. This equality follows from applying Corollary 11.2.5 to the (closed) curve made up of γ_1 followed by "γ_2 backwards" (this curve is homotopic to a constant, since U is simply connected). Define $F(Q) = \oint_\gamma f(z)\,dz$ for any curve γ from P to Q. It follows, as in the proof of Morera's theorem (Theorem 3.1.4), that F is holomorphic and $F' = f$ on U.

Thus one has the chain of implications for a domain $U \subset \mathbb{C}$ but unequal to $\mathbb{C}$: homeomorphism to the disc $D \Rightarrow$ (topological) simple connectivity $\Rightarrow$ holomorphic simple connectivity $\Rightarrow$ conformal equivalence to the disc D $\Rightarrow$ homeomorphism to the disc D, so that all these properties are logically equivalent to each other. [*Note:* It is surprisingly hard to prove that "simple connectivity" $\Rightarrow$ "homeomorphism to D" without using complex analytic methods.] This in particular proves Theorem 6.4.2 and also establishes the following assertion, which is the result usually called the Riemann mapping theorem:

Theorem 11.3.1 (Riemann mapping theorem). *If U is a connected, simply connected open subset of $\mathbb{C}$, then either $U = \mathbb{C}$ or U is conformally equivalent to the unit disc D.*

11.4. Simple Connectivity and Connectedness of the Complement

If γ is a closed curve in $\mathbb{C}$, then the winding number, or index, $\operatorname{Ind}_\gamma(a)$ is a continuous function of the point $a \in \mathbb{C} \setminus \gamma$. In particular, if $\gamma : [0,1] \to U$ is a closed curve in an open and connected set U, then $\operatorname{Ind}_\gamma(a)$ is continuous on $\mathbb{C} \setminus U$. It follows that the winding number $\operatorname{Ind}_\gamma(a)$ is constant on each connected component of $\mathbb{C} \setminus U$. In particular, if C_1 is an unbounded component of $\mathbb{C} \setminus U$, then $\operatorname{Ind}_\gamma(a) = 0$ for all $a \in C_1$ (cf. Exercise 9). If U is bounded, then $\mathbb{C} \setminus U$ has exactly one unbounded component. Hence if U is bounded and $\mathbb{C} \setminus U$ has only one component, then that component is unbounded and $\operatorname{Ind}_\gamma(a) \equiv 0$ for all $a \in \mathbb{C} \setminus U$.

In Section 11.2, we saw that if U is simply connected, then $\operatorname{Ind}_\gamma(a) = 0$ for all $a \in \mathbb{C} \setminus U$ (Lemma 11.2.3). It is natural to suspect that there is some relationship between the two possible reasons why $\operatorname{Ind}_\gamma(a) = 0$ for all $a \in \mathbb{C} \setminus U$, one reason being that U is simply connected, the other that $\mathbb{C} \setminus U$ has only one component. [This intuition is increased by noting that the annulus is not simply connected and its complement has two components. Moreover the curves which are not homotopic to constants are ones that, intuitively, go around the bounded component of the complement of the domain.] It is, in fact, true that these conditions are related and indeed equivalent for bounded domains. The following theorem makes the expectation precise:

Theorem 11.4.1. *If U is a bounded, open, connected subset of $\mathbb{C}$, then the following properties of U are equivalent:*

(a) *U is simply connected;*

(b) *$\mathbb{C} \setminus U$ is connected;*

(c) *for each closed curve γ in U and $a \in \mathbb{C} \setminus U$, $\operatorname{Ind}_\gamma(a) = 0$.*

We already know, from Section 11.2, that **(a)** implies **(c)**. Also, **(b)** implies **(c)** by the reasoning already noted: If $\mathbb{C} \setminus U$ is connected, then $\operatorname{Ind}_\gamma(a)$, being continuous and integer-valued, is constant on $\mathbb{C} \setminus U$. But $\operatorname{Ind}_\gamma(a) = 0$ for all $a \in \mathbb{C} \setminus U$ with $|a| > \sup\{|z| : z \in U\}$ by Exercise 9. Therefore **(c)** holds if **(b)** does.

To prove that **(c)** implies **(a)**, combine Theorem 11.2.4 with Lemma 4.5.2 to see that **(c)** implies holomorphic simple connectivity. Then, as explained in the second paragraph of Section 11.3, simple connectivity follows.

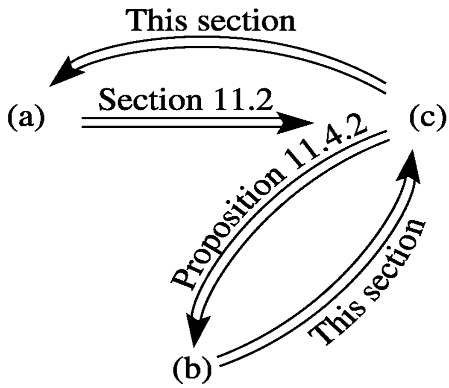

Figure 11.2

The remaining assertion, that **(c)** $\Rightarrow$ **(b)** (which will complete the proof), is the most difficult. We prove this next. By now the logic of the proof is probably obscure. Figure 11.2 outlines the reasoning.

The following proposition gives the critical step **(c)** $\Rightarrow$ **(b)**:

Proposition 11.4.2. *If U is a bounded, connected open set and if $\mathbb{C} \setminus U$ is not connected, then there is a closed curve γ lying in U such that* $\mathrm{Ind}_\gamma(a) \neq 0$ *for some $a \in \mathbb{C} \setminus U$.*

Proof of Proposition 11.4.2. If the closed set $\mathbb{C} \setminus U$ is not connected, then it is expressible as the union of two disjoint, nonempty, closed sets C_1 and C_2. [In detail: By the definition of connectedness, the fact that $\mathbb{C} \setminus U$ is not connected means that $\mathbb{C} \setminus U = C_1 \cup C_2$ where C_1, C_2 are disjoint, nonempty, and closed in $\mathbb{C} \setminus U$ in the relative topology of $\mathbb{C} \setminus U$. But, since $\mathbb{C} \setminus U$ is a closed set in $\mathbb{C}$, the sets C_1 and C_2 are necessarily closed in $\mathbb{C}$ also.] Precisely one of these two sets can be unbounded, since $\{z \in \mathbb{C} : |z| \geq 1 + \sup_{w \in U} |w|\}$ is connected (Exercise 18(a)). Let C_2 be the unbounded set and let C_1 be the bounded one. Choose $a \in C_1$. Now define

$$d = \inf\{|z - w| : z \in C_1 \text{ and } w \in C_2\}.$$

Notice that $d > 0$ because C_1, being closed and bounded, is compact and C_2 is closed (Exercise 17).

