Chapter 1

Introduction

Welcome to the world of super mathematics! In this short chapter we give some of the basic definitions concerning Lie superalgebras. We also state the Classification Theorem for finite dimensional classical simple Lie superalgebras.

1.1. Basic Definitions

Let K be a field. We assume that the characteristic of K is different from $2, 3$. Unless otherwise stated, all vector spaces, Lie algebras, etc., are defined over K. A $\mathbb{Z}_2$-*graded vector space* is merely a direct sum of vector spaces $V = V_0 \oplus V_1$. We call elements of V_0 (resp. V_1) *even* (resp. *odd*). Nonzero elements of $V_0 \cup V_1$ are *homogeneous* and for homogeneous $v \in V_i$, we set $\overline{v} = i$, the *degree* of v. First we mention an important convention which we use throughout this book.

Degree Convention 1.1.1. *If v is an element of a $\mathbb{Z}_2$-graded vector space and $\overline{v}$ appears in some formula or expression, then v is assumed to be homogeneous.*

This convention simplifies the notation in many formulas beginning with the very definition of a Lie superalgebra. Background on $\mathbb{Z}_2$-graded structures is contained in Section A.1 of Appendix A.

A *Lie superalgebra* is a $\mathbb{Z}_2$-graded vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ together with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(a) $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ for $\alpha, \beta \in \mathbb{Z}_2$ ($\mathbb{Z}_2$-grading),

(b) $[a, b] = -(-1)^{\overline{a}\overline{b}}[b, a]$ (graded skew-symmetry),

(c) $(-1)^{\overline{ac}}[a,[b,c]]+(-1)^{\overline{ab}}[b,[c,a]]+(-1)^{\overline{bc}}[c,[a,b]]=0$ (graded Jacobi identity),

for all $a, b, c \in \mathfrak{g}$.[1]

The related notion of a $\mathbb{Z}$-graded Lie algebra was introduced in [**MM65**]. A *$\mathbb{Z}$-graded Lie algebra* is a direct sum $\mathfrak{g} = \bigoplus_{\alpha\in\mathbb{Z}} \mathfrak{g}(\alpha)$ together with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ such that

(a) $[\mathfrak{g}(\alpha), \mathfrak{g}(\beta)] \subseteq \mathfrak{g}(\alpha+\beta)$,
(b) $[a,b] = -(-1)^{\alpha\beta}[b,a]$,
(c) $(-1)^{\alpha\gamma}[a,[b,c]] + (-1)^{\alpha\beta}[b,[c,a]] + (-1)^{\beta\gamma}[c,[b,a]] = 0$,

for all $a \in \mathfrak{g}(\alpha), b \in \mathfrak{g}(\beta), c \in \mathfrak{g}(\gamma)$.

Any $\mathbb{Z}$-graded Lie algebra can be made into a Lie superalgebra by setting

$$\mathfrak{g}_0 = \bigoplus_{\alpha\in\mathbb{Z}} \mathfrak{g}(2\alpha), \quad \mathfrak{g}_1 = \bigoplus_{\alpha\in\mathbb{Z}} \mathfrak{g}(2\alpha+1). \tag{1.1.1}$$

Scheunert calls a Lie superalgebra with a $\mathbb{Z}$-grading satisfying (1.1.1) a *consistently graded Lie superalgebra,* [**Sch79**].

There is another formulation of the graded Jacobi identity which is used frequently. Let A be a not necessarily associative $\mathbb{Z}_2$-graded algebra over K. A K-linear map $\partial : A \longrightarrow A$ of degree α [2] is a *left superderivation* provided

$$\partial(bc) = \partial(b)c + (-1)^{\alpha\bar{b}} b\partial(c)$$

for all $b, c \in A$. Similarly ∂ is a *right superderivation of degree* α if

$$\partial(bc) = b\partial(c) + (-1)^{\alpha\bar{c}} \partial(b)c$$

for all $b, c \in A$.

Lemma 1.1.2. *Assume that A is a $\mathbb{Z}_2$-graded algebra whose product*

$$(a,b) \longrightarrow [a,b]$$

is graded skew-symmetric. Then the following are equivalent.

(a) *A satisfies the graded Jacobi identity.*
(b) *For all $a \in A$ the map $\operatorname{ad} a : b \longrightarrow [a,b]$ is a left superderivation of degree $\bar{a}$.*
(c) *For all $a \in A_\alpha$ the map $\operatorname{ad}' a : b \longrightarrow [b,a]$ is a right superderivation of degree $\bar{a}$.*

[1] Note that if char $K = 2$ or 3, we encounter difficulties with this definition. For example in characteristic 3, if $a = b = c \in \mathfrak{g}_1$, then (c) holds vacuously.

[2] For the degree of a linear map between two $\mathbb{Z}$-graded vector spaces in general, when defined, see (A.4.7) in Appendix A.

Proof. Exercise 1.4.2. □

Henceforth a *superderivation* will mean a left superderivation. Superderivations of degree 0 (resp. 1) are also called even (resp. odd) superderivations.

If $\mathfrak{g}$ is a Lie superalgebra, then a $\mathfrak{g}$-*module* is a vector space V together with a bilinear map, $\mathfrak{g} \times V \to V$, denoted $(x, v) \to xv$ such that

$$x(yv) - (-1)^{\overline{xy}} y(xv) = [x, y]v$$

for all $x, y \in \mathfrak{g}$. We do not assume that V is $\mathbb{Z}_2$-graded.

Let $\mathfrak{g}$ be a Lie superalgebra. If V and W are subspaces of $\mathfrak{g}$, we write $[V, W]$ for the subspace spanned by all $[v, w]$ with $v \in V, w \in W$. A $\mathbb{Z}_2$-graded subspace $\mathfrak{a}$ of $\mathfrak{g}$ is an *ideal* if $[\mathfrak{a}, \mathfrak{g}] \subset \mathfrak{a}$. The *derived series* $\mathfrak{g}^{(i)}$ of $\mathfrak{g}$ is defined by setting

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \text{ if } i \geq 0.$$

The *lower central series* $\mathfrak{g}^{[i]}$ of $\mathfrak{g}$ is defined by

$$\mathfrak{g}^{[0]} = \mathfrak{g}, \quad \mathfrak{g}^{[i+1]} = [\mathfrak{g}, \mathfrak{g}^{[i]}] \text{ if } i \geq 0.$$

We say that $\mathfrak{g}$ is *solvable* (resp. *nilpotent*) if $\mathfrak{g}^{(n)} = 0$ (resp. $\mathfrak{g}^{[n]} = 0$) for large n and that $\mathfrak{g}$ is *abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$.

1.2. Simple Lie Superalgebras

We say the Lie superalgebra $\mathfrak{g}$ is *simple* if it is not abelian and the only $\mathbb{Z}_2$-graded ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$.

Lemma 1.2.1. *Let $\mathfrak{g}$ be a Lie superalgebra such that $\mathfrak{g}_0 \neq 0 \neq \mathfrak{g}_1$. Then $\mathfrak{g}$ is simple if and only if the following conditions hold.*

(a) *If $\mathfrak{a}$ is a nonzero $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$ such that $[\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{a}]] \subseteq \mathfrak{a}$, then $[\mathfrak{g}_1, \mathfrak{a}] = \mathfrak{g}_0$.*

(b) *$\mathfrak{g}_1$ is a faithful $\mathfrak{g}_0$-module under the adjoint action.*

(c) $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$.

Proof. Assume that $\mathfrak{g}$ is simple and that $\mathfrak{a}$ is as in (a). Let

$$\mathfrak{b} = \{x \in \mathfrak{g}_0 | [x, \mathfrak{g}_1] = 0\}$$

and

$$\mathfrak{c} = [\mathfrak{g}_0, \mathfrak{g}_1].$$

It is readily checked that each of

$$[\mathfrak{g}_1, \mathfrak{a}] \oplus \mathfrak{a}, \quad \mathfrak{b}, \quad \text{and} \quad \mathfrak{g}_0 \oplus \mathfrak{c}$$

is an ideal of $\mathfrak{g}$, and properties (a)–(c) follow from this.

Conversely suppose that (a)–(c) hold and that I is a graded ideal of $\mathfrak{g}$. If $I_1 = 0$, then $I_0 \neq 0$ and $[I_0, \mathfrak{g}_1] = 0$, contradicting (b). Thus $I_1 \neq 0$ and I_1 is a $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$ such that $[\mathfrak{g}_1, [\mathfrak{g}_1, I_1]] \subset I_1$. Thus by (c) and (a), I contains $[\mathfrak{g}_1, I_1] = \mathfrak{g}_0$ and $[\mathfrak{g}_0, \mathfrak{g}_1] = \mathfrak{g}_1$. □

Define a *left ideal* in a Lie superalgebra $\mathfrak{g}$ to be a subspace $\mathfrak{a}$ of $\mathfrak{g}$ such that $[\mathfrak{g}, \mathfrak{a}] \subseteq \mathfrak{a}$. Left ideals are not assumed to be $\mathbb{Z}_2$-graded subspaces. The remaining results in this section are adapted from [**Sch79**].

Proposition 1.2.2. *The only left ideals in a simple Lie superalgebra* $\mathfrak{g}$ *are* 0 *and* $\mathfrak{g}$.

To prove this, we need a lemma.

Lemma 1.2.3. *Let* $\mathfrak{g}$ *be a simple Lie superalgebra, and suppose that* $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ *is an odd linear map such that*

$$\tau([a, b]) = [a, \tau(b)] \quad \textit{for } a, b \in \mathfrak{g}. \tag{1.2.1}$$

Then $\tau = 0$.

Proof. Since the kernel and image of τ are ideals, τ is either zero or bijective. For a contradiction assume the latter. If a, b are homogeneous of the same degree, then

$$[\tau(a), \tau(b)] = \tau([\tau(a), b]) = -\tau([b, \tau(a)]) = -\tau^2([b, a]). \tag{1.2.2}$$

However one side of this equation is symmetric in a, b and the other side is skew-symmetric. Thus, since τ is bijective, $[a, b] = 0$ if a, b are homogeneous of the same degree. Next if a, b are homogeneous of different degree, then $a, \tau(b)$ have the same degree, so $\tau([a, b]) = [a, \tau(b)] = 0$. Therefore $[\mathfrak{g}, \mathfrak{g}] = 0$, a contradiction. □

Proof of Proposition 1.2.2. The linear map $\gamma : \mathfrak{g} \longrightarrow \mathfrak{g}$ defined by $\gamma(x) = (-1)^{\bar{x}} x$ is an automorphism of $\mathfrak{g}$. If $x \in \mathfrak{g}$, then the component of x in $\mathfrak{g}$ of degree $\alpha \in \mathbb{Z}_2$ is $(x + (-1)^\alpha \gamma(x))/2$. If $\mathfrak{a}$ is a nonzero left ideal in $\mathfrak{g}$, then so is $\gamma(\mathfrak{a})$. Assume that $\mathfrak{a}$ is different from 0 and $\mathfrak{g}$. Then $\mathfrak{a} \cap \gamma(\mathfrak{a}) = 0$, and $\mathfrak{a} + \gamma(\mathfrak{a}) = \mathfrak{g}$. Thus $\mathfrak{g}$ is the direct sum of $\mathfrak{a}$ and $\gamma(\mathfrak{a})$. It follows that for $\alpha \in \mathbb{Z}_2$,

$$\mathfrak{g}_\alpha = \{x + (-1)^\alpha \gamma(x) | x \in \mathfrak{a}\}. \tag{1.2.3}$$

Define a linear map $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ by

$$\tau(x) = x, \quad \tau(\gamma(x)) = -\gamma(x) \quad \text{for} \quad x \in \mathfrak{a}.$$

From (1.2.3) we deduce that $\tau(\mathfrak{g}_\alpha) = \mathfrak{g}_{\alpha+1}$ for $\alpha \in \mathbb{Z}_2$. Since $\mathfrak{a}$ and $\gamma(\mathfrak{a})$ are left ideals, it follows that $\tau([a, b]) = [a, \tau(b)]$ for $a, b \in \mathfrak{g}$. The existence of a map τ with these properties contradicts Lemma 1.2.3. □

From now on we consider only graded ideals in Lie superalgebras. We mention several further general properties of simple Lie superalgebras. A bilinear form (,) on $\mathfrak{g}$ is *invariant* if $([a,b],c) = (a,[b,c])$ for all $a,b,c \in \mathfrak{g}$, (see (A.2.28) for the general case of an invariant bilinear form on a $\mathfrak{g}$-module.)

Proposition 1.2.4. *Let $\mathfrak{g}$ be a simple Lie superalgebra.*

(a) *Any invariant bilinear form on $\mathfrak{g}$ is either nondegenerate or equal to zero.*

(b) *Any invariant bilinear form on $\mathfrak{g}$ is supersymmetric, that is, $(a,b) = (-1)^{\overline{ab}}(b,a)$ for $a,b \in \mathfrak{g}$.*

(c) *Any two nonzero invariant bilinear forms on $\mathfrak{g}$ are proportional.*

(d) *The invariant bilinear forms on $\mathfrak{g}$ are either all odd or all even.*

Proof. If ψ is any invariant bilinear form, then the radical

$$\{b \in \mathfrak{g} | \psi(a,b) = 0 \text{ for all } a \in \mathfrak{g}\}$$

of ψ is a left ideal, and thus (a) follows from Proposition 1.2.2. The proofs of (b) and (c) are outlined in the exercises. To prove (d) consider bilinear forms ψ and ψ' of different degrees such that ψ is nondegenerate. There is a unique odd linear map $\tau : \mathfrak{g} \longrightarrow \mathfrak{g}$ such that $\psi'(a,b) = \psi(a,\tau(b))$ for $a,b \in \mathfrak{g}$. Then for $a,b,c \in \mathfrak{g}$,

$$\psi(a,\tau([b,c])) = \psi'(a,[b,c]) = \psi'([a,b],c) = \psi([a,b],\tau(c)) = \psi(a,[b,\tau(c)]).$$

Since ψ is nondegenerate, it follows that $[b,\tau(c)] = \tau([b,c])$. Hence by Lemma 1.2.3, $\tau = 0$, so $\psi' = 0$. □

Proposition 1.2.5. *Let $\mathfrak{g}$ be a finite dimensional simple Lie superalgebra with $\mathfrak{g}_1 \neq 0$, and suppose $\mathfrak{g}_1 = \mathfrak{g}_1^1 + \mathfrak{g}_1^2$ is the sum of two proper $\mathfrak{g}_0$-invariant subspaces. Then $\mathfrak{g}_1$ is the direct sum of $\mathfrak{g}_1^1$ and $\mathfrak{g}_1^2$, each $\mathfrak{g}_1^i$ is a simple $\mathfrak{g}_0$-module, and we have*

$$[\mathfrak{g}_1^1, \mathfrak{g}_1^1] = [\mathfrak{g}_1^2, \mathfrak{g}_1^2] = 0, \quad [\mathfrak{g}_1^1, \mathfrak{g}_1^2] = \mathfrak{g}_0. \tag{1.2.4}$$

The proof uses the next result, known as the *modular law.*

Lemma 1.2.6. *If A, B, and C are subgroups of an abelian group D and $B \subseteq A$, then $A \cap (B + C) = (A \cap C) + B$.*

A special case of the proposition requires no finite dimensionality assumption.