Now cover the plane with a grid of squares of side $d/10$ (and sides parallel to the coordinate axes) and suppose without loss of generality that a is the center of one of these squares. Now look at the set S of those closed squares from the grid that have a nonempty intersection with C_1. Note that no closed square in the set S can have a nonempty intersection with C_2 because if a $\frac{d}{10} \times \frac{d}{10}$ square intersected both C_1 and C_2, then there would be points in C_1 and C_2 that are within distance $\sqrt{2}d/10$ (i.e., the length of the diagonal of that square). That would be a contradiction to the choice of d.

Now let S_1 be the set of squares in the collection S which can be reached from the square containing a by a chain of squares in S, each having an edge in common with the previous one (Figure 11.3). Orient all the squares in S_1 as shown in Figure 11.4 (counterclockwise), and consider all the edges which do not belong to two squares in S_1. This set of edges is clearly the union of a finite number of oriented, piecewise linear, closed curves $\gamma_j,\ j = 1, \ldots, k$, in U; the orientations of these curves are determined by the orientations of the squares, with the curves disjoint except possibly for isolated vertices in common—Figure 11.5. [*Exercise:* Show by example how a γ_j can occur that is not the entire boundary of a component of S.]

Now consider the sum of the integrals over the oriented squares:

$$\sum_{Q \in S_1} \frac{1}{2\pi i} \oint_{\partial Q} \frac{1}{\zeta - a}\, d\zeta.$$

This sum equals 1 because for the square Q_a containing a we have

$$\frac{1}{2\pi i} \oint_{\partial Q_a} \frac{1}{\zeta - a}\, d\zeta = 1, \tag{$*$}$$

while for all the other squares we have

$$\frac{1}{2\pi i} \oint_{\partial Q} \frac{1}{\zeta - a}\, d\zeta = 0.$$

On the other hand, because a common edge of two squares in S_1 is counted once in one direction and once in the other, the sum

$$\sum_{Q \in S_1} \frac{1}{2\pi i} \oint_{\partial Q} \frac{1}{\zeta - a}\, d\zeta = \sum_j \frac{1}{2\pi i} \oint_{\gamma_j} \frac{1}{\zeta - a}\, d\zeta\,.$$

In particular, by $(*)$, it must be that

$$\sum_j \frac{1}{2\pi i} \oint_{\gamma_j} \frac{1}{\zeta - a}\, d\zeta \neq 0.$$

Hence at least one term of this sum is nonzero, that is, some γ_j is a closed curve in U with nonzero winding number about the point $a \in C_1$. This is the desired conclusion. □

The reader should think carefully about the argument just given. The point, philosophically, is that the use of the squares enables us to make a specific problem out of what at first appears to be a rather nebulous intuition. In particular, it is not necessary to know any results about boundaries of general regions in order to establish the proposition.

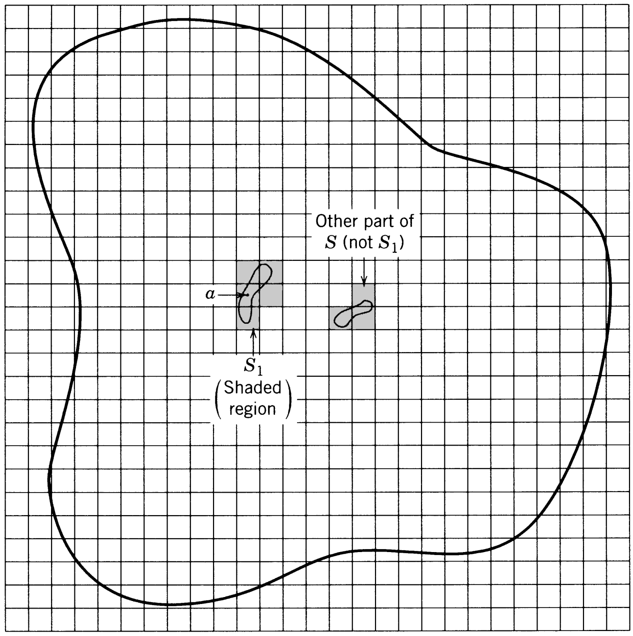

Figure 11.3

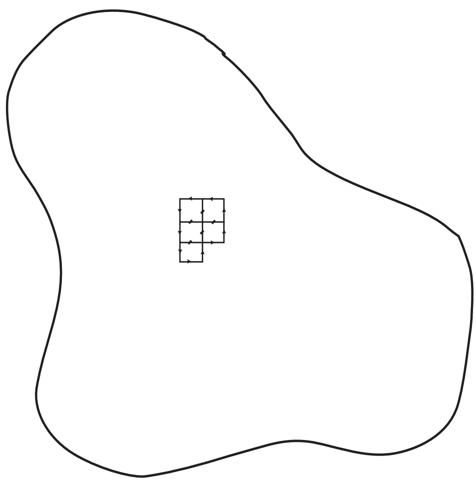

Figure 11.4

It is interesting to note that an independently proved result in Chapter 12 will show directly that **(b)** implies **(a)** in Theorem 11.4.1. We already know that if U is analytically simply connected, then it must be simply connected. Moreover, if every holomorphic function f on U has a holomorphic antiderivative, then U is holomorphically simply connected by definition.

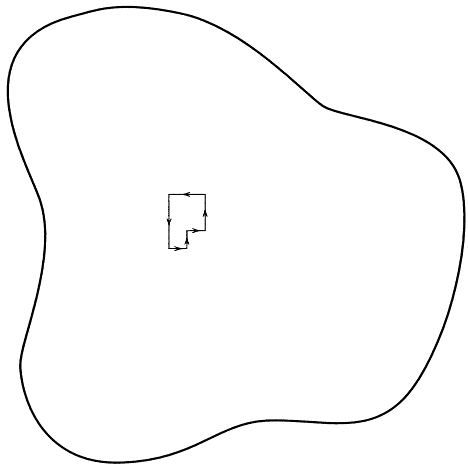

Figure 11.5

Finally we know that f has a holomorphic antiderivative if

$$\oint_\gamma f(z)\, dz = 0 \qquad (**)$$

for every piecewise smooth closed curve γ in U.

The advantage of condition $(**)$ is that it is closed under normal limits. In particular, we shall apply the following result, which is a corollary of a theorem that will be proved independently in Section 12.1 (see Exercise 38):

Proposition 11.4.3 (Corollary of Runge's theorem)**.** *If U is a bounded, connected open set in $\mathbb{C}$ with $\mathbb{C} \setminus U$ connected, then each holomorphic function $f : U \to \mathbb{C}$ is the limit, uniformly on compact subsets of U, of some sequence $\{P_j\}$ of polynomials.*

This theorem is a generalization of the fact that a holomorphic function on a disc can be approximated uniformly on compact sets by the partial sums of its power series.

Now assume **(b)** in Theorem 11.4.1. For each fixed, closed curve γ,

$$\oint_\gamma P(\zeta)\, d\zeta = 0$$

for any polynomial P (because P certainly has an antiderivative). Therefore

$$\oint_\gamma f(\zeta)\, d\zeta = 0$$

for each function f that is the uniform limit on γ of a sequence of polynomials. In particular, if $\mathbb{C} \setminus U$ is connected, then

$$\oint_\gamma f(\zeta)\, d\zeta = 0$$

for every holomorphic function $f : U \to \mathbb{C}$. Thus **(b)** implies **(a)** as required.

11.5. Multiply Connected Domains Revisited

We developed earlier the intuitive idea that a (topologically) simply connected domain was one with "no holes." This chapter has given that notion a precise form: **(i)** the absence of a hole in the sense of there being no points in the complement for a closed curve in the domain to wrap around is exactly the same as **(ii)** the absence of a hole in the sense of there being no bounded component of the complement of the (bounded) domain. But what about domains that *do* have holes, that is, that *do* have nonempty bounded components of their complement?

It is natural, in case the complement $\mathbb{C} \setminus U$ of a bounded domain U has a finite number k of bounded components, to say that U has k holes. [As before, $\mathbb{C} \setminus U$ has exactly one unbounded component.] In more customary if less picturesque terminology, U is said to have *connectivity* $k + 1$ (i.e., the number of components of $\mathbb{C} \setminus U$). Then it is also tempting to ask whether the number of holes can be determined by looking at $\pi_1(U)$. For $k = 0$, we know this to be true: U has no holes, that is, $k = 0$ if and only if $\pi_1(U)$ is trivial, that is, $\pi_1(U)$ equals the group with one element (the identity). But can we tell the difference between a domain with, say, two holes and a domain with three holes just by examining their fundamental groups? We know that the fundamental groups will then both be nontrivial, but do they somehow encode exactly how many holes there are?

The answer to this question is "yes." But this assertion is not so easy to prove. The whole situation is really part of the subject of algebraic topology, not of complex analysis. Therefore we do not want to go into all the details. But, just so you will know the facts, we shall summarize the key points that topologists have established:

If U is a bounded domain with k holes (i.e., $\mathbb{C} \setminus U$ has k bounded components), then U is homeomorphic to the open disc with k points removed. The disc with k points removed is not homeomorphic to the disc with ℓ points removed if $k \neq \ell$. Indeed, if $k \neq \ell$, then the disc with k points removed has a different fundamental group than the disc with ℓ points removed. [Here "different" means that the two groups are not isomorphic as groups.]

We can actually describe what these fundamental groups are. For this, recall that the *free group* F generated by a set A (or the *free group* on A) is defined to be the set of all finite "words" made up of elements of A, that is, finite sequences with ± 1 exponents:

$$a_1^{\pm 1} a_2^{\pm 1} \cdots a_n^{\pm 1}.$$

Here n is arbitrary but finite. A word is not allowed if a^{+1} is followed by a^{-1} (same a) or if a^{-1} is followed by a^{+1}.

These words are combined—this is the group operation—by juxtaposition:

$$\left(a_1^{\pm 1} a_2^{\pm 1} \cdots a_m^{\pm 1}\right) \cdot \left(b_1^{\pm 1} b_2^{\pm 1} \cdots b_n^{\pm 1}\right) = a_1^{\pm 1} a_2^{\pm 1} \cdots a_m^{\pm 1} b_1^{\pm 1} b_2^{\pm 1} \cdots b_n^{\pm 1}$$

(with a's, b's $\in A$), subject to the rule that if $a_m = b_1$, and if a_m and b_1 appear with oppositely signed exponents, then they shall be "cancelled." If, in this situation, a_{m-1} and b_2 are equal but have oppositely signed exponents, then they are to be cancelled as well; and so forth. For instance,

$$(a_1 a_2 a_3)(a_3^{-1} a_2^{-1} a_1) = a_1 a_1.$$

(For convenience, $a_1 a_1$ is usually written a_1^2.) It is easy to see that the allowable words and the composition described form a group. In an obvious sense, the group just described is generated by the elements of A with no "relations" assumed: hence it is called the "free group" generated by A.

With this terminology in mind, we can now describe the fundamental groups of domains with holes: The fundamental group of a bounded domain with k holes is (isomorphic to) the free group on (a set with) k generators.

It is not *quite* obvious that the free group F_k on k generators is not isomorphic to the free group F_ℓ on ℓ generators when $k \neq \ell$. To see that these groups are not isomorphic, the thing to do is to look at

$$F_k/[F_k, F_k]$$

and

$$F_\ell/[F_\ell, F_\ell].$$

(Recall from group theory that if G is a group, then the *commutator subgroup* $[G, G]$ of G is the subgroup generated by the elements

$$\{g_1 g_2 g_1^{-1} g_2^{-1} : g_1, g_2 \in G\}.)$$

You can check for yourself that

$$F_k/[F_k, F_k]$$

is isomorphic to the additive group

$$\{(x_1, \ldots, x_k) \in \mathbb{R}^k : x_j \in \mathbb{Z}, \text{ all } j\}.$$

From this, it is easy to see that

$$F_k/[F_k, F_k]$$

is not isomorphic to $F_\ell/[F_\ell, F_\ell]$ if $k \neq \ell$. So F_k cannot be isomorphic to F_ℓ either.