Lemma 1.2.7. *Let $\mathfrak{g}$ be a simple Lie superalgebra with $\mathfrak{g}_1 \neq 0$, and suppose $\mathfrak{g}_1 = \bigoplus_{i=1}^r \mathfrak{g}_1^i$ for nonzero $\mathfrak{g}_0$-invariant subspaces $\mathfrak{g}_1^i$. Then $r = 1$ or 2 and if $r = 2$, then (1.2.4) holds.*

Proof. If $r = 1$, there is nothing to prove. Suppose first that $r = 2$. Set $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ where $\mathfrak{a}_0 = [\mathfrak{g}_1^1, \mathfrak{g}_1^1]$ and $\mathfrak{a}_1 = [\mathfrak{g}_1^1, [\mathfrak{g}_1^1, \mathfrak{g}_1^1]]$. Then we claim that $\mathfrak{a}$ is an ideal of $\mathfrak{g}$. Clearly $\mathfrak{a}$ is $\mathfrak{g}_0$-invariant and $[\mathfrak{g}_1^1, \mathfrak{a}] \subseteq \mathfrak{a}$. Now using the Jacobi identity,

$$[[\mathfrak{g}_1^1, \mathfrak{g}_1^1], \mathfrak{g}_1^2] \subseteq [\mathfrak{g}_1^1, [\mathfrak{g}_1^1, \mathfrak{g}_1^2]] \subseteq [\mathfrak{g}_1^1, \mathfrak{g}_0] \subseteq \mathfrak{g}_1^1.$$

Since $\mathfrak{g}_1^2$ is $\mathfrak{g}_0$-invariant, this implies

$$[[\mathfrak{g}_1^1, \mathfrak{g}_1^1], \mathfrak{g}_1^2] \subseteq [\mathfrak{g}_0, \mathfrak{g}_1^2] \subseteq \mathfrak{g}_1^1 \cap \mathfrak{g}_1^2 = 0.$$

Hence again by the Jacobi identity,

$$\begin{aligned}[\mathfrak{g}_1^2, \mathfrak{a}_1] = [\mathfrak{g}_1^2, [\mathfrak{g}_1^1, [\mathfrak{g}_1^1, \mathfrak{g}_1^1]]] &\subseteq [\mathfrak{g}_1^1, [\mathfrak{g}_1^2, [\mathfrak{g}_1^1, \mathfrak{g}_1^1]]] + [[\mathfrak{g}_1^2, \mathfrak{g}_1^1], [\mathfrak{g}_1^1, \mathfrak{g}_1^1]] \\ &\subseteq [\mathfrak{g}_0, [\mathfrak{g}_1^1, \mathfrak{g}_1^1]] \subseteq [\mathfrak{g}_1^1, \mathfrak{g}_1^1] = \mathfrak{a}_0,\end{aligned}$$

and clearly $[\mathfrak{g}_1^1, \mathfrak{a}_1] \subseteq [\mathfrak{g}_1^1, \mathfrak{g}_1^1]$. Thus $\mathfrak{a}$ is an ideal in $\mathfrak{g}$. It is a proper ideal since $[\mathfrak{g}_1^1, [\mathfrak{g}_1^1, \mathfrak{g}_1^1]] \subseteq [\mathfrak{g}_1^1, \mathfrak{g}_0] \subseteq \mathfrak{g}_1^1 \neq \mathfrak{g}_1$. Thus $[\mathfrak{g}_1^1, \mathfrak{g}_1^1] = 0$ and similarly $[\mathfrak{g}_1^2, \mathfrak{g}_1^2] = 0$. Since $[\mathfrak{g}_1, [\mathfrak{g}_1, \mathfrak{g}_1]] \subseteq \mathfrak{g}_1$, it follows from Lemma 1.2.1(a) that $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$, and thus $[\mathfrak{g}_1^1, \mathfrak{g}_1^2] = \mathfrak{g}_0$. Finally if $r > 2$, then for $1 \le s \le r$ we have $\mathfrak{g}_1 = \mathfrak{g}_1^s \oplus \bigoplus_{i \neq s} \mathfrak{g}_1^i$, so by the case $r = 2$, $[\bigoplus_{i \neq s} \mathfrak{g}_1^i, \bigoplus_{i \neq s} \mathfrak{g}_1^i] = 0$. This implies that $[\mathfrak{g}_1^s, \mathfrak{g}_1^t] = 0$ for $1 \le s, t \le r$ and hence that $[\mathfrak{g}_1, \mathfrak{g}_1] = 0$, a contradiction, since as shown above $[\mathfrak{g}_1, \mathfrak{g}_1] = \mathfrak{g}_0$. □

Proof of Proposition 1.2.5. For $i = 1, 2$ define a sequence of subspaces $\{\mathfrak{a}_n^i\}_{n \ge -1}$ of $\mathfrak{g}$ by

$$\mathfrak{a}_{-1}^i = \mathfrak{g}_1, \quad \mathfrak{a}_0^i = \mathfrak{g}_0, \quad \mathfrak{a}_1^i = \mathfrak{g}_1^i,$$

and then inductively $\mathfrak{a}_n^i = [\mathfrak{g}_1^i, \mathfrak{a}_{n-1}^i]$ for $n \ge 2$. Note that $\mathfrak{a}_n^i \subseteq \mathfrak{g}_0$ for n even and $\mathfrak{a}_n^i \subseteq \mathfrak{g}_1$ for n odd. By induction we see that

(1.2.5) $$\mathfrak{a}_n^i \text{ is } \mathfrak{g}_0\text{-invariant}, \ \mathfrak{a}_{n+2}^i \subseteq \mathfrak{a}_n, \ \text{and } [\mathfrak{g}_1, \mathfrak{a}_{n+1}^i] \subseteq \mathfrak{a}_n^i.$$

Since $\mathfrak{g}$ is finite dimensional, there is an integer $m \ge 1$ such that $\mathfrak{a}_{2m+2}^i = \mathfrak{a}_{2m}^i$ for $i = 1, 2$. Hence, by (1.2.5), $\mathfrak{a}_{2m}^i \oplus \mathfrak{a}_{2m+1}^i$ is a graded ideal of $\mathfrak{g}$ which must be zero since $\mathfrak{a}_{2m+1}^i \subseteq \mathfrak{g}_1^1 \neq \mathfrak{g}_1$. It follows that $\mathfrak{a}_M^i = 0$ for some M. Next for $r \ge 0$ define

$$\mathfrak{b}_0^r = \sum_{s=0}^{r} \mathfrak{a}_{2r-2s}^1 \cap \mathfrak{a}_{2s}^2$$

and

$$\mathfrak{b}_1^r = \sum_{s=0}^{r+1} \mathfrak{a}_{2r-2s+1}^1 \cap \mathfrak{a}_{2s-1}^2.$$

By (1.2.5), $\mathfrak{b}^r = \mathfrak{b}_0^r \oplus \mathfrak{b}_1^r$ is a graded ideal of $\mathfrak{g}$ for all $r \ge 0$. Note that $\mathfrak{b}_0^0 = \mathfrak{g}_0$ and that $\mathfrak{b}_1^0 = \mathfrak{g}_1^1 + \mathfrak{g}_1^2 = \mathfrak{g}_1$. Now since $\sum_{s=1}^{r+1} \mathfrak{a}_{2r-2s+1}^1 \cap \mathfrak{a}_{2s-1}^2 \subset \mathfrak{a}_1^2$, we have $\mathfrak{b}_1^r \subseteq (\mathfrak{a}_{2r+1}^1 \cap \mathfrak{a}_{-1}^2) + \mathfrak{a}_1^2$, and because $\mathfrak{a}_1^2 \subseteq \mathfrak{a}_{-1}^2$, the modular law gives

$$\mathfrak{b}_1^r \subseteq \mathfrak{a}_{-1}^2 \cap (\mathfrak{a}_{2r+1}^1 + \mathfrak{a}_1^2) \subseteq \mathfrak{a}_{2r+1}^1 + \mathfrak{g}_1^2.$$

Similarly

$$\mathfrak{b}_1^r \subseteq (\mathfrak{a}_{2r+1}^1 + \mathfrak{g}_1^2) \cap (\mathfrak{a}_{2r+1}^2 + \mathfrak{g}_1^1). \tag{1.2.6}$$

It follows that $\mathfrak{b}_1^r \subseteq \mathfrak{g}_1^1 \cap \mathfrak{g}_1^2$ for all sufficiently large r, and then $\mathfrak{b}^r = 0$ since $\mathfrak{b}^r$ is an ideal. Let N be minimal such that $\mathfrak{b}^N = 0$. Then $\mathfrak{b}^{N-1} = \mathfrak{g}$. Therefore $\mathfrak{a}_{2N-1}^1 \neq 0$, since otherwise $\mathfrak{b}_1^{N-1} \subseteq \mathfrak{g}_1^2$ by (1.2.6). Likewise $\mathfrak{a}_{2N-1}^2 \neq 0$. Also the sums defining $\mathfrak{b}^{N-1}$ are direct; for example, for $1 \leq p \leq N-1$,

$$(\mathfrak{a}_{2N-2p-2}^1 \cap \mathfrak{a}_{2p}^2) \cap \sum_{s=0}^{p-1} (\mathfrak{a}_{2N-2s-2}^1 \cap \mathfrak{a}_{2s}^2) \subseteq \mathfrak{a}_{2N-2p}^1 \cap \mathfrak{a}_{2p}^2 = 0.$$

Since $\mathfrak{a}_{2N-1}^1 \cap \mathfrak{a}_{-1}^2$ and $\mathfrak{a}_{2N-1}^2 \cap \mathfrak{a}_{-1}^1$ are nonzero, the other terms defining $\mathfrak{b}_1^{N-1}$ are zero by Lemma 1.2.7. It follows from the minimality of N that $N = 1$. From $\mathfrak{b}^0 = \mathfrak{g}$, we see that $\mathfrak{g}_1 = \mathfrak{a}_1^1 + \mathfrak{a}_1^2$, and from $\mathfrak{b}^1 = 0$ that $\mathfrak{a}_1^1 \cap \mathfrak{a}_1^2 = 0$. Thus $\mathfrak{g}_1$ is the direct sum of $\mathfrak{a}_1^1 = \mathfrak{g}_1^1$ and $\mathfrak{a}_1^2 = \mathfrak{g}_1^2$. It only remains to show that $\mathfrak{g}_1^1$ and $\mathfrak{g}_1^2$ are simple $\mathfrak{g}_0$-modules. However if $\mathfrak{c}$ is a proper submodule of, say, $\mathfrak{g}_1^1$, then $\mathfrak{g}_1$ is the sum of the two proper submodules $\mathfrak{c} \oplus \mathfrak{g}_1^2$ and $\mathfrak{g}_1^1$. Therefore, by what we have already shown, this sum must be direct, and so $\mathfrak{c} = 0$. □

From now on, unless otherwise indicated, we assume that all Lie superalgebras are finite dimensional over K. We say that $\mathfrak{g}$ is *classical simple* if $\mathfrak{g}$ is simple and $\mathfrak{g}_1$ is a completely reducible $\mathfrak{g}_0$-module.

Corollary 1.2.8. *Let $\mathfrak{g}$ be a classical simple Lie superalgebra, and suppose the center $\mathfrak{z} = \mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ is nontrivial. Then $\mathfrak{z}$ is one-dimensional and $\mathfrak{g}_1$ is the direct sum of two simple submodules:*

$$\mathfrak{g}_1 = \mathfrak{g}_1^1 \oplus \mathfrak{g}_1^2.$$

Furthermore there is a unique element $z \in \mathfrak{z}$ such that $\mathfrak{z} = Kz$ and

$$[z, x] = (-1)^i x \tag{1.2.7}$$

for all $x \in \mathfrak{g}_1^i$.

Proof. Suppose that $\mathfrak{g}_1$ is simple as a $\mathfrak{g}_0$-module. Then if $z \in \mathfrak{z}$, there exists $\alpha \in K$ such that $[z, x] = \alpha x$ for all $x \in \mathfrak{g}_1$, since $\mathfrak{g}_1$ is finite dimensional. Hence if $x, y \in \mathfrak{g}_1$, we have

$$0 = [z, [x, y]] = 2\alpha[x, y],$$

and so $\alpha = 0$. However Lemma 1.2.1 requires that $\mathfrak{g}_1$ be a faithful $\mathfrak{g}_0$-module, so this gives $\mathfrak{z} = 0$ against our assumption.