It turns out that a closed curve γ (at a base point) in a domain U is homologous to 0 in the sense we have already defined if and only if

$$[\gamma] \in [\pi_1(U), \pi_1(U)].$$

For a bounded domain, with k holes, $k \geq 2$, the group $[\pi_1(U), \pi_1(U)] \neq 0$; equivalently, $\pi_1(U) \cong F_k$ is not commutative in this case. So, for $k \geq 2$, there are always curves that are homologous to 0 but not homotopic to the constant curve at the base point. This point is discussed in more detail in the exercises.

Even a little more is true: Define an equivalence relation on the closed curves γ starting and ending at the base point P by

$$\gamma_1 \sim \gamma_2$$

if $\mathrm{Ind}_{\gamma_1}(Q) = \mathrm{Ind}_{\gamma_2}(Q)$ for every $Q \in \mathbb{C} \setminus U$. Equivalently, $\gamma_1 \overset{H}{\sim} \gamma_2$ (read γ_1 is homologous to γ_2) if $\gamma_1 \cdot \gamma_2^{-1}$ is homologous to 0, where $\gamma_2^{-1}(t) = \gamma_2(1-t)$, as in finding homotopy inverses (and $\cdot$ denotes the composition of curves defined in Section 11.1). The equivalence classes relative to this equivalence relation are called *homology classes* (actually, homology classes at P—the base point usually is not important and is therefore ignored). We shall write $[\gamma]_h$ for the homology class of γ.

We can define a group structure on the set of homology classes by defining a composition $+_h$:

$$[\gamma_1]_h +_h [\gamma_2]_h \overset{\mathrm{def}}{=} [\gamma_1 \cdot \gamma_2]_h,$$

where $\cdot$ is the composition of paths used to defined π_1. We used a plus sign for this composition because the group of homology classes is always abelian, that is,

$$[\gamma_1]_h +_h [\gamma_2]_h = [\gamma_2]_h +_h [\gamma_1]_h.$$

This is actually easy to see since $\mathrm{Ind}_{\gamma_1 \cdot \gamma_2}(Q) = \mathrm{Ind}_{\gamma_2 \cdot \gamma_1}(Q)$. The group of homology classes of curves is denoted $H_1(U)$.

The group of homology classes H_1 and the group of homotopy classes π_1 are related:

$$H_1(U) \cong \pi_1(U)/[\pi_1(U), \pi_1(U)].$$

With this striking fact, we conclude our brief tour of the key ideas of elementary algebraic topology and invite you to consult [MAS] or [GRH] for a more detailed view of these matters. Much of what we have said can be generalized to arbitrary topological spaces, and it is worthwhile to find out how.

Exercises

1. Provide the details of the proof of the Cauchy integral formula (using Theorem 11.2.4) for multiply connected domains, using the fact that $[f(\zeta) - f(z)]/[\zeta - z]$ is a holomorphic function of $\zeta \in U$ for each fixed $z \in U$.

2. A set $S \subseteq \mathbb{R}^N$ is called *path connected* if, for any two points $P, Q \in S$, there is a continuous function (i.e., a *path*) $\gamma : [0,1] \to S$ such that $\gamma(0) = P$ and $\gamma(1) = Q$. Prove that any connected open set in $U \subseteq \mathbb{R}^2$ is in fact path connected. [*Hint:* Show that the set of points S that can be connected by a continuous path to a fixed point P in U is both open and closed in U.]

3. Let $U = \{z \in \mathbb{C} : 1/2 < |z| < 2\}$. Consider the two paths $\gamma_1(t) = e^{2\pi i t}$ and $\gamma_2(t) = e^{4\pi i t}$, $0 \leq t \leq 1$. Prove that γ_1 and γ_2 are not homotopic. [*Hint:* You could use the index as a homotopy invariant. Or you could construct your own invariant in terms of crossings of the segment $\{0+it : 1/2 < t < 2\}$.]

4. Is the union of two topologically simply connected open sets in the plane also topologically simply connected? How about the intersection? What if the word "plane" is replaced by "Riemann sphere"?

5. Let $U_1 \subseteq U_2 \subseteq \cdots \subseteq \mathbb{C}$ be topologically simply connected open sets. Define $\mathcal{U} = \bigcup_j U_j$. Prove that $\mathcal{U}$ is then topologically simply connected. What if the hypothesis of openness is removed?

6. Use your intuition to try and guess the form of π_1 (the first homotopy group) of the set consisting of the plane minus two disjoint closed discs.

7. Let S be the unit sphere in $\mathbb{R}^3$. Calculate the first homotopy group of S. [*Hint:* First show that any closed curve is homotopic to a closed curve that is a finite union of arcs of great circles. Do so by subdividing an arbitrary given closed curve and using the fact that two curves with the same endpoints, both curves lying in a fixed open hemisphere, are homotopic with fixed endpoints.]

8. Refer to Exercise 2 for terminology. Show that if a planar set is path connected, then it must be connected. Show that the set $\{(0,y) : y \in [-1,1]\} \cup \{(x, \sin 1/x) : x \in (0,1]\}$ is connected but not path connected.

9. Suppose that $A \subseteq \mathbb{C}$ is a connected, unbounded subset of $\mathbb{C}$ and that γ is a closed curve in $\mathbb{C}$ that does not intersect A. Show that $\operatorname{Ind}_\gamma(a) = 0$ for all $a \in A$. [*Hint:* The quantity $\operatorname{Ind}_\gamma(a)$ is a continuous, integer-valued function on the connected set A and is hence constant on A. Also, for $a \in A$, with $|a|$ large enough, the index $\operatorname{Ind}_\gamma(a)$ is smaller than 1 and hence must be equal to 0.]

10. Let S be a subset of the plane that is compact and topologically simply connected. Does it follow that the *interior* of S is topologically simply connected? [*Hint:* Look at the complement of the set.]

11. Let T be an open subset of the plane that is topologically simply connected. Does it follow that the *closure* of T is topologically simply connected?

12. Let $f(z) = 1/[(z+1)(z-2)(z+3i)]$. Write a Cauchy integral formula for f, where at least a part of the integration takes place over the circle with center 0 and radius 5, and $|z| < 5$.

13. Let $g(z) = \tan z$. Write a Cauchy integral formula for g, where at least a part of the integration takes place over the circle with center at 0 and radius 10, and $|z| < 10$.

14. Let S and T be topologically simply connected subsets of the plane. Does it follow that $S \setminus T$ is topologically simply connected?

15. Let S be a topologically simply connected subset of $\mathbb{C}$ and let f be a complex-valued, continuous function on S. Does it follow that $f(S)$ is topologically simply connected?

16. Let S be a subset of $\mathbb{C}$ and let $f : \mathbb{C} \to \mathbb{C}$ be an onto, continuous function. Does it follow that $f^{-1}(S)$ is topologically simply connected if S is?

17. Prove that if C_1, C_2 are disjoint, nonempty closed sets in $\mathbb{C}$ and if C_1 is bounded, then $\inf\{|z-w| : z \in C_1, w \in C_2\} > 0$.

18. **(a)** Assume that the domain U is bounded. Prove that there can be just one connected component of the complement of U that is unbounded. [*Hint:* $\{z \in \mathbb{C} : |z| \geq 1 + \sup_{w \in U} |w|\}$ is connected.]

(b) Assume further that the complement of the bounded domain U has only finitely many connected components, and let C_1 be the union of the *bounded* components of the complement. Let C_2 be the unbounded component of the complement. Define

$$d = \inf\{|z-w| : z \in C_1 \text{ and } w \in C_2\}\,.$$

Use Exercise 17 to prove that $d > 0$.

19. Show that part **(a)** of Exercise 18 fails if we remove the hypothesis that U is bounded.

20. Let U be a domain and $P \in U$. Let γ be a closed curve in U that begins and ends at P. Show that $\gamma \cdot \gamma^{-1}$ and $\gamma^{-1} \cdot \gamma$ are homotopic to the constant curve at P. [*Hint:* For the first of these,

$$H(s,t) = \begin{cases} \gamma(2ts) & \text{if} \quad 0 \le t \le 1/2 \\ \gamma(2s(1-t)) & \text{if} \quad 1/2 < t \le 1. \end{cases}$$

21. Let U be a domain. Fix a point $P \in U$. Let $\gamma_1, \gamma_2, \gamma_3$ be closed curves based at P. Show that if γ_1 is homotopic to γ_2 and γ_2 is homotopic to γ_3, then γ_1 is homotopic to γ_3. [*Hint:* If H_1 is a homotopy from γ_1 to γ_2 and H_2 is a homotopy from γ_2 to γ_3, then define

$$H(s,t) = \begin{cases} H_1(2s,t) & \text{if} \quad 0 \le s \le 1/2 \\ H_2(2s-1,t) & \text{if} \quad 1/2 < s \le 1. \end{cases}$$

22. Let U be a domain. Suppose that $\Gamma : [0,1] \to U$ is a continuous curve with $\Gamma(0) = P \in U$ and $\Gamma(1) = Q \in U$. For each continuous, closed curve $\gamma : [0,1] \to U$ with $\gamma(0) = \gamma(1) = P$, define $\Gamma^{-1} \cdot \gamma \cdot \Gamma : [0,1] \to U$ by

$$(\Gamma^{-1} \cdot \gamma \cdot \Gamma)(t) = \begin{cases} \Gamma(1-4t) & \text{if} \quad 0 \le t \le 1/4 \\ \gamma(2(t-1/4)) & \text{if} \quad 1/4 < t \le 3/4 \\ \Gamma(4(t-3/4)) & \text{if} \quad 3/4 < t \le 1. \end{cases}$$

(a) Show that $\Gamma^{-1} \cdot \gamma \cdot \Gamma$ is a continuous curve that starts and ends at Q.

(b) Show that, with Γ fixed, the equivalence class $[\Gamma^{-1} \cdot \gamma \cdot \Gamma]$ of $\Gamma^{-1} \cdot \gamma \cdot \Gamma$ in $\pi_1(U)$ based at Q depends only on the equivalence class $[\gamma]$ of γ in $\pi_1(U)$ based at P.

(c) Show that the mapping

$$[\gamma] \mapsto [\Gamma^{-1} \cdot \gamma \cdot \Gamma]$$

is a group homomorphism.

(d) Show that the composition of the group homomorphisms $[\gamma] \mapsto [\Gamma^{-1} \cdot \gamma \cdot \Gamma]$ followed by $[\delta] \mapsto [\Gamma^{-1} \cdot \delta \cdot \Gamma]$ (for δ a closed curve at Q) is the identity mapping from $\pi_1(U)$ based at P to itself.

(e) Show that $\pi_1(U)$ based at P is isomorphic to $\pi_1(U)$ based at Q for all points P, Q in U.

23. Let $U \subseteq \mathbb{C}$ be a connected open set and $\gamma : [0,1] \to U$ a closed curve in U. We define the statement "γ is freely homotopic to a constant in U" to mean: There is a continuous function $H : [0,1] \times [0,1] \to U$ such that **(1)** $H(1,t)$ is a constant, independent of $t \in [0,1]$; **(2)** $H(s,0) = H(s,1)$ for all $s \in [0,1]$; and **(3)** $H(0,t) = \gamma(t)$ for all $t \in [0,1]$. (One thinks of $H(s,\,\cdot\,)$ as a family of closed curves, the curve corresponding to $s = 0$ being γ and the curve corresponding to $s = 1$ being a constant.) Clearly, if γ is homotopic at $\gamma(0)$ to the constant curve at $\gamma(0)$, in the sense introduced in Section 11.1, then γ is freely homotopic to a constant.

Prove the converse: If γ is freely homotopic to a constant, then γ is homotopic at $\gamma(0)$ to a constant. [*Hint:* The bottom edge of the square can be deformed with fixed endpoints to the curve formed by the other three sides in succession. The H-image of the three-sided curve is itself homotopic at $\gamma(0)$ to the constant curve at $\gamma(0)$.]

24. In the text we defined $\pi_1(U)$ for any (open) domain U. The same construction can be applied to define $\pi_1(X, P)$ for any topological space X with $P \in X$. Here P is a "base point." Think this generalization through, and show that if X is path connected (see Exercise 2—the definition there applies to any topological space), then $\pi_1(X, P)$ and $\pi_1(X, Q)$ are isomorphic groups for all $P, Q \in X$ (see Exercise 22 for a clue).