Since $\mathfrak{g}_1$ is completely reducible, it follows from Proposition 1.2.5 that $\mathfrak{g}_1 = \mathfrak{g}_1^1 \oplus \mathfrak{g}_1^2$ is a direct sum of two simple submodules. Now for nonzero $z \in \mathfrak{z}$ there exist $\alpha_1, \alpha_2 \in K$ such that $[z, x] = \alpha_i x$, for all $x \in \mathfrak{g}_1^i$. Hence for

$x \in \mathfrak{g}_1^1, y \in \mathfrak{g}_1^2$ we have $0 = [z, [x, y]] = (\alpha_1 + \alpha_2)[x, y]$, so as $[\mathfrak{g}_1^1, \mathfrak{g}_1^2] = \mathfrak{g}_0$ by Proposition 1.2.5 we have $\alpha_1 + \alpha_2 = 0$. Since $\mathfrak{g}_1$ is faithful as a $\mathfrak{g}_0$-module, we have $\alpha_1 \neq 0$. If $w \in \mathfrak{z}$, there exists $\beta_i \in K$ such that $[w, x] = \beta_i x$, for all $x \in \mathfrak{g}_1^i$. Hence $[\alpha_1 w - \beta_1 z, x] = 0$ for all such x, and faithfulness implies that $\dim \mathfrak{z} = 1$. Finally we can replace z by z/α_1 to ensure that (1.2.7) holds. □

Theorem 1.2.9. *The simple Lie superalgebra $\mathfrak{g}$ is classical simple if and only if $\mathfrak{g}_0$ is reductive.*

Comments on the Proof. We assume that $\mathfrak{g}_1 \neq 0$. By Lemma 1.2.1, $\mathfrak{g}_1$ is a faithful $\mathfrak{g}_0$-module. Thus if $\mathfrak{g}_1$ is a completely reducible $\mathfrak{g}_0$-module, it follows from the proof of [**Hum72**, Proposition 19.1] that $\mathfrak{g}_0$ is reductive.

Conversely assume that $\mathfrak{g}_0$ is reductive. If $\mathfrak{g}_0$ is semisimple, then $\mathfrak{g}_1$ is completely reducible, as are all finite dimensional $\mathfrak{g}_0$-modules. Given this, the proof of complete reducibility is surprisingly difficult when the center of $\mathfrak{g}_0$ is nontrivial. We refer to [**Sch79**, Theorem 1, page 101] for full details. □

A classical simple Lie superalgebra $\mathfrak{g}$ is called *basic* if $\mathfrak{g}$ admits an even nondegenerate $\mathfrak{g}$-invariant bilinear form. See Section 8.3 for more on basic classical simple Lie superalgebras.

1.3. Classification of Classical Simple Lie Superalgebras

From this point on, unless otherwise stated, we work over an algebraically closed base field K of characteristic zero. We now state the classification of finite dimensional classical simple Lie superalgebras due to Kac, and independently to Nahm, Rittenberg, and Scheunert, [**SNR76a**], [**SNR76b**]. Special cases of the classification were also obtained by other authors; see [**Kac77a**, pages 47–48] for details. As is the case for semisimple Lie algebras there are a number of infinite families and a finite number of exceptions. One difference is the existence of the infinite family of algebras $D(2, 1; \alpha)$ depending on the continuous parameter α.

Theorem 1.3.1. *Let $\mathfrak{g}$ be a finite dimensional classical simple Lie superalgebra. Then either $\mathfrak{g}$ is a simple Lie algebra or $\mathfrak{g}$ is isomorphic to one of the following algebras:*

$$A(m, n) = \mathfrak{sl}(m+1, n+1) \quad \text{with } m > n \geq 0,$$

$$A(n, n) = \mathfrak{psl}(n+1, n+1) \quad \text{with } n \geq 1,$$

$$B(m, n) = \mathfrak{osp}(2m+1, 2n) \quad \text{with } m \geq 0, n > 0,$$

$$C(n) = \mathfrak{osp}(2, 2n-2) \quad \text{with } n \geq 2,$$

$$D(m, n) = \mathfrak{osp}(2m, 2n) \quad \text{with } m \geq 2, n \geq 1,$$

$$D(2,1;\alpha) = \Gamma(1, -1-\alpha, \alpha) \quad \alpha \neq 0, -1,$$

$$\mathfrak{p}(n), \quad n \geq 2; \quad \mathfrak{q}(n), \quad n \geq 2,$$

$$G(3), \textit{ a simple algebra of dimension } 31,$$

$$F(4), \textit{ a simple algebra of dimension } 40.$$

The Lie superalgebras appearing in the theorem will be described in Chapters 2 and 4. We remark that Kac classifies all finite dimensional simple Lie superalgebras. The remaining algebras which are not classical are often called *simple superalgebras of Cartan type.* This refers to the fact that they are Lie superalgebra analogs of some infinite dimensional Lie algebras studied by Cartan; see [**Car09**], [**Car53**], and also [**FS88**] for some finite dimensional simple Lie algebra analogs in positive characteristic.

Among the Lie superalgebras listed in Theorem 1.3.1 we have the following isomorphisms:

(a) $A(1,0) \cong C(2)$,
(b) $D(2,1) \cong D(2,1,1)$.

In addition there are some isomorphisms between the various $D(2,1;\alpha)$ as described in Section 4.2. There are no further isomorphisms between the algebras listed in the theorem. The simplicity of the algebras in Theorem 1.3.1 can be proved by verifying the hypotheses of Lemma 1.2.1.

1.4. Exercises

1.4.1. Let $A = A_0 \oplus A_1$ be a $\mathbb{Z}_2$-graded associative algebra. Show that A becomes a Lie superalgebra if we define for all $a, b \in A$,

$$[a,b] = ab - (-1)^{\overline{a}\overline{b}} ba.$$

We say that A is *supercommutative* if $[a,b] = 0$ for all a, b.

1.4.2. Prove Lemma 1.1.2.

1.4.3. Let $\mathfrak{k}$ be a Lie superalgebra, and let A be a supercommutative associative algebra over K. Set

$$\mathfrak{k}_A = (\mathfrak{k} \otimes A)_0 = (\mathfrak{k}_0 \otimes A_0) \oplus (\mathfrak{k}_1 \otimes A_1).$$

Show that $\mathfrak{k}_A$ becomes a Lie algebra when we define

$$[x \otimes a, y \otimes b] = [x,y] \otimes ab$$

for $x, y \in \mathfrak{k}, \ a, b \in A$.

1.4.4. Let A be a $\mathbb{Z}_2$-graded algebra and set

$$\mathrm{Sder}(A)_\alpha = \{\partial \in \mathrm{End}_K A | \partial \text{ is a superderivation of degree } \alpha\}.$$

Show that $\mathrm{Sder}(A) = \mathrm{Sder}(A)_0 \oplus \mathrm{Sder}(A)_1$ is a Lie subsuperalgebra of $\mathrm{End}_K A$.

1.4.5. Prove Proposition 1.2.4(b). Hint: If (,) is an invariant bilinear form on $\mathfrak{g}$, show that $(a, [b, c]) = (-1)^{\overline{a}\overline{b}+\overline{a}\overline{c}}([b, c], a)$ for $a, b, c \in \mathfrak{g}$.

1.4.6. Prove Proposition 1.2.4(c). Hint: If (,) is an invariant bilinear form on $\mathfrak{g}$, show that the map $\phi : \mathfrak{g} \longrightarrow \mathfrak{g}^*$ defined by $\phi(x)(y) = (x, y)$ is a homomorphism of $\mathfrak{g}$-modules. Use Schur's Lemma.

1.4.7. Prove the modular law, Lemma 1.2.6.

Chapter 8

Enveloping Algebras of Classical Simple Lie Superalgebras

In this chapter we set up some of the notation that we will use in subsequent chapters to study enveloping algebras of classical simple Lie superalgebras. Until further notice $\mathfrak{g}$ will be a classical simple Lie superalgebra or a Lie superalgebra of Type **A**.

8.1. Root Space and Triangular Decompositions

In Subsection A.1.3 we recall the root space and triangular decompositions of a reductive Lie algebra $\mathfrak{k}$. We want similar decompositions for $\mathfrak{g}$. First let $\mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}_0$. From the Classification Theorem 1.3.1 and the description of the classical simple Lie superalgebras in Chapter 2, it follows that the adjoint action of $\mathfrak{h}_0$ on $\mathfrak{g}$ is semisimple. This can be shown independently of the classification; see Theorem 1.2.9. Let $\mathfrak{h} = \mathfrak{g}^0$ be the centralizer of $\mathfrak{h}_0$ in $\mathfrak{g}$. We denote the set of roots of $\mathfrak{g}_i$ with respect to $\mathfrak{h}_0$ by Δ_i for $i = 0, 1$ and put $\Delta = \Delta_0 \cup \Delta_1$. Thus

$$\Delta_i = \{\alpha \in \mathfrak{h}_0^* | \alpha \neq 0, \mathfrak{g}_i^\alpha \neq 0\}. \tag{8.1.1}$$

Having chosen $\mathfrak{h}_0$, we have a canonical *root space decomposition* (compare equation (2.1.1))

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}^\alpha.$$

By a *triangular decomposition* of $\mathfrak{g}$ we mean a vector space decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+ \tag{8.1.2}$$

with $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ a BPS-subalgebra and $\mathfrak{n}^- \oplus \mathfrak{h}$ the BPS-subalgebra opposite $\mathfrak{b}$.

Lemma 8.1.1. *If* (8.1.2) *is a triangular decomposition, then*

(a) $\mathfrak{n}^-, \mathfrak{n}^+, \mathfrak{h} = \mathfrak{g}^0$ *are* $\mathbb{Z}_2$*-graded subalgebras of* $\mathfrak{g}$ *with* $\mathfrak{n}^\pm$ *nilpotent,*

(b) $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h}_0 \oplus \mathfrak{n}_0^+$ *is a triangular decomposition of the reductive Lie algebra* $\mathfrak{g}_0$ *as in* (A.1.6).

Proof. Since for any roots α and β, we have $[\mathfrak{g}^\alpha, \mathfrak{g}^\beta] \subseteq \mathfrak{g}^{\alpha+\beta}$ and since the set of roots is finite, it follows that $\mathfrak{n}^\pm$ are nilpotent. This easily gives (a), and (b) follows from Theorem A.1.1. □

Given the triangular decomposition (8.1.2), let Δ^+ be the set of roots of $\mathfrak{n}^+$, and set $Q = \mathbb{Z}\Delta$, $Q^+ = \mathbb{N}\Delta^+$. Define a partial order $\geq$ on $\mathfrak{h}^*$ by setting

$$\lambda \geq \mu \text{ if and only if } \lambda - \mu \in Q^+. \tag{8.1.3}$$

This extends the definitions given in Section 5.1. There are several ways that triangular decompositions may arise. The first imitates the situation for reductive Lie algebras.

Lemma 8.1.2. *If* $\alpha_1, \ldots, \alpha_n$ *is a basis of simple roots for* $\mathfrak{g}$, *then there is a triangular decomposition of* $\mathfrak{g}$ *with* $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \pm Q^+} \mathfrak{g}^\alpha$.

Proof. This follows from Lemma 3.2.2. □

Lemma 8.1.3. *If* $\mathfrak{g} = \mathfrak{g}(A, \tau)$ *is the contragredient Lie superalgebra defined by* (5.2.1), *then* $\mathfrak{g}$ *has a basis of simple roots. Moreover the antiautomorphism of* $\mathfrak{g}$ *defined in Lemma* 5.2.5 *is the identity on* $\mathfrak{h}$ *and interchanges* $\mathfrak{n}^-$ *and* $\mathfrak{n}^+$.

Proof. By the definition of realizations, the roots $\alpha_1, \ldots, \alpha_n$ are linearly independent in $\mathfrak{h}^*$. Thus from (5.2.2), $\{\alpha_1, \ldots, \alpha_n\}$ is a basis of simple roots. The remaining assertion is clear. □

A classical simple Lie superalgebra $\mathfrak{g}$ is said to be of *Type I* if

$$\mathfrak{g}_1 = \mathfrak{g}_1^+ \oplus \mathfrak{g}_1^-, \tag{8.1.4}$$

a direct sum of two simple $\mathfrak{g}_0$-submodules. The Type I Lie superalgebras consist of the series $A(m,n), C(n)$, and $\mathfrak{p}(n)$. By checking each case, or by using Proposition 1.2.5, we have that

$$[\mathfrak{g}_1^+, \mathfrak{g}_1^+] = [\mathfrak{g}_1^-, \mathfrak{g}_1^-] = 0. \tag{8.1.5}$$

Also for any Lie superalgebra $\mathfrak{g}$ of Type **A** we have a decomposition as in (8.1.4) such that (8.1.5) holds. Let $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$ be a triangular decomposition of $\mathfrak{g}_0$, and set

$$\mathfrak{n}^+ = \mathfrak{n}_0^+ \oplus \mathfrak{g}_1^+, \qquad \mathfrak{n}^- = \mathfrak{n}_0^- \oplus \mathfrak{g}_1^- \tag{8.1.6}$$

and

$$\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+, \qquad \mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1^+. \tag{8.1.7}$$

Then the following holds.

Lemma 8.1.4.

(a) $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ *is a triangular decomposition of* $\mathfrak{g}$.

(b) $I = \mathfrak{g}_1^+ U(\mathfrak{p}) = U(\mathfrak{p})\mathfrak{g}_1^+$ *is a nilpotent ideal of* $U(\mathfrak{p})$ *such that*

$$U(\mathfrak{p})/I \cong U(\mathfrak{g}_0).$$

(c) *As vector spaces,* $U(\mathfrak{g}) = U(\mathfrak{g}_1^-) \otimes U(\mathfrak{p})$.

Proof. If $\mathfrak{g} = \mathfrak{psl}(2,2)$, then (a) is easy to check directly using results from Section 3.6. Otherwise (a) follows from the explicit description of Borel subalgebras given in Chapter 3, together with Proposition 4.6.1. Part (b) follows from Lemma 6.1.7. To prove (c), note that $\mathfrak{g} = \mathfrak{g}_1^- \oplus \mathfrak{p}$ and use Lemma 6.1.4. □

Remarks 8.1.5.

(a) The choice of which $\mathfrak{g}_0$-submodule of $\mathfrak{g}_1$ to call $\mathfrak{g}_1^+$ and which to call $\mathfrak{g}_1^-$ in (8.1.4) is arbitrary but can have some significant consequences. For example if $\mathfrak{g} = \mathfrak{p}(n)$, then $\dim \mathfrak{g}_1^+ \neq \dim \mathfrak{g}_1^-$.