25. Let S and T be topologically simply connected sets in the plane. Does it follow that $S \times T$ is topologically simply connected?

26. Let $\Psi : X \to Y$ be a continuous mapping from a topological space X to a topological space Y. Let P be a point of X. Show that Ψ "induces" a group homomorphism Ψ_* from $\pi_1(X, P)$ to $\pi_1(Y, \Psi(P))$ by $\Psi_*([\gamma]) = [\Psi \circ \gamma]$. [*Hint:* Part of the problem, of course, is to show that Ψ_* is well defined, i.e., that if $\gamma_1 \sim \gamma_2$, then $\Psi_*([\gamma_1]) = \Psi_*([\gamma_2])$.]

27. Let $U = \{z \in \mathbb{C} : 1 < |z| < 3\}$.

(a) Show that every closed curve $\gamma : [0, 1] \to U$ with $\gamma(0) = \gamma(1) = 2$ is homotopic at the point $P = 2$ to a closed curve $\widehat{\gamma} : [0, 1] \to U$ such that $\widehat{\gamma}([0, 1]) \subseteq \{z : |z| = 2\}$.

(b) Show that if γ_1, γ_2 (as in part **(a)**) are homotopic at the point 2 in U, then $\widehat{\gamma}_1$ and $\widehat{\gamma}_2$ are homotopic at the point 2 in $\{z : |z| = 2\}$.

(c) Deduce that $\pi_1(U)$ is isomorphic to $\pi_1(S^1)$, where $S^1 = \{z : |z| = 2\}$.

28. The goal of this exercise is to prove that $\pi_1(S^1)$ is isomorphic to the group of integers $\mathbb{Z}$ under addition. Here $S^1 \equiv \{z \in \mathbb{C} : |z| = 1\}$, that is, the unit circle in the plane.

(a) Let $\gamma : [0, 1] \to S^1$ be a continuous curve with $\gamma(0) = 1$. Show that there is one and only one continuous function $A_\gamma : [0, 1] \to \mathbb{R}$ such that $A_\gamma(0) = 0$ and $e^{2\pi i A_\gamma(t)} = \gamma(t)$ for all $t \in [0, 1]$. In accordance with our previous terminology we call $A_\gamma(1)$ the *winding number* of γ. [*Hint:* For existence, subdivide the interval $[0, 1]$ into subintervals with a partition $0 = t_0 < t_1 < t_2 < \cdots < t_k = 1$ such that, for each $j = 0, 1, \ldots, k-1$, the image $\gamma([t_j, t_{j+1}])$ is contained in some arc of the unit circle having length less than π. Select "branches" of the angle function on each such circular arc, and

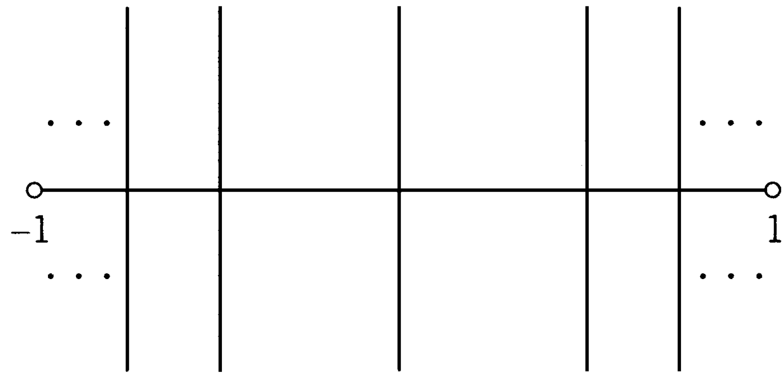

Figure 11.6

patch these together, adjusting by multiples of 2π when necessary, to make them fit together continuously.]

(b) Suppose that γ_1 and γ_2 are as in part **(a)** and that, moreover, $\gamma_1(1) = \gamma_2(1) = 1$. Prove that γ_1 and γ_2 are homotopic at 1 (as closed curves in S^1) if and only if $A_{\gamma_1}(1) = A_{\gamma_2}(1)$. [*Hint:* If γ_1 and γ_2 are homotopic, then $A_{\gamma_1}(1) = A_{\gamma_2}(1)$ because the winding number cannot "jump" when the curve is deformed continuously. For a homotopy when $A_{\gamma_1}(1) = A_{\gamma_2}(1)$, use

$$\exp\left[2\pi i((1-s)A_{\gamma_1}(t) + sA_{\gamma_2}(t))\right]$$

for $s, t \in [0, 1]$.

(c) Conclude that $\pi_1(S^1) \simeq \mathbb{Z}$.

29. Use the ideas and results of Exercises 27 and 28 to show that $\pi_1(\{z \in \mathbb{C} : R_1 < |z| < R_2\}) \cong \mathbb{Z}$ for any R_1, R_2 such that $0 \leq R_1 < R_2 \leq +\infty$. [*Hint:* See Exercise 27.]

30. The purpose of the present exercise is to construct a curve that is homologous to zero but not homotopic to zero. The construction is rather elaborate. You should consider it as an open-ended invitation to learn more about topology.

An open interval $I \subseteq \mathbb{R}$ can be subdivided into infinitely many subintervals with the property that the endpoints of the subintervals accumulate only at the endpoints of I. For instance, the interval $(-1, 1)$ can be subdivided into $(0, 1/2), (1/2, 3/4), (3/4, 7/8), \ldots$ and $(-1/2, 0)$, $(-3/4, -1/2)$, $(-7/8, -3/4), \ldots$. With this subdivision process, which can be scaled to apply to any interval, we construct a subset of $\mathbb{R}^2$ as follows:

Begin with $(-1, 1)$ subdivided as described in the last paragraph. Attach to each point that occurs as an endpoint in the subdivision a vertical, open segment centered at that point and having length one (Figure

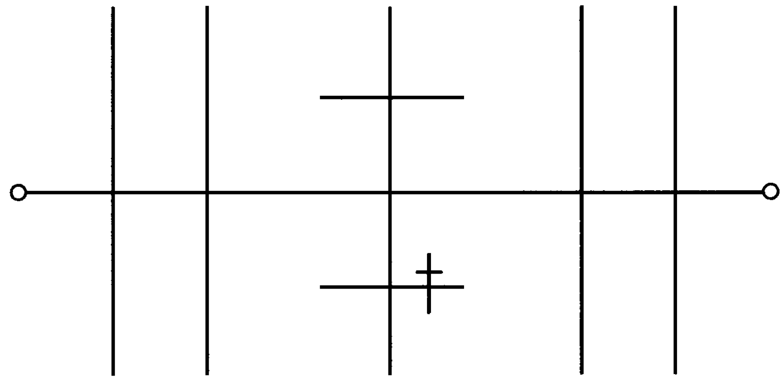

Figure 11.7

11.6). Now subdivide each of these vertical segments as in the preceding paragraph. On each vertical segment J, attach at each subdivision point a small horizontal open segment, centered on J; the segments J should be so short that no two horizontal segments intersect.

Continue this process by subdividing each new horizontal segment, and adding even smaller vertical segments that are centered on the horizontal segments at each subdivision point. Some steps of the process are shown in Figure 11.7.

The result of repeating this construction infinitely many times is an "infinite graph" G (in the sense of graph theory) with four edges meeting at each vertex. Now define a map $\pi : G \to \mathbb{R}^2$ whose image is the union of two circles that osculate at a point. For specificity, let the two circles be

$$\{(x, y) \in \mathbb{R}^2 : (x + 1)^2 + y^2 = 1\} \ \cup \ \{(x, y) \in \mathbb{R}^2 : (x - 1)^2 + y^2 = 1\}.$$

Denote this union by $S = S_1 \cup S_2$. We define the map as follows: All vertices in the graph get mapped to the common point $(0, 0)$ of the two circles. A horizontal segment between two adjacent vertices of G is to be "wrapped" counterclockwise around S_2. A vertical segment between two adjacent vertices of G is to be "wrapped" counterclockwise around S_1. Now prove the following:

(a) The mapping π just described is a continuous function from G (with the topology induced from $\mathbb{R}^2$) onto S.

(b) If $t \to \gamma(t) \in \mathbb{R}^2$, $t \in [0, 1]$, is a continuous curve with $\gamma(t)$ lying in S for all $t \in [0, 1]$ and with $\gamma(0) = (0, 0)$, then there is a unique curve $t \mapsto \widehat{\gamma}(t) \in G$ such that $\widehat{\gamma}(0) = (0, 0)$ and $\pi(\widehat{\gamma}(t)) = \gamma(t)$ for all $t \in [0, 1]$. [*Hint:* Look carefully at how you did Exercise 28 **(a)** and imitate that idea.]

(c) If γ_1, γ_2 are two curves as in part **(b)** and if $[\gamma_1] = [\gamma_2] \in \pi_1(S)$, then $\widehat{\gamma}_1(1) = \widehat{\gamma}_2(1)$.

(d) If γ_1, γ_2 are as in parts **(b)**, **(c)** and if $\widehat{\gamma}_1(1) = \widehat{\gamma}_2(1)$, then $[\gamma_1] = [\gamma_2]$ in $\pi_1(S)$.

(e) Deduce that if α_1 is a curve that goes once counterclockwise around S_2 and α_2 is a curve that goes once counterclockwise around S_1 (with $\alpha_j(0) = \alpha_j(1) = (0,0)$, $j = 1, 2$), then

$$[\alpha_1 \cdot \alpha_2] \neq [\alpha_2 \cdot \alpha_1].$$

(f) Conclude from **(e)** that $[\alpha_1 \cdot \alpha_2 \cdot \alpha_1^{-1} \cdot \alpha_2^{-1}] \neq [e]$.

(g) Show that there is a continuous mapping $R : \mathbb{R}^2 \setminus \{(-1,0),(+1,0)\} \to S$ with the property that R is restricted to S is the identity.

(h) Deduce from **(g)** that $[\alpha_1 \cdot \alpha_2 \cdot \alpha_1^{-1} \cdot \alpha_2^{-1}] \neq [e]$ in the homotopy group $\pi_1(\mathbb{R}^2 \setminus \{(-1,0),(+1,0)\})$. That is, show that $\alpha_1 \cdot \alpha_2 \cdot \alpha_1^{-1} \cdot \alpha_2^{-1}$ is not homotopic to a constant mapping in $\mathbb{R}^2 \setminus \{(-1,0),(+1,0)\}$. [*Hint:* Such a homotopy, if it existed, could be composed with R to give a homotopy taking place in the union of the two circles S.]

(i) Show that $\alpha_1 \cdot \alpha_2 \cdot \alpha_1^{-1} \cdot \alpha_2^{-1}$ *is* homologous to zero in $\mathbb{R}^2 \setminus \{(-1,0),(1,0)\}$.

31. Let $U = \{z : |z| < 1, z \neq 1/2, z \neq -1/2\}$. Let $f : U \to \mathbb{C}$ be holomorphic. Prove that there is a holomorphic function $F : U \to \mathbb{C}$ such that $F' = f$ if and only if

$$\oint_{|\zeta - 1/2| = 1/4} f(\zeta)\, d\zeta = 0 \qquad \text{and} \qquad \oint_{|\zeta + 1/2| = 1/4} f(\zeta)\, d\zeta = 0.$$

* **32.** Let U be a connected open set in $\mathbb{C}$. Define an equivalence relation $\sim$ on the set of holomorphic functions on U by $f_1 \sim f_2$ if there is a holomorphic function $F : U \to \mathbb{C}$ such that $F' = f_1 - f_2$ on all of U. Prove the following statements about $\sim$:

(a) The relation $\sim$ is indeed an equivalence relation.

(b) The set of equivalence classes induced by $\sim$ forms a vector space over $\mathbb{C}$ under the operations $[f] + [g] = [f + g]$ and $\alpha[f] = [\alpha f]$, $\alpha \in \mathbb{C}$.

(c) Use Exercise 31 to prove that, if $U = \{z : |z| < 1, z \neq 1/2, z \neq -1/2\}$, then the linear space constructed in **(b)** is a two-dimensional vector space over the field $\mathbb{C}$ of scalars.

** **(d)** Use Runge's theorem to convince yourself that if U is a bounded, connected open set in $\mathbb{C}$ such that $\mathbb{C} \setminus U$ has $k+1$ components, then the vector space has dimension k over $\mathbb{C}$.