(b) If $\mathfrak{g} = \mathfrak{psl}(n,n)$ or $\mathfrak{p}(n)$, the triangular decomposition of $\mathfrak{g}$ given by Lemma 8.1.4 does not arise from a basis of simple roots.

Proposition 8.1.6.

(a) *If* $\mathfrak{g}$ *is a classical simple Lie superalgebra or a Lie superalgebra of Type* **A**, *then* $\mathfrak{g}$ *has a triangular decomposition.*

(b) *If* $\mathfrak{g} = \mathfrak{g}(A,\tau)$ *or* $\mathfrak{q}(n)$, *there is an antiautomorphism* $x \longrightarrow {}^t x$ *of* $\mathfrak{g}$ *which is the identity on* $\mathfrak{h}$, *sends* $\mathfrak{g}^\alpha$ *to* $\mathfrak{g}^{-\alpha}$ *for all roots* α, *and interchanges* $\mathfrak{n}^-$ *and* $\mathfrak{n}^+$ *for some triangular decomposition* $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$. *This antiautomorphism sends* ${}^t x$ *to* x *for all* $x \in \mathfrak{g}$.

Proof. For (a) note that by the Classification Theorem, the only case not covered by Lemma 8.1.2 or Lemma 8.1.4 is the Lie superalgebra $\mathfrak{q}(n)$. In this case the result follows easily from Lemma 2.4.1. The statement in (b) follows from Lemma 2.4.1 or Lemma 8.1.3. □

The map in Proposition 8.1.6(b) extends to an antiautomorphism of $U(\mathfrak{g})$, which we also denote by $x \longrightarrow {}^t x$.

Next we examine some consequences of the assumption that $\mathfrak{g}$ has a triangular decomposition.

Lemma 8.1.7. *Suppose that the Lie superalgebra* $\mathfrak{g}$ *has a triangular decomposition as in* (8.1.2). *Then as vector spaces we have*

(8.1.8) $$U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{b}),$$

(8.1.9) $$U(\mathfrak{b}) = U(\mathfrak{h}) \otimes U(\mathfrak{n}^+),$$

and

(8.1.10) $$U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+).$$

Proof. Equations (8.1.8) and (8.1.9) follow immediately from Lemma 6.1.4. To prove (8.1.10), suppose $x_1, \ldots, x_m,\ h_1, \ldots, h_n$, and $y_1, \ldots, y_p$ are homogeneous bases for $\mathfrak{n}^-, \mathfrak{h}$, and $\mathfrak{n}^+$ respectively. Then the set of all monomials

$$x_1^{a_1} \ldots x_m^{a_m} h_1^{b_1} \ldots h_n^{b_n} y_1^{c_1} \ldots y_p^{c_p}$$

where the exponents a_i, b_i, c_i satisfy the same conditions as in Theorem 6.1.2 form a basis for $U(\mathfrak{g})$. Equation (8.1.10) results since the set of all such monomials with $a_i = c_i = 0$ for all i form a basis for $U(\mathfrak{h})$ while the remaining monomials form a basis for $\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+$. □

We call the projection $\zeta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h}) = S(\mathfrak{h})$ relative to the decomposition equation (8.1.10) the *Harish-Chandra projection*.

8.2. Verma Modules and the Category $\mathcal{O}$

8.2.1. Verma Modules. Verma modules for semisimple Lie algebras were introduced in Verma's thesis; see [**Ver68**]. Before we can define Verma modules for classical simple Lie superalgebras, we need to know about some peculiar behavior of the Lie superalgebra $\mathfrak{g} = \mathfrak{q}(n)$. We consider the subalgebras of $\mathfrak{g}$ defined in (2.4.6)–(2.4.8). The next result is taken from [**Pen86**, Proposition 1].

Proposition 8.2.1.

(a) *For any* $\lambda \in \mathfrak{h}_0^*$ *there exists a unique graded simple* $\mathfrak{b}$*-module* V_λ *such that* $\mathfrak{n}^+ V_\lambda = 0$ *and* $hv = \lambda(h)v$ *for all* $h \in \mathfrak{h}_0^*, v \in V_\lambda$.

(b) *Any finite dimensional graded simple* $\mathfrak{b}$*-module is isomorphic to* V_λ *for some* $\lambda \in \mathfrak{h}_0^*$.

Proof. (a) For $\lambda \in \mathfrak{h}_0^*$, define a symmetric bilinear form f_λ on $\mathfrak{h}_1$ by $f_\lambda(x, y) = \lambda([x, y])$. Let $\mathfrak{h}_1^\perp = \{x \in \mathfrak{h}_1 | f_\lambda(x, \mathfrak{h}_1) = 0\}$ be the radical of f_λ and let

$$\mathfrak{a}_\lambda = \operatorname{Ker} \lambda \oplus \mathfrak{h}_1^\perp.$$

Then $\mathfrak{a}_\lambda$ is an ideal in $\mathfrak{h}$ and we set $\mathfrak{c}_\lambda = \mathfrak{h}/\mathfrak{a}_\lambda$. We can regard λ as a linear form on $(\mathfrak{c}_\lambda)_0$. If $\lambda \neq 0$, we can find $z \in (\mathfrak{c}_\lambda)_0$ such that $\lambda(z) = 1$. The factor algebra $A_\lambda = U(\mathfrak{c}_\lambda)/(z-1)$ of $U(\mathfrak{h})$ depends only on λ. Let q_λ be the quadratic form on $(\mathfrak{c}_\lambda)_1$ defined by

$$q_\lambda(x + \mathfrak{h}_1^\perp) = \frac{1}{2} f_\lambda(x, x).$$

Then q_λ is nonsingular, and A_λ is isomorphic to the Clifford algebra of q_λ. By Theorem A.3.6 this Clifford algebra is a central simple graded algebra, so it has a unique graded simple module V_λ. We can regard V_λ as a $U(\mathfrak{b})$-module, and it has the required properties. If $\lambda = 0$, then $\mathfrak{a}_\lambda = \mathfrak{h}$, and we let V_0 be the trivial $U(\mathfrak{b})$-module and $A_0 = U(\mathfrak{h})/\mathfrak{h}U(\mathfrak{h})$.

(b) Now let V be any finite dimensional graded simple $\mathfrak{b}$-module and let $\phi : \mathfrak{b} \longrightarrow \mathfrak{gl}(V)$ be the representation afforded by V. We claim that if $x \in \mathfrak{n}^+$, then $\phi(x)$ is nilpotent. It suffices to show this for $x \in \mathfrak{n}_0^+$, since for $x \in \mathfrak{n}_1^+$ we have

$$\phi(x)^2 = \frac{1}{2}\phi([x, x]).$$

By Lie's Theorem every $\mathfrak{b}_0$-composition factor of V is one-dimensional, and hence annihilated by $\mathfrak{n}_0^+ = [\mathfrak{b}_0^+, \mathfrak{b}_0^+]$, so the claim follows. From Engel's Theorem for Lie superalgebras [**Sch79**, page 236], $V' = \{v \in V | \mathfrak{n}^+ v = 0\} \neq 0$. Since $\mathfrak{n}^+$ is an ideal in $\mathfrak{b}, V'$ is a graded submodule of V and so $V = V'$. Because $\mathfrak{h}_0$ is central in $\mathfrak{h}$, there exists $\lambda \in \mathfrak{h}_0^*$ such that $hv = \lambda v$ for all $h \in \mathfrak{h}$ by an easy adaptation of Schur's Lemma. Thus V is an $(\mathfrak{h}/\operatorname{Ker}\lambda)$-module. If $\mathfrak{h}_1^\perp$ is defined as in the proof of (a), then the image $\phi(\mathfrak{h}_1^\perp)$ of $\mathfrak{h}_1^\perp$ in $\mathfrak{gl}(V)$ is central. Since ϕ is graded simple and $\phi(\mathfrak{h}_1^\perp)$ consists of odd elements, it follows that $\phi(\mathfrak{h}_1^\perp) = 0$. Hence V is a graded simple A_λ-module, so $V \cong V_\lambda$. □

Let $\mathfrak{g}$ be a classical simple Lie superalgebra or a Lie superalgebra of Type **A** and fix a triangular decomposition as in (8.1.2).

Lemma 8.2.2.

(a) *For $\lambda \in \mathfrak{h}_0^*$, there is a unique finite dimensional graded simple $\mathfrak{b}$-module V_λ such that $\mathfrak{n}^+ V_\lambda = 0$ and $hv = \lambda(h)v$ for all $h \in \mathfrak{h}_0$ and $v \in V_\lambda$.*

(b) *If $\mathfrak{g} \neq \mathfrak{q}(n)$, then V_λ is one-dimensional.*

Proof. If $\mathfrak{g} = \mathfrak{q}(n)$, this was proved in Proposition 8.2.1. In all other cases $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}^+$ an ideal in $\mathfrak{b}$ and $\mathfrak{h} = \mathfrak{h}_0$ is abelian. The same reasoning used in the proof of Proposition 8.2.1(b) shows that $\mathfrak{n}^+ V = 0$ for any finite dimensional graded simple $\mathfrak{b}$-module V, and the result follows easily. □

Fix a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

of $\mathfrak{g}$, and set $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}^+$. Similarly let $\mathfrak{k}$ be a reductive Lie algebra, and fix a triangular decomposition as in (A.1.6). Let V_λ be as in Lemma 8.2.2, and let Kv_λ be the one-dimensional $\mathfrak{b}_0$-module with $\mathfrak{n}_0^+ v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$ for $h \in \mathfrak{h}_0$. We define *Verma modules* for $\mathfrak{k}$ and $\mathfrak{g}$ by

$$M(\lambda) = U(\mathfrak{k}) \otimes_{U(\mathfrak{b}_0)} Kv_\lambda, \tag{8.2.1}$$

$$\widetilde{M}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\lambda. \tag{8.2.2}$$

Next we give some of the most basic properties on Verma modules.

Lemma 8.2.3.

(a) *The module $M(\lambda)$ (resp. $\widetilde{M}(\lambda)$) has a unique maximal submodule (resp. a unique maximal $\mathbb{Z}_2$-graded submodule).*

(b) $\widetilde{M}(\lambda) = U(\mathfrak{n}^-)V_\lambda$*; this is a free $U(\mathfrak{n}^-)$-module with basis a vector space basis for V_λ.*

(c) $M(\lambda) = U(\mathfrak{n}_0^-)v_\lambda$*; this is a free $U(\mathfrak{n}_0^-)$-module with basis v_λ.*

(d) *There is a surjective map of $U(\mathfrak{g})$-modules*

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M(\lambda) \longrightarrow \widetilde{M}(\lambda).$$[1]

(e) $\mathrm{End}_{U(\mathfrak{g})} \widetilde{M}(\lambda) \cong \mathrm{End}_{U(\mathfrak{h})} V_\lambda$, *and* $\mathrm{End}_{U(\mathfrak{k})} M(\lambda) \cong K$.

Proof. (a) Let N be a $\mathbb{Z}_2$-graded submodule of $\widetilde{M}(\lambda)$. Then N is a direct sum of weight spaces. If $N^\lambda \neq 0$, then since V_λ is a simple $\mathfrak{b}$-module and λ is the highest weight of $\widetilde{M}(\lambda)$, we have $V_\lambda \subseteq N$, so $N = \widetilde{M}(\lambda)$, because $\widetilde{M}(\lambda)$ is generated by V_λ. Hence if N is a proper $\mathbb{Z}_2$-graded submodule, then $N \subseteq \bigoplus_{\mu \neq \lambda} \widetilde{M}(\lambda)^\mu = M^-$. Therefore the sum of all proper $\mathbb{Z}_2$-graded submodules of $\widetilde{M}(\lambda)$ is contained in M^-, so this sum is the unique maximal $\mathbb{Z}_2$-graded submodule. This proves the result for $\widetilde{M}(\lambda)$, and the proof for $M(\lambda)$ is similar.

[1] This statement can also be proved using Frobenius reciprocity, as can Lemma 8.2.4(a).

(b) By (8.1.8), $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$ as vector spaces. Therefore

$$\begin{aligned}\widetilde{M}(\lambda) &= U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\lambda \\ &= U(\mathfrak{n}^-) \otimes U(\mathfrak{b}) \otimes_{U(\mathfrak{b})} V_\lambda \\ &= U(\mathfrak{n}^-) \otimes V_\lambda.\end{aligned}$$

Now (b) is clear and the same proof works for (c).

(d) By (c) any element of $M(\lambda)$ can be written uniquely in the form bv_λ with $b \in U(\mathfrak{n}_0^-)$. Thus if v is any nonzero homogeneous element of V_λ, we can define a map ϕ of $U(\mathfrak{g})$-modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{g}_0)} M(\lambda) \longrightarrow \widetilde{M}(\lambda)$ by $\phi(a \otimes bv_\lambda) = abv$ for $a \in U(\mathfrak{g})$ and $b \in U(\mathfrak{n}_0^-)$. Now $V_\lambda = U(\mathfrak{b})v$ since V_λ is a graded simple $U(\mathfrak{b})$-module, so from (8.2.2) we have $\widetilde{M}(\lambda) = U(\mathfrak{g})v$. Thus the map ϕ is surjective. For the proof of (e) see Exercise 8.7.4. □

By Lemma 8.2.3, the module $M(\lambda)$ (resp. $\widetilde{M}(\lambda)$) has a unique simple (resp. graded-simple) quotient which we denote by $L(\lambda)$ (resp. $\widetilde{L}(\lambda)$). Any nonzero factor module of $M(\lambda)$ or $\widetilde{M}(\lambda)$ is called a *module generated by a highest weight vector with weight* λ.

8.2.2. Highest Weight Modules in the Type I Case. At this point we prove a result that will be used to study primitive ideals in the Type I case; see Theorem 15.2.5. We use the notation of Lemma 8.1.4.

Lemma 8.2.4.

(a) *For $\lambda \in \mathfrak{h}^*$ we regard $M(\lambda)$ as a $U(\mathfrak{p})$-module with I acting trivially. Then*

$$\widetilde{M}(\lambda) \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M(\lambda).$$

(b) *As a $U(\mathfrak{g})$-module, $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L(\lambda)$ has a unique simple factor module which is isomorphic to $\widetilde{L}(\lambda)$.*

(c) *As a $U(\mathfrak{p})$-module, $\widetilde{L}(\lambda)$ has a unique simple submodule which is isomorphic to $L(\lambda)$.*

Proof. There is a well-defined map of $U(\mathfrak{g})$-modules $\phi : U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M(\lambda) \longrightarrow \widetilde{M}(\lambda)$ given by $\phi(a \otimes bv_\lambda) = abv$. Now by Lemma 8.1.4 we have $U(\mathfrak{g}) = U(\mathfrak{g}_1^-) \otimes U(\mathfrak{p})$, so since $\mathfrak{g}_1^- \oplus \mathfrak{n}_0^- = \mathfrak{n}^-$ and $M(\lambda) = U(\mathfrak{n}_0^-)v_\lambda$, we have

$$\begin{aligned}U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M(\lambda) &= U(\mathfrak{g}_1^-) \otimes U(\mathfrak{p}) \otimes_{U(\mathfrak{p})} M(\lambda) \\ &= U(\mathfrak{g}_1^-) \otimes U(\mathfrak{n}_0^-)v_\lambda \\ &= U(\mathfrak{n}^-) \otimes Kv_\lambda \\ &= \widetilde{M}(\lambda),\end{aligned}$$

and ϕ is an isomorphism.

(b) holds as $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L(\lambda)$ is a factor module of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M(\lambda) \cong \widetilde{M}(\lambda)$.

(c) Suppose that $L(\nu)$ is a simple $U(\mathfrak{p})$-submodule of $\widetilde{L}(\lambda)$. Then there is a nonzero homomorphism of $U(\mathfrak{g})$-modules from $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L(\nu)$ to $\widetilde{L}(\lambda)$. Since $\widetilde{L}(\lambda)$ is simple, we see that $\widetilde{L}(\lambda)$ is a factor module of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L(\nu)$, and hence using (a), $\widetilde{L}(\lambda)$ is also a factor module of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M(\nu) \cong \widetilde{M}(\nu)$. However $\widetilde{M}(\nu)$ has a unique simple factor module $\widetilde{L}(\lambda)$. Therefore $\widetilde{L}(\nu) \cong \widetilde{L}(\lambda)$ and it follows that $\nu = \lambda$. □

We call the $U(\mathfrak{g})$-module $K(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L(\lambda)$ the *Kac module* with highest weight λ.

8.2.3. The Category $\mathcal{O}$. In the study of Verma modules for $U(\mathfrak{k})$ it is convenient to consider the category $\mathcal{O}$ introduced by Bernšteĭn, Gel′fand, and Gel′fand, [**BGG71**], [**BGG75**], [**BGG76**]. By definition, objects in the category $\mathcal{O}$ are $U(\mathfrak{k})$-modules with the following properties.

(a) $M = \bigoplus_{\mu \in \mathfrak{h}_0^*} M^\mu$.

(b) For all $v \in M, \dim U(\mathfrak{n}_0^+)v < \infty$.

(c) M is a finitely generated $U(\mathfrak{k})$-module.

Morphisms are just $U(\mathfrak{k})$-module homomorphisms. Modules satisfying condition (a) are called *weight modules*. When $\mathfrak{k} = \mathfrak{g}_0$, we also consider the category $\widetilde{\mathcal{O}}$ of graded $U(\mathfrak{g})$-modules which belong to the category $\mathcal{O}$ when regarded as $U(\mathfrak{k})$-modules by restriction. We emphasize that the definition of the category $\widetilde{\mathcal{O}}$ depends only on a triangular decomposition of $\mathfrak{g}_0$. However whenever we refer to highest weight modules, we implicitly fix a triangular decomposition of $\mathfrak{g}$ as in (8.1.2).

It is easy to see that a submodule of a homomorphic image of an object in the category $\mathcal{O}$ is again an object in $\mathcal{O}$. We show below that any object in $\mathcal{O}$ has finite length. First we identify the simple modules in the categories $\mathcal{O}$ and $\widetilde{\mathcal{O}}$.

Lemma 8.2.5. *If L is a simple object in $\widetilde{\mathcal{O}}$ (resp. $\mathcal{O}$), then $L \cong \widetilde{L}(\lambda)$ (resp. $L \cong L(\lambda)$) for some $\lambda \in \mathfrak{h}^*$.*

Proof. We give the proof for $L \in \widetilde{\mathcal{O}}$ since it is slightly more involved. Since L is $\mathbb{Z}_2$-graded, there exists $\mu \in \mathfrak{h}^*$ such that L^μ contains a nonzero homogeneous element v. Since $N = U(\mathfrak{n}^+)v$ is finite dimensional, there exists $\lambda \in \mathfrak{h}^*$ such that $N^\lambda \neq 0$, but $N^{\lambda+\alpha} = 0$ for all positive roots α. Now N^λ is a finite dimensional $\mathbb{Z}_2$-graded $\mathfrak{b}$-module, so if $\mathfrak{g} = \mathfrak{q}(n)$, N^λ contains a $\mathfrak{b}$-submodule isomorphic to V_λ, by Proposition 8.2.1. If $\mathfrak{g} \neq \mathfrak{q}(n)$, then V_λ is one-dimensional, and then obviously N^λ contains a copy of V_λ. Since $U(\mathfrak{g})V_\lambda$ is a $\mathfrak{g}$-submodule of L, it equals L by simplicity. Then the result follows since $\widetilde{M}(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} V_\lambda$ maps onto L, and $\widetilde{M}(\lambda)$ has a unique simple $\mathbb{Z}_2$-graded factor module. □

Lemma 8.2.6. *Let M be a module in the category $\mathcal{O}$.*

(a) *If M is nonzero, then M contains a highest weight vector.*

(b) *There is a finite series of submodules*

$$M = M_s \supset M_{s-1} \supset \cdots \supset M_1 \supset M_0 = 0$$

such that M_i/M_{i-1} is a highest weight module for $1 \le i \le s$.

(c) *For all $\mu \in \mathfrak{h}_0^*$ the weight space M^μ is finite dimensional.*

Proof. Part (a) follows from Engel's Theorem and the fact that $\dim U(\mathfrak{n}_0^+)v < \infty$ for $v \in M$. For (b) suppose that M_{i-1} has been constructed and $M_{i-1} \neq M$. Then by (a) there exists $v_i \in M$ such that the image of v_i in M/M_{i-1} is a highest weight vector, and we set $M_i = U(\mathfrak{k})v_i + M_{i-1}$. Since M is Noetherian, this process terminates, so $M = M_s$ for some s. By (b) the proof of (c) reduces to the case where M is a highest weight module. For highest weight modules, the result follows from the well-known fact that the dimension of the weight spaces of a Verma module for $\mathfrak{k}$ are given by a partition function; see Section 8.4 or [**Hum72**, Section 24]. □

8.2.4. Central Characters and Blocks. Denote the center of $U(\mathfrak{g})$ (resp. $U(\mathfrak{k})$) by $Z(\mathfrak{g})$ (resp. $Z(\mathfrak{k})$). For simplicity assume that $\mathfrak{g} \neq \mathfrak{q}(n)$ (see [**Ser83**] for this case). Then in (8.2.2), V_λ is one-dimensional. Thus if $z \in Z(\mathfrak{g})$ and $v \in V_\lambda$, zv is a highest weight vector of weight λ so we have

$$zv = \chi_\lambda(z)v, \tag{8.2.3}$$

for some $\chi_\lambda(z) \in K$. The map $\chi_\lambda : Z(\mathfrak{g}) \longrightarrow K$ is an algebra homomorphism called *the central character of* $U(\mathfrak{g})$ *afforded by the* $U(\mathfrak{g})$*-module* $\widetilde{M}(\lambda)$. Because z is central in $U(\mathfrak{g})$ and $\widetilde{M}(\lambda)$ is generated by v, it follows that z acts on $\widetilde{M}(\lambda)$ as the scalar $\chi_\lambda(z)$. Similarly, if $z \in Z(\mathfrak{g}_0)$, z acts on $M(\lambda)$ as $\chi^0_\lambda(z)$ and the map $\chi^0_\lambda : Z(\mathfrak{g}_0) \longrightarrow K$ is called *the central character of* $U(\mathfrak{g}_0)$ *afforded by the* $U(\mathfrak{g}_0)$*-module* $M(\lambda)$.

Now let $\zeta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h}) = S(\mathfrak{h})$ be the Harish-Chandra projection; see (8.1.10). Suppose that

$$M = U(\mathfrak{g})v_\lambda \tag{8.2.4}$$

is a $U(\mathfrak{g})$-module generated by a highest weight vector v_λ with weight λ. For $h \in \mathfrak{h}$ we have $hv_\lambda = \lambda(h)v_\lambda$. We set $h(\lambda) = \lambda(h)$ and view $S(\mathfrak{h})$ as the algebra of polynomial functions on $\mathfrak{h}^*$. Then clearly $fv_\lambda = f(\lambda)v_\lambda$ for all $f \in S(\mathfrak{h})$. Set $M^- = \bigoplus_{\mu \neq \lambda} M^\mu$. Then $\mathfrak{n}^+ v_\lambda = 0$ and $\mathfrak{n}^- M \subseteq M^-$. Therefore $(\mathfrak{n}^- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^+)v_\lambda \subseteq M^-$, so for any $z \in U(\mathfrak{g})$ we have

$$zv_\lambda \equiv \zeta(z)v_\lambda = \zeta(z)(\lambda)v_\lambda \bmod M^-. \tag{8.2.5}$$

Corollary 8.2.7. *If $\zeta : U(\mathfrak{g}) \longrightarrow U(\mathfrak{h}) = S(\mathfrak{h})$ is the Harish-Chandra projection, then for all $z \in Z(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$ we have*

$$\chi_\lambda(z) = \zeta(z)(\lambda).$$

Proof. Combine (8.2.3) and (8.2.5). □

For each central character χ of $U(\mathfrak{k})$ and M an object of $\mathcal{O}$ we define

$$M_\chi = \{m \in M | (\operatorname{Ker} \chi)^n m = 0 \text{ for some } n \geq 0\}. \tag{8.2.6}$$

For distinct central characters $\chi_0, \ldots, \chi_p$ with $p \geq 1$, we have for large n,

$$(\operatorname{Ker} \chi_0)^n + \bigcap_{i=1}^{p} (\operatorname{Ker} \chi_i)^n = Z(\mathfrak{k}), \tag{8.2.7}$$

by the Chinese Remainder Theorem (note that the left side cannot be contained in any maximal ideal). Therefore the sum $\sum M_\chi$ is direct where χ runs over all central characters of $U(\mathfrak{k})$. Since the category $\mathcal{O}$ is closed under images, it follows that

$$M = \bigoplus M_\chi. \tag{8.2.8}$$

Moreover this sum is finite since M is finitely generated. In addition for $\lambda \in \mathfrak{h}^*$, we set

$$M_{(\lambda)} = M_{\chi_\lambda}. \tag{8.2.9}$$

Let $\mathcal{O}_\lambda$ be the full subcategory of the category $\mathcal{O}$ consisting of all modules M such that $M = M_{(\lambda)}$. The categories $\mathcal{O}_\lambda$ are called the *blocks* of the category $\mathcal{O}$.

8.2.5. Contravariant Forms.

Let ζ be the Harish-Chandra projection (8.1.10).

Lemma 8.2.8. *Assume $\mathfrak{g} = \mathfrak{g}(A, \tau)$ or $\mathfrak{q}(n)$ and the triangular decomposition satisfies the conditions of Proposition* 8.1.6(b). *Then:*

(a) *The decomposition* (8.1.10) *is stable under the antiautomorphism $x \longrightarrow {}^t x$.*

(b) $\zeta(u) = \zeta({}^t u)$ *for all $u \in U(\mathfrak{g})$.*

Proof. Exercise 8.7.1. □

Following Jantzen [**Jan79**, Section 1.6], we say that a bilinear form F defined on a $U(\mathfrak{g})$-module M is *contravariant* if

(a) $F(x,y) = F(y,x)$,

(b) $F(ux,y) = F(x, {}^t u y)$,

for all $x, y \in M$ and $u \in U(\mathfrak{g})$.

Corollary 8.2.9. *The bilinear form F on $U(\mathfrak{g})$ with values in $S(\mathfrak{h})$ given by*

$$F(x,y) = \zeta({}^t x y) \quad \text{for } x, y \in U(\mathfrak{g})$$

is contravariant.

Proof. Exercise 8.7.2. ☐

Now suppose again that M is as in (8.2.4).

Lemma 8.2.10. *For $y \in U(\mathfrak{g})$ the following conditions are equivalent.*

(a) *$y v_\lambda$ is contained in a proper submodule of M.*

(b) *$U(\mathfrak{g}) y v_\lambda \subseteq M^-$.*

(c) *$\zeta({}^t x y)(\lambda) = 0$ for all $x \in U(\mathfrak{g})$.*

Proof. Clearly (a) and (b) are equivalent, and the equivalence of (b) and (c) results from setting $z = {}^t x y$, for $x \in U(\mathfrak{g})$, in (8.2.7). ☐

Corollary 8.2.11. *There is a well-defined K-valued contravariant bilinear form $(\ ,\ )_M$ on M given by the formula*

$$(x v_\lambda, y v_\lambda)_M = \zeta({}^t x y)(\lambda)$$

for $x, y \in U(\mathfrak{g})$. The radical of $(\ ,\ )_M$ is the largest proper submodule of M.

Proof. See Exercise 8.7.5. ☐

This applies in particular if $M = \widetilde{M}(\lambda)$ is a Verma module, and we set $F^\lambda = F_{\widetilde{M}(\lambda)}$. Recall that $\widetilde{M}(\lambda) = U(\mathfrak{n}^-) v_\lambda$, a free $U(\mathfrak{n}^-)$-module. The bilinear forms F^λ are related to the $S(\mathfrak{h})$-valued bilinear form F introduced in Corollary 8.2.9. Indeed we have

$$F(x,y)(\lambda) = F^\lambda(x v_\lambda, y v_\lambda) = \zeta({}^t x y)(\lambda) \tag{8.2.10}$$

for all $\lambda \in \mathfrak{h}^*$ and $x, y \in U(\mathfrak{n}^-)$.

8.2.6. Base Change. Much of the foregoing is easily extended to a more general situation. What follows is largely based on [**Jan79**, Kapitel 4]. This material is used in the construction of the Jantzen filtration. For a vector space V over K and a commutative K-algebra A, except where noted, we set $V_A = V \otimes A$. If $\mathfrak{k}$ is a Lie superalgebra, then $\mathfrak{k}_A$ is a Lie superalgebra over A and we define the enveloping algebra $U(\mathfrak{k}_A)$ to be $U(\mathfrak{k})_A = U(\mathfrak{k}) \otimes A$.

Now suppose that $\mathfrak{g}$ is classical simple and $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Mainly for notational reasons we assume that $\mathfrak{g} \neq \mathfrak{q}(n)$, so that $\mathfrak{h} = \mathfrak{h}_0$. Fix a triangular decomposition $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ of $\mathfrak{g}$, and set $\mathfrak{b} = \mathfrak{n}^- \oplus \mathfrak{h}$. Then $\mathfrak{g}_A = \mathfrak{n}_A^- \oplus \mathfrak{h}_A \oplus \mathfrak{n}_A^+$ is a *triangular decomposition* of $\mathfrak{g}_A$. Note that $\mathfrak{h}_A^*$ is naturally paired with $\mathfrak{h}_A$, and we can consider $\mathfrak{g}_A$-modules with highest weights in $\mathfrak{h}_A^* = \mathfrak{h}^* \otimes A$. Let M be a $U(\mathfrak{g}_A)$-module and let $\lambda \in \mathfrak{h}_A^*$. Set

$$M^\lambda = \{m \in M | hm = \lambda(h)m \text{ for all } h \in \mathfrak{h}_A\}.$$

If M is torsion free as an A-module and $\lambda \neq \mu \in \mathfrak{h}_A^*$, then $M^\lambda \cap M^\mu = 0$, and more generally the sum of weight spaces $\bigoplus_{\mu \in \mathfrak{h}_A^*} M^\mu$ is direct. We say that M is a *weight module* if

$$M = \bigoplus_{\mu \in \mathfrak{h}_A^*} M^\mu. \tag{8.2.11}$$

Equation (8.2.11) is called the *weight space decomposition* of M. For $\lambda \in \mathfrak{h}_A^*$, let $A_\lambda = Av_\lambda$ be the $U(\mathfrak{b}_A)$-module which is isomorphic to A as an A-module and which satisfies $\mathfrak{n}_A^+ v_\lambda = 0$ and $ha = \lambda(h)a$ for $h \in \mathfrak{h}_A$ and $a \in A_\lambda$. Then the Verma module with highest weight λ is defined to be

$$\widetilde{M}(\lambda)_A = U(\mathfrak{g}_A) \otimes_{U(\mathfrak{b}_A)} A_\lambda. \tag{8.2.12}$$

The weight space decomposition of $\widetilde{M}(\lambda)_A$ is given by

$$\widetilde{M}(\lambda)_A = \bigoplus_{\eta \in Q^+} \widetilde{M}(\lambda)_A^{\lambda - \eta}.$$

If B is a field extension of K, then $\widetilde{M}(\lambda)_B$ has a unique simple factor module which we denote by $\widetilde{L}(\lambda)_B$. Note however that $\widetilde{L}(\lambda)_B$ need not be obtained from a $U(\mathfrak{g})$-module by extension of scalars. If A is a domain with field of fractions B, then $\widetilde{L}(\lambda)_B$ is torsion free as an A-module, and we set $\widetilde{L}(\lambda)_A = U(\mathfrak{g}_A)v_\lambda$. Note that $\widetilde{L}(\lambda)_A \otimes B = \widetilde{L}(\lambda)_B$.

Consider the Harish-Chandra projection $\zeta_A : U(\mathfrak{g}_A) \longrightarrow U(\mathfrak{h}_A) = S(\mathfrak{h}_A)$ relative to the decomposition

$$U(\mathfrak{g}_A) = U(\mathfrak{h}_A) \oplus (\mathfrak{n}_A^- U(\mathfrak{g}_A) + U(\mathfrak{g}_A)\mathfrak{n}_A^+). \tag{8.2.13}$$

Just as in Corollary 8.2.9, the $S(\mathfrak{h}_A)$-valued bilinear form $F \otimes A$ on $U(\mathfrak{g}_A)$ with values given by

$$(F \otimes A)(x, y) = \zeta_A({}^t xy) \quad \text{for } x, y \in U(\mathfrak{g})$$

is contravariant. Next suppose that

$$M = U(\mathfrak{g}_A)v_\lambda \tag{8.2.14}$$

is a $U(\mathfrak{g}_A)$-module generated by a highest weight vector v_λ with weight λ. Then Corollary 8.2.11 holds verbatim except that the contravariant bilinear form $(\ ,\ )_M$ on M is now A-valued.

Let $\phi : A \longrightarrow A'$ be a map of commutative K-algebras. For a vector space V, define $V_\phi : V_A \longrightarrow V_{A'}$ by $V_\phi(v \otimes a) = v \otimes \phi(a)$. If V has an additional structure, such as that of a Lie superalgebra or associative algebra, then V_ϕ respects this structure. Now suppose that M is as in (8.2.14) and let ψ denote the natural map $\psi : M \longrightarrow M' = M \otimes_A A'$. Then we have

$$\phi((x, y)_M) = (\psi(x), \psi(y))'_M.$$

Given $\lambda \in \mathfrak{h}_A^*$, abbreviate $\mathfrak{h}_\phi^*(\lambda)$ to λ'. Then ϕ induces a map from the $\mathfrak{b}_A$-module A_λ to the $\mathfrak{b}_{A'}$-module $A_{\lambda'}$ that is compatible with $\mathfrak{b}_\phi : \mathfrak{b}_A \longrightarrow \mathfrak{b}_{A'}$. The same holds for the map $U(\mathfrak{g})_\phi : U(\mathfrak{g}_A) \longrightarrow U(\mathfrak{g}_{A'})$, if we consider the action of $\mathfrak{b}_A$ (resp. $\mathfrak{g}_A$) and $\mathfrak{b}_{A'}$ (resp. $\mathfrak{g}_{A'}$) via right (resp. left) multiplication. Thus ϕ induces a map $\widetilde{M}(\lambda)_\phi : \widetilde{M}(\lambda)_A \longrightarrow \widetilde{M}(\lambda')_{A'}$ sending uv_λ to $(U(\mathfrak{g})_\phi u)v_{\lambda'}$.

8.2.7. Further Properties of the Category $\mathcal{O}$. In the proof of the next result, we use the fact that a submodule of a finitely generated module over the Noetherian ring $U(\mathfrak{g}_0)$ is again finitely generated.

Theorem 8.2.12. *If M is an object in the category $\mathcal{O}$, then M has finite length.*

Proof. By (8.2.8), we may assume that $M = M_\chi$ where $\chi = \chi_\lambda^0$. Since $U(\mathfrak{g}_0)$ is Noetherian and M is finitely generated, M is a Noetherian $U(\mathfrak{g}_0)$-module. If N is a nonzero submodule of M, then N has a maximal submodule N', and $N/N' \cong L(\mu)$ for some $\mu \in \mathfrak{h}^*$ by Lemma 8.2.5. Since $L(\mu), M(\lambda)$ have the same central character, we have $\mu \in W \circ \lambda$ by Theorem A.1.8. If the result is false, M would contain an infinite chain of submodules $N_0 \supseteq N_1 \supseteq \dots$ with N_i/N_{i+1} simple for each i. Since W is finite, infinitely many of these quotients would be isomorphic to one another. This would imply that M has a weight of infinite multiplicity, a contradiction to Lemma 8.2.6. □

Corollary 8.2.13. *Any object in the category $\widetilde{\mathcal{O}}$ of graded $U(\mathfrak{g})$-modules has finite length.*

Proof. Immediate. □

Lemma 8.2.14. *The Verma module $\widetilde{M}(\lambda)$ has a finite composition series as a $\mathbb{Z}_2$-graded module with composition factors of the form $\widetilde{L}(\mu)$ where $\chi_\lambda = \chi_\mu$ and $\mu \leq \lambda$. Furthermore $\widetilde{L}(\lambda)$ is a composition factor of $\widetilde{M}(\lambda)$ with multiplicity one.*

Proof. By Corollary 8.2.13, $\widetilde{M}(\lambda)$ has a finite composition series. Suppose that $\widetilde{L}(\mu)$ is a composition factor. Since any central element z of $U(\mathfrak{g})$ acts on $\widetilde{M}(\lambda)$ as the scalar $\chi_\lambda(z)$, it follows that $\chi_\lambda = \chi_\mu$. Finally it was shown in the proof of Lemma 8.2.3 that any proper $\mathbb{Z}_2$-graded submodule of $\widetilde{M}(\lambda)$ is contained in $\bigoplus_{\mu\neq\lambda} \widetilde{M}(\lambda)^\mu$. The remaining statements follow from this. □

Lemma 8.2.15. *If M, V are modules in the category $\widetilde{\mathcal{O}}$ with V finite dimensional, then $M \otimes V$ is a module in the category $\widetilde{\mathcal{O}}$.*

Proof. This follows from the corresponding result in the semisimple case; see [**Hum08**, Theorem 1.1]. □

For a tensor product of a Verma module with a finite dimensional module we can say more.

Lemma 8.2.16. *Assume $\mathfrak{g} \neq \mathfrak{q}(n)$.*

(a) *Let V be a finite dimensional $U(\mathfrak{g})$-module, and order the set of weights $\nu_1, \dots, \nu_s$ of V so that $\nu_i > \nu_j$ implies $i < j$. If $\lambda \in \mathfrak{h}^*$, then $\widetilde{M}(\lambda) \otimes V$ has series*

$$\widetilde{M}(\lambda) \otimes V = M_s \supset M_{s-1} \supset \cdots \supset M_0 = 0$$

such that $M_i/M_{i-1} \cong \widetilde{M}(\lambda + \nu_i)$ for $1 \leq i \leq s$.

(b) *Let V be a finite dimensional $U(\mathfrak{k})$-module, and order the set of weights $\nu_1, \dots, \nu_s$ of V as in* (a). *Then $M(\lambda) \otimes V$ has a finite series of submodules*

$$M(\lambda) \otimes V = M_s \supset M_{s-1} \supset \cdots \supset M_0 = 0$$

such that $M_i/M_{i-1} \cong M(\lambda + \nu_i)$ for $1 \leq i \leq s$.

Proof. (a) The argument is well known so we merely sketch it; see [**Dix96**, Lemma 7.6.14] or [**Jan79**, Satz 2.2] for more details. Let $x_1, \dots, x_s$ be a basis for V, where x_i has weight ν_i. Then $a_i = v_\lambda \otimes x_i \in Kv_\lambda \otimes V$ has weight $\lambda + \nu_i$. Let M_i be the submodule generated by $a_1, \dots, a_i$. Then M_i/M_{i-1} is generated by the image of a_i, which is a highest weight vector, so M_i/M_{i-1} is an image of $\widetilde{M}(\lambda+\nu_i)$. A computation using the PBW Theorem for $U(\mathfrak{n}^-)$ completes the proof of (a) and the proof of (b) is similar. For a different proof of (b) see [**Hum08**, Theorem 3.6]. □

For a much more detailed study of the category $\mathcal{O}$ we refer to [**Hum08**]. We note in particular that the category $\mathcal{O}$ has enough projectives and that there are only finitely many indecomposable projectives in each block. It follows that each block is equivalent to the category of finite dimensional modules over a finite dimensional algebra, namely the endomorphism ring of a projective generator. Furthermore each block has a natural graded structure, and the combinatorics of graded translation functors and graded projectives can be described using Hecke algebras and Kazhdan-Lusztig theory [**BGS96**], [**Str03a**], [**Soe90a**]. For explicit examples in low ranks see [**MS08**], [**Str03b**].

8.3. Basic Classical Simple Lie Superalgebras and a Hypothesis

8.3.1. Basic Lie Superalgebras. Recall that a classical simple Lie superalgebra $\mathfrak{g}$ is called basic if $\mathfrak{g}$ admits an even nondegenerate $\mathfrak{g}$-invariant bilinear form. Such a form is necessarily supersymmetric; see Proposition 1.2.4. By Theorem 5.4.1 the contragredient Lie superalgebra $\mathfrak{g} = G(A, \tau)/C$ is basic. On the other hand, by Exercises 2.7.11 and 2.7.13, if $\mathfrak{g} \cong \mathfrak{p}(n)$ or $\mathfrak{q}(n)$, then $\mathfrak{g}$ does not have an even nondegenerate invariant bilinear form. Thus the basic classical simple Lie superalgebras are precisely the finite dimensional simple Lie superalgebras that can be constructed as contragredient Lie superalgebras. In particular for $\mathfrak{g}$ basic we have $\mathfrak{h} = \mathfrak{h}_0$. In addition if $\mathfrak{g} = \mathfrak{gl}(m, n)$, then by Exercise 2.7.1 we can define a nondegenerate $\mathfrak{g}$-invariant supersymmetric even bilinear form on $\mathfrak{g}$ using the supertrace. Hence although $\mathfrak{g}$ is not simple, it has many properties in common with basic classical simple Lie superalgebras. Thus in the following we assume that $\mathfrak{g}$ is either basic classical simple or $\mathfrak{g} = \mathfrak{gl}(m, n)$.

In Section A.1 we defined the Weyl group of a reductive Lie algebra $\mathfrak{k}$ using the Killing form on $\mathfrak{k}$. However for $\mathfrak{g}$ as above it is preferable to use a bilinear form $(\ ,\ )$ with properties as in the preceding paragraph, and we shall always do so; see also Remark A.1.7. Let $\mathfrak{h} = \mathfrak{h}_0$ be a Cartan subalgebra of $\mathfrak{g}$ and define the set of roots of Δ as in (8.1.1). Set

(8.3.1) $$\overline{\Delta}_0 = \{\alpha \in \Delta_0 | \alpha/2 \notin \Delta_1\}, \quad \overline{\Delta}_1 = \{\alpha \in \Delta_1 | 2\alpha \notin \Delta_0\}.$$

If $\alpha \in \Delta_1$, we say that α is *isotropic* if $(\alpha, \alpha) = 0$.

Lemma 8.3.1. *We have the following.*

(a) $(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = 0$ *unless* $\alpha + \beta = 0$.

(b) *The restriction of* $(\ ,\ )$ *to* $\mathfrak{g}^\alpha \times \mathfrak{g}^{-\alpha}$ *is nondegenerate. In particular the restriction of* $(\ ,\ )$ *to* $\mathfrak{h}$ *is nondegenerate.*

Proof. (a) Suppose $x \in \mathfrak{g}^\alpha$ and $y \in \mathfrak{g}^\beta$. If $\alpha + \beta \neq 0$, there exists $h \in \mathfrak{h}$, such that $(\alpha + \beta)(h) \neq 0$. Then we have

$$([h, x], y) = -([x, h], y) = -(x, [h, y]).$$

This implies $(\alpha + \beta)(h)(x, y) = 0$, so $(x, y) = 0$.

(b) The first statement follows immediately from (a), and the second holds because $\mathfrak{h} = \mathfrak{g}^0$. □

Next recall the elements $h_\alpha \in \mathfrak{h}^*$ defined in (5.4.2).

Lemma 8.3.2.

(a) *If* $x \in \mathfrak{g}^\alpha$, $y \in \mathfrak{g}^{-\alpha}$, *then* $[x, y] = (x, y)h_\alpha$.

(b) $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}] = Kh_\alpha$.

(c) *An odd root* α *is isotropic if and only if* 2α *is not a root, that is,* $\alpha \in \overline{\Delta}_1$.

Proof. (a) For $h \in \mathfrak{h}$ we have, using invariance of the form $(\ , \ )$, that

$$\begin{aligned} (h, [x, y]) &= ([h, x], y) \\ &= \alpha(h)(x, y) \\ &= (h, h_\alpha)(x, y) \\ &= (h, (x, y)h_\alpha). \end{aligned}$$

This says that $[x, y] - (x, y)h_\alpha$ is orthogonal to $\mathfrak{h}$, so the result follows from (b).

(b) If $\mathfrak{g} = \mathfrak{psl}(2, 2)$, the result can be checked directly. Otherwise $[\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ can have dimension at most one, since each of $\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}$ is one-dimensional. Hence the result follows from (a) and Lemma 8.3.1(b).

(c) Suppose that $(\alpha, \alpha) \neq 0$. Since $(\ , \ )$ is nondegenerate and $(\mathfrak{g}^\alpha, \mathfrak{g}^\beta) = 0$ unless $\alpha + \beta = 0$, we can choose $x \in \mathfrak{g}^\alpha, y \in \mathfrak{g}^{-\alpha}$ such that $(x, y) \neq 0$. Thus by (a) and the Jacobi identity

$$\begin{aligned} [[x, x], y] &= -2[[x, y], x] \\ &= -2(x, y)[h_\alpha, x] \\ &= -2(x, y)(\alpha, \alpha)x. \end{aligned}$$

Thus, $0 \neq [x, x] \in \mathfrak{g}^{2\alpha}$. Conversely if 2α is an even root, then $(\alpha, \alpha) = (2\alpha, 2\alpha)/4 \neq 0$. □

By the lemma, the set of nonisotropic roots is

$$\Delta_{\text{nonisotropic}} = \Delta_0 \cup (\Delta_1 \backslash \overline{\Delta}_1).$$

For $\alpha \in \Delta_{\text{nonisotropic}}$ we set $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ and define the *reflection* s_α in the hyperplane orthogonal to α by

$$s_\alpha(\lambda) = \lambda - (\lambda, \alpha^\vee)\alpha. \tag{8.3.2}$$

This definition extends (A.1.2). As usual the *Weyl group* W is the subgroup of $GL(\mathfrak{h}^*)$ generated by all reflections.

Now consider a triangular decomposition for the classical simple Lie superalgebra $\mathfrak{g}$ (resp. reductive Lie algebra $\mathfrak{k}$) as in (8.1.2) (resp. (A.1.6)). Let $\mathfrak{b}, \mathbf{b}$ be Borel subalgebras of $\mathfrak{g}, \mathfrak{k}$ respectively. We allow the case $\mathfrak{k} = \mathfrak{g}_0$, in which case we assume $\mathfrak{b}_0 = \mathbf{b}$. Let ρ_0 be the half sum of the positive roots of $\mathbf{b}$, $\rho_1(\mathfrak{b})$ the half sum of the positive odd roots of $\mathfrak{b}$, and $\rho(\mathfrak{b}) = \rho_0 - \rho_1(\mathfrak{b})$. When $\mathfrak{b}$ is fixed, we set $\rho = \rho(\mathfrak{b})$. Define translated actions of the Weyl group W on $\mathfrak{h}^*$ by

$$w \circ \lambda = w(\lambda + \rho_0) - \rho_0, \tag{8.3.3}$$

$$w \,.\, \lambda = w(\lambda + \rho) - \rho \tag{8.3.4}$$

for $w \in W$ and $\lambda \in \mathfrak{h}^*$.

8.3.2. A Hypothesis.

Remark 8.3.3. The Lie superalgebras $\mathfrak{psl}(n, n)$ and $\mathfrak{sl}(n, n)$ have the unpleasant property that the simple roots for the distinguished Borel subalgebra are not linearly independent. This causes some technical difficulties, for example with character formulas and with the Šapovalov determinant. The simplest solution is to work instead with $\mathfrak{gl}(n, n)$. To justify this, we make some remarks about the relation between representations of $\mathfrak{psl}(n, n)$ and $\mathfrak{sl}(n, n)$, and between representations of $\mathfrak{sl}(m, n)$ and $\mathfrak{gl}(m, n)$ (m is not necessarily equal to n).[2] First note that since $\mathfrak{psl}(n, n)$ is an image of $\mathfrak{sl}(n, n)$, we can regard any $\mathfrak{psl}(n, n)$-module as an $\mathfrak{sl}(n, n)$-module. Thus nothing is lost in studying representations of $\mathfrak{sl}(n, n)$ instead of studying representations of $\mathfrak{psl}(n, n)$.

Note that $z = \sum_{i=1}^{m+n} e_{i,i}$ is central in $\mathfrak{gl}(m, n)$, and if $m \neq n$ we have a direct sum of ideals,

$$\mathfrak{gl}(m, n) = \mathfrak{sl}(m, n) \oplus Kz,$$

and therefore

$$U(\mathfrak{gl}(m, n)) = U(\mathfrak{sl}(m, n))[z].$$

This makes it very easy to pass between representations of $\mathfrak{gl}(m, n)$ and representations of $\mathfrak{sl}(m, n)$. It is not always as clear what to do if $m = n$, but we give two examples below.

[2]We remark that a comparison between the category $\mathcal{O}$ (and parabolic category $\mathcal{O}$) for the Lie algebras $\mathfrak{sl}(n)$ and $\mathfrak{gl}(n)$ can be found in [**Kla11**, Section 1.4].

Let $\mathfrak{g} = \mathfrak{gl}(m,n), \mathfrak{g}' = \mathfrak{sl}(m,n)$, let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices, and let $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$. Suppose $\lambda \in \mathfrak{h}^*$, and let λ' denote the restriction of λ to $\mathfrak{h}'$. Let $\widetilde{M}^{\mathfrak{g}}(\lambda)$ and $\widetilde{M}^{\mathfrak{g}'}(\lambda')$ be the Verma modules for $\mathfrak{g}$ and $\mathfrak{g}'$ constructed using the distinguished Borel subalgebra. Then as a $\mathfrak{g}'$-module $\widetilde{M}^{\mathfrak{g}}(\lambda) \cong \widetilde{M}^{\mathfrak{g}'}(\lambda')$, and for α a root of $\mathfrak{g}$, we have $(\lambda - \alpha)' = \lambda' - \alpha'$, so the weight spaces of $\widetilde{M}^{\mathfrak{g}}(\lambda)$ restricted to $\mathfrak{g}'$ are just the weight spaces of $\widetilde{M}^{\mathfrak{g}'}(\lambda')$.

Now consider the simple highest weight module $\widetilde{L}(\lambda)$ for $\mathfrak{g}'$. If $m \neq n$, we set

$$z = (n\sum_{i=1}^{m} e_{i,i} + m\sum_{i=m+1}^{m+n} e_{i,i})/(n-m) \in \mathfrak{g}'.$$

Then z is central in $\mathfrak{g}'_0$ and $[z, e_\beta] = e_\beta$ for any odd positive root β. It follows that as a $\mathfrak{g}'_0$-module we have a decomposition

$$L = \bigoplus_{i \geq 0} L(i) \tag{8.3.5}$$

where

$$L(i) = \{x \in L | zx = (\lambda(z) - i)x\}.$$

This is useful if we want to study the primitive ideals of $U(\mathfrak{g}'_0)$ that are minimal over $U(\mathfrak{g}'_0) \cap \operatorname{ann}_{U(\mathfrak{g}')} \widetilde{L}(\lambda)$. Note that

$$L(i) = \operatorname{span}\{e_{-\pi}v_\lambda | \pi(\beta) = 1 \text{ for exactly } i \text{ odd positive roots } \beta\}.$$

When $m = n$, we take this as the definition of $L(i)$, and then (8.3.5) still holds. Let $Q_0^+ = \sum_{\alpha \in \Delta_0^+} \mathbb{N}\alpha$; compare (A.1.5). Then for $w \in W$, we have $w \circ (\lambda - Q_0^+) \subseteq \lambda - Q_0^+$ and $w\beta \in \beta + Q_0^+$, where β is the unique odd simple root. It follows that if $i \neq j$, then any weight in $L(i)$ cannot be in the W-orbit of a weight of $L(j)$.

In view of Remark 8.3.3 we will often invoke the following hypothesis in the rest of this book. When this is done, we will use the notation established in this chapter without further comment.

Hypothesis 8.3.4. The Lie superalgebra $\mathfrak{g}$ is either basic classical simple of type *different from A* or $\mathfrak{g} = \mathfrak{gl}(m,n)$. Equivalently $\mathfrak{g}$ can be constructed as a contragredient Lie superalgebra $\mathfrak{g}(A, \tau)$; see Exercise 5.6.12 and the remarks before Lemma 8.3.1.

If Hypothesis 8.3.4 holds, then as we have remarked above, $\mathfrak{g}$ has a nondegenerate $\mathfrak{g}$-invariant supersymmetric even bilinear form $(\ , \ )$. In addition $\mathfrak{g}$ has a basis of simple roots, denoted by Π or by $\Pi_{\mathfrak{b}}$ if we need to call attention to the corresponding Borel subalgebra $\mathfrak{b}$. Set $Q^+ = \sum_{\gamma \in \Pi} \mathbb{N}\gamma$; compare (A.1.5).

8.4. Partitions and Characters

Assume Hypothesis 8.3.4. For $\lambda \in \mathfrak{h}^*$, set $D(\lambda) = \lambda - Q^+$ and let $\mathcal{E}$ be the set of functions on $\mathfrak{h}^*$ which are zero outside of a finite union of sets of the form $D(\lambda)$. Elements of $\mathcal{E}$ can be written as formal linear combinations $\sum_{\lambda \in \mathfrak{h}^*} c_\lambda \epsilon^\lambda$ where $\epsilon^\lambda(\mu) = \delta_{\lambda\mu}$. We can make $\mathcal{E}$ into an algebra using the convolution product

$$(fg)(\lambda) = \sum_{\mu+\nu=\lambda} f(\mu)g(\nu).$$

Next we use partitions to introduce some functions in $\mathcal{E}$. If $\eta \in Q^+$, a *partition* of η is a map $\pi : \Delta^+ \longrightarrow \mathbb{N}$ such that $\pi(\alpha) = 0$ or 1 for all $\alpha \in \Delta_1^+$ and

$$\sum_{\alpha \in \Delta^+} \pi(\alpha)\alpha = \eta. \tag{8.4.1}$$

For π a partition, set $|\pi| = \sum_{\alpha \in \Delta^+} \pi(\alpha)$. We denote by $\mathbf{P}(\eta)$ the set of partitions of η, and for $\alpha \in \Delta_1^+$ we define

$$\mathbf{P}_\alpha(\eta) = \{\pi \in \mathbf{P}(\eta) | \pi(\alpha) = 0\}.$$

Set $\mathbf{p}(\eta) = |\mathbf{P}(\eta)|$ and $\mathbf{p}_\alpha(\eta) = |\mathbf{P}_\alpha(\eta)|$. The *partition function* p is defined by $p = \sum \mathbf{p}(\eta)\epsilon^{-\eta}$. Thus $p(\mu) = \sum \mathbf{p}(\eta)\epsilon^{-\eta}(\mu) = \mathbf{p}(-\mu)$. Also,

$$p = \prod_{\alpha \in \Delta_1^+} (1 + \epsilon^{-\alpha}) / \prod_{\alpha \in \Delta_0^+} (1 - \epsilon^{-\alpha}). \tag{8.4.2}$$

Similarly if $\alpha \in \Delta_1^+$, we define $p_\alpha = \sum \mathbf{p}_\alpha(\eta)\epsilon^{-\eta}$. Then we have

$$p_\alpha = p/(1 + \epsilon^{-\alpha}). \tag{8.4.3}$$

Hence

$$\mathbf{p}(\eta) = \mathbf{p}_\alpha(\eta) + \mathbf{p}_\alpha(\eta - \alpha).$$

Partitions are useful because they can be used to index a basis for $U(\mathfrak{n}^\pm)$, as in the next lemma. First however we need a suitable basis for $\mathfrak{n}^\pm$. From the description of the root systems given in earlier chapters, $\Delta_0 \cap \Delta_1$ is empty and $\dim \mathfrak{g}^\alpha = 1$ for all $\alpha \in \Delta^+$. For all $\alpha \in \Delta^+$, we choose elements $e_\alpha \in \mathfrak{g}^\alpha$, $e_{-\alpha} \in \mathfrak{g}^{-\alpha}$ such that

$$[e_\alpha, e_{-\alpha}] = h_\alpha.$$

This is possible by Lemma 8.3.2. Fix an ordering on the set Δ^+, and for π a partition, set

$$e_{-\pi} = \prod_{\alpha \in \Delta^+} e_{-\alpha}^{\pi(\alpha)}, \tag{8.4.4}$$

the product being taken with respect to this order. In addition set

$$e_\pi = {}^t e_{-\pi} = \prod_{\alpha \in \Delta^+} e_\alpha^{\pi(\alpha)} \tag{8.4.5}$$

where the product is taken in the opposite order.

Lemma 8.4.1. *The elements* $e_{\pm\pi}$ *with* $\pi \in \mathbf{P}(\eta)$ *form a basis of* $U(\mathfrak{n}^\pm)^{\pm\eta}$. *Thus* $\dim U(\mathfrak{n}^\pm)^{\pm\eta} = \mathbf{p}(\eta)$.

Proof. This follows easily from the PBW Theorem. □

Corollary 8.4.2. *Any element of* $U(\mathfrak{g})$ *can be written uniquely as a finite sum of the form*

$$\sum_{\gamma,\mu \in Q^+} \sum_{\substack{\nu \in \mathbf{P}(\mu) \\ \tau \in \mathbf{P}(\gamma)}} e_{-\tau} \phi_{\tau,\nu} \, {}^t e_{-\nu} \tag{8.4.6}$$

with $\phi_{\tau,\nu} \in S(\mathfrak{h})$.

Proof. Combine Lemma 8.4.1 with Lemma 8.1.7. □

See Exercise 8.7.6 for the analog of Lemma 8.4.1 for homogenized enveloping algebras.

Remark 8.4.3. Sometimes, for example in Chapter 9, it is helpful to modify the notation used in Lemma 8.4.1. To do this, we suppose that $\eta \in Q^+$ and define $\overline{\mathbf{P}}(\eta)$ to be the set of all maps $\pi : \Delta^+ \longrightarrow \mathbb{N}$ such that (8.4.1) holds, and in addition $\pi(\alpha) = 0$ or 1 for all $\alpha \in \overline{\Delta}_1$, and $\pi(\alpha) = 0$ for all $\alpha \in \Delta_0 \backslash \overline{\Delta}_0$. Clearly there is a bijection $f : \mathbf{P}(\eta) \longrightarrow \overline{\mathbf{P}}(\eta)$ defined by

(a) $f(\pi)(\alpha) = \pi(\alpha)$ for $\alpha \in \overline{\Delta}_0^+ \cup \overline{\Delta}_1^+$,

(b) $f(\pi)(\alpha) = 0$ for $\alpha \in \Delta_0^+ \backslash \overline{\Delta}_0^+$,

(c) $f(\pi)(\alpha) = \pi(\alpha) + 2\pi(2\alpha)$ for $\alpha \in \Delta_1^+ \backslash \overline{\Delta}_1^+$.

Now suppose the set Δ^+ is ordered in such a way that 2α follows α whenever $\alpha \in \Delta_1^+ \backslash \overline{\Delta}_1^+$. If $\sigma = f(\pi)$, we define $e_{\pm\sigma}$ as in (8.4.4) and (8.4.5) with π replaced by σ. It is clear that $e_{\pm\sigma}$ is a nonzero constant multiple of $e_{\pm\pi}$.

Lemma 8.4.4. *The elements* $e_{\pm\sigma}$ *with* $\sigma \in \overline{\mathbf{P}}(\eta)$ *form a basis of* $U(\mathfrak{n}^-)^{\pm\eta}$.

Proof. This follows easily from Lemma 8.4.1 and the above remarks. □

Example 8.4.5. Suppose that $\mathfrak{g} = \mathfrak{osp}(1,2)$ and use the notation of Exercise A.4.4. Then $x^2 = -e$, where x (resp. e) is a basis for $\mathfrak{n}_1^+$ (resp. $\mathfrak{n}_0^+$). The bases of $U(\mathfrak{n}^+)$ given by Lemmas 8.4.1 and 8.4.4 are $\{e^k, xe^k | k \in \mathbb{N}\}$ and $\{x^k | k \in \mathbb{N}\}$ respectively.

If M is an object of $\mathcal{O}$, the *character* $\operatorname{ch} M$ of M is defined by

$$\operatorname{ch} M = \sum \dim M^\eta \, \epsilon^\eta.$$

Since $\widetilde{M}(\lambda)^{\lambda-\mu}$ has a basis consisting of all $e_{-\pi}v_\lambda$ with $\pi \in \mathbf{P}(\mu)$, it follows that

$$\operatorname{ch} \widetilde{M}(\lambda) = \epsilon^\lambda p. \tag{8.4.7}$$

Note that $M(\lambda), \widetilde{M}(\lambda) \in \mathcal{E}$, so $\mathcal{E}$ is useful in calculations involving characters; see for example Section 14.4 on the Kac-Weyl character formula. Also if $M \in \mathcal{O}$ and E is a finite dimensional simple module, we have

$$\operatorname{ch}(M \otimes E) = \operatorname{ch} M \operatorname{ch} E \in \mathcal{E}.$$

The *Grothendieck groups* of the categories $\mathcal{O}, \widetilde{\mathcal{O}}$ are denoted by $K(\mathcal{O})$ and $K(\widetilde{\mathcal{O}})$ respectively. The group $K(\mathcal{O})$ is defined as follows. For $M \in \mathcal{O}$, write $[M]$ for the isomorphism class of M. Then $K(\mathcal{O})$ is the free abelian group generated by the symbols $[M]$ with relations $[M] = [M'] + [M'']$ whenever

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is a short exact sequence in $\mathcal{O}$. For any such sequence, we have

$$\operatorname{ch} M = \operatorname{ch} M' + \operatorname{ch} M'' \in \mathcal{E}.$$

It follows from Lemma 8.2.5 and Theorem 8.2.12 that $\operatorname{ch} M \in \mathcal{E}$ for any module $M \in \mathcal{O}$. The same results imply that $K(\mathcal{O})$ is free abelian on the $[L(\lambda)]$ with $\lambda \in \mathfrak{h}^*$. Let $C(\mathcal{O})$ be the additive subgroup of $\mathcal{E}$ generated by the characters $\operatorname{ch} L(\lambda)$ for $\lambda \in \mathfrak{h}^*$. Then it is easy to show the following; see [**Jan79**, Satz 1.11].

Theorem 8.4.6. *There is an isomorphism from the group $K(\mathcal{O})$ to $C(\mathcal{O})$ sending $[M]$ to $\operatorname{ch} M$ for all modules $M \in \mathcal{O}$.*

The group $K(\widetilde{\mathcal{O}})$ is defined similarly, and of course there are analogous results for the category $\widetilde{\mathcal{O}}$.

8.5. The Casimir Element

Assume Hypothesis 8.3.4. We show how to construct a central element Ω of $U(\mathfrak{g})$ known as the *Casimir element.* Assume that $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ are bases of $\mathfrak{g}$ such that $(x_i, y_j) = \delta_{i,j}$ and x_i, y_i are homogeneous elements of $\mathfrak{g}$ of the same degree β_i. Fix $x \in \mathfrak{g}^\alpha$ and write

$$[x, x_i] = \sum_j a_{ij} x_j, \quad [x, y_i] = \sum_j b_{ij} y_j.$$

Then

$$\begin{aligned} a_{ik} &= \sum_j a_{ij}(x_j, y_k) = ([x, x_i], y_k) \\ &= -(-1)^{\alpha\beta_i}([x_i, x], y_k) = -(-1)^{\alpha\beta_i}(x_i, [x, y_k]) \\ &= -(-1)^{\alpha\beta_i}\sum_j (x_i, b_{kj}y_j) = -(-1)^{\alpha\beta_i} b_{ki}. \end{aligned}$$

Now set $\Omega = \sum(-1)^{\beta_i} x_i y_i \in U(\mathfrak{g})$.

Lemma 8.5.1. *The Casimir element Ω is central in $U(\mathfrak{g})$.*

Proof. Retaining the notation above, we have for $x \in \mathfrak{g}^\alpha$

$$\begin{aligned} x\Omega - \Omega x &= \sum_i (-1)^{\beta_i}[x, x_i]y_i + \sum_i (-1)^{\beta_i(\alpha+1)} x_i[x, y_i] \\ &= \sum_{i,j} (-1)^{\beta_i} a_{ij} x_j y_i + \sum_{i,j} (-1)^{\beta_i(\alpha+1)} b_{ij} x_i y_j \\ &= -\sum_{i,j} (-1)^{\beta_j(\alpha+1)} b_{ij} x_i y_j + \sum_{i,j} (-1)^{\beta_i(\alpha+1)} b_{ij} x_i y_j = 0. \end{aligned}$$

Here we used the fact that if $b_{ij} \neq 0$, then $\beta_j \equiv \beta_i + \alpha \bmod 2$ and hence $\beta_j(\alpha + 1) \equiv \beta_i(\alpha + 1) \bmod 2$. □

Remark 8.5.2. It is possible to give a more conceptual proof of the lemma using the fact that

$$\mathfrak{g} \otimes \mathfrak{g} \cong \mathfrak{g} \otimes \mathfrak{g}^* \cong \mathrm{End}(\mathfrak{g}).$$

Then up to scalar, Ω corresponds to the identity map in $\mathrm{End}(\mathfrak{g})$. From this it follows that Ω is independent of the choice of dual bases.

Next we compute the action of the Casimir element Ω on the Verma module $\widetilde{M}(\lambda)$. To do this, we choose the dual bases in a special way.

Lemma 8.5.3. *For all $\lambda \in \mathfrak{h}^*$, Ω acts on the Verma module $\widetilde{M}(\lambda)$ as the scalar $(\lambda + 2\rho, \lambda)$, that is,*

$$\chi_\lambda(\Omega) = (\lambda + 2\rho, \lambda).$$

Proof. By Lemma 8.3.1 the restriction of (,) to $\mathfrak{h}$ is nondegenerate. Let $h_1, \dots, h_m$ and $k_1, \dots, k_m$ be dual bases of $\mathfrak{h}$ with respect to (,), so that $(h_i, k_j) = \delta_{ij}$. We claim that

$$(\mu, \mu) = \sum_i \mu(h_i)\mu(k_i) \quad \text{for all } \mu \in \mathfrak{h}^*. \tag{8.5.1}$$

To see this, write $h_\mu = \sum_j a_j h_j$. Then $\mu(h_i) = (h_\mu, h_i) = \sum_j a_j(h_j, h_i)$, and $\mu(k_i) = (h_\mu, k_i) = \sum_j a_j(h_j, k_i) = a_i$. Thus

$$(\mu, \mu) = (h_\mu, h_\mu) = \sum_{i,j} a_i a_j (h_i, h_j) = \sum_i \mu(h_i)\mu(k_i).$$

For all $\alpha \in \Delta^+$, the elements $e_\alpha \in \mathfrak{g}^\alpha, e_{-\alpha} \in \mathfrak{g}^{-\alpha}$ are chosen so that $[e_\alpha, e_{-\alpha}] = h_\alpha$. It follows from Lemma 8.3.1 that we obtain bases $\{x_i\}, \{y_j\}$ for $\mathfrak{g}$ such that $(x_i, y_j) = \delta_{i,j}$ by taking the elements

$$\{h_i, k_i\},\ \{e_\alpha, e_{-\alpha}\}_{\alpha\in\Delta_0}$$

and

$$\{e_\alpha, e_{-\alpha}\}_{\alpha\in\Delta_1^+},\ \{e_{-\alpha}, -e_\alpha\}_{\alpha\in\Delta_1^+}.$$

Thus we can write

$$\begin{aligned}\Omega &= \sum_{i=1}^m h_i k_i + \sum_{\alpha\in\Delta_0^+}(e_\alpha e_{-\alpha} + e_{-\alpha}e_\alpha) \\ &\quad + \sum_{\alpha\in\Delta_1^+}(e_{-\alpha}e_\alpha - e_\alpha e_{-\alpha}).\end{aligned}$$

We have $\Omega v_\lambda = \chi_\lambda(\Omega)v_\lambda$ from the definition of χ_λ. Furthermore

$$(\sum h_i k_i)v_\lambda = \sum \lambda(h_i)\lambda(k_i)v_\lambda = (\lambda, \lambda)v_\lambda$$

by (8.5.1). Now since $e_\alpha v_\lambda = 0$ for all $\alpha \in \Delta^+$, we have

$$(e_\alpha e_{-\alpha})v_\lambda = (\pm e_{-\alpha}e_\alpha + h_\alpha)v_\lambda = (\alpha, \lambda)v_\lambda.$$

Thus since $2\rho_0 = \sum_{\alpha\in\Delta_0^+}\alpha,\ 2\rho_1 = \sum_{\alpha\in\Delta_1^+}\alpha$, and $\rho = \rho_0 - \rho_1$, we obtain

$$\begin{aligned}\Omega v_\lambda &= (\lambda, \lambda)v_\lambda + (\sum_{\alpha\in\Delta_0^+} e_\alpha e_{-\alpha} - \sum_{\alpha\in\Delta_1^+} e_\alpha e_{-\alpha})v_\lambda \\ &= (\lambda + 2\rho, \lambda)v_\lambda,\end{aligned}$$

as required. □

Corollary 8.5.4. *If $\alpha \in \Pi$ is a simple root, then $2(\rho, \alpha) = (\alpha, \alpha)$.*

Proof. If $\widetilde{M}(0)$ denotes the Verma module with highest weight vector v of weight 0, it is easy to see that $e_{-\alpha}v$ is another highest weight vector of weight $-\alpha$. Thus there is a nonzero homomorphism $\widetilde{M}(-\alpha) \longrightarrow \widetilde{M}(0)$. Therefore the Casimir element acts as the same scalar on both Verma modules. Hence from Lemma 8.5.3,

$$0 = (2\rho, 0) = (-\alpha + 2\rho, -\alpha),$$

and the result follows from this. □

Note that these definitions imply that

$$h_\lambda v = (\lambda, \