* **33.** Let U be a connected open set in $\mathbb{C}$. Define an equivalence relation $\sim$ on the set of all harmonic functions from U to $\mathbb{R}$ by $h_1 \sim h_2$ if $h_1 - h_2 : U \to \mathbb{R}$ has a harmonic conjugate (i.e., $h_1 - h_2 = \operatorname{Re} F$ for some holomorphic function $F : U \to \mathbb{R}$). Prove the following statements:

(a) The relation $\sim$ is indeed an equivalence relation.
(b) The set of equivalence classes forms a vector space over the scalar field $\mathbb{R}$ with operations $[h_1] + [h_2] = [h_1 + h_2]$ and $\alpha[h] = [\alpha h]$, $\alpha \in \mathbb{R}$.
(c) Show that the dimension of this new vector space is k for the domain

$$U = D(0,1) \setminus \{z_1, \dots, z_k\},$$

that is, the disc with k points removed. Do this by showing that the functions $u_j : z \mapsto \ln |z - z_j|$, $j = 1, \dots, k$, form a basis. [*Hint:* Refer to Exercise 6 of Chapter 7 for the case of one deleted point.]

* **34.** Let U be a connected open set in $\mathbb{C}$. Define an equivalence relation $\sim$ on the set of all (real) C^∞ differentials (i.e., differentials with C^∞ coefficients) $\omega = A(x, y)dx + B(x, y)dy$ with $\partial B/\partial x = \partial A/\partial y$ by $\omega_1 \sim \omega_2$ if $\omega_1 - \omega_2 = df$ for some C^∞ function $f : U \to \mathbb{R}$. Prove the following statements:
(a) The relation $\sim$ is indeed an equivalence relation.
(b) The equivalence classes under this relation form a vector space over $\mathbb{R}$ in a natural way (refer to Exercises 32 and 33).
(c) The vector space has dimension k when the region U is a disc with k points deleted. What differentials can you use as an explicit basis?

* **35.** Prove the following result, which is commonly known as the Lebesgue covering lemma: If K is a compact set in a Euclidean space $\mathbb{R}^n$, $n \geq 1$, and if $\{U_\lambda, \lambda \in \Lambda\}$ is a collection of open sets in $\mathbb{R}^n$ such that $K \subset \bigcup_{\lambda \in \Lambda} U_\lambda$, then there is an $\epsilon > 0$ such that, for each $k \in K$, there is a set U_λ with $B(k, \epsilon) \subset U_\lambda$. Here $B(y, r)$ is the Euclidean ball with center y and radius r. [*Hint:* Suppose not. Then there is, for $j = 1, 2, 3, \dots$, a $k_j \in K$ such that $B(k_j, 1/j)$ fails to be contained in any U_λ. Choose a subsequence $\{k_{j_\ell}\}$ that converges to a point, call it k_0, in K. Then derive a contradiction from the fact that $k_0 \in U_{\lambda_0}$ for some index λ_0.] Note that the Lebesgue covering lemma actually applies to compact metric spaces in general, by the same proof.

* **36.** Suppose that U is a connected, open set in $\mathbb{C}$, $f : U \to \mathbb{C}$ is a holomorphic function, and $\gamma : [0, 1] \to U$ is a continuous curve (note that γ is *not* assumed to be piecewise C^1). Carry out in detail the following program to define $\oint_\gamma f$:
(a) Choose, by applying Exercise 35, a positive integer N such that, for $j = 0, 1, 2, \dots, N-1$, $\gamma([j/N, (j+1)/N]) \subset$ some open disc D_j contained in U.

(b) Let $F_j : D_j \to \mathbb{C}$ be a holomorphic function such that $F_j' = f$ on D_j. Then each

$$\oint_\gamma f = \sum_{j=0}^{N-1} \big[F_j(\gamma((j+1)/N)) - F_j(\gamma(j/N))\big].$$

(c) Suppose that $0 = t_0 < t_1 < \cdots < t_M = 1$ is any partition of $[0,1]$ with the property that $\gamma([t_j, t_{j+1}]) \subset$ some open disc D_j contained in U. Verify that

$$\oint_\gamma f = \sum_{j=0}^{M-1} \big[F(\gamma(t_{j+1})) - F(\gamma(t_j))\big]\,.$$

In particular, verify that the definition of $\oint_\gamma f$ in part **(b)** is independent of the choice of the integer N.

(d) Check that if γ is a piecewise C^1 curve, then the definition of $\oint_\gamma f$ that we have formulated in part **(b)** agrees with the definition from Chapter 2; that is, check that the new definition gives the same answer as $\int_0^1 f(\gamma(t)) \cdot \gamma'(t)\, dt$.

*** 37.** Construct an alternative proof of Corollary 11.2.5 by completing the following outline:

Let $H : [0,1]\times[0,1] \to U$ be a homotopy through closed curves, that is, H is continuous and $H(s,0) = H(s,1)$ for all $s \in [0,1]$. Suppose that $f : U \to \mathbb{C}$ is holomorphic. We want to prove that

$$\oint_{H(0,\cdot)} f \;=\; \oint_{H(1,\cdot)} f.$$

Choose, using Exercise 35, a positive integer N so large that, for each j,k with $j = 0,1,2,\ldots,N-1$, $k = 0,1,2,\ldots,N-1$, it holds that $H\big([j/N,(j+1)/N]\times[k/N,(k+1)/N]\big)$ is contained in some open disc contained in U. Then the counterclockwise integral of f around the curve given by the H-image of the boundary of the square $[j/N,(j+1)/N] \times [k/N,(k+1)/N]$ equals 0.

Sum over j,k and note that the inside edges cancel to deduce that the integral of f around the boundary of $[0,1] \times [0,1]$ equals 0. Then deduce the desired conclusion of the theorem.

38. (a) Prove: If U is a bounded, connected open set in $\mathbb{C}$ with $\mathbb{C} \setminus U$ connected, then the set $K_\delta = \{z \in U : \text{dist}(z, \mathbb{C}\setminus U) \geq \delta\}$, with $\delta > 0$ sufficiently small, is compact and $\mathbb{C} \setminus K_\delta$ is connected.

(b) Deduce Proposition 11.4.3 from Corollary 12.1.2 in the next chapter.

39. Justify in detail the application of Fubini's Theorem in the discussion preceding Theorem 11.2.6. For this, you will need to think carefully about the definition of the integral of a holomorphic function along a continuous curve given earlier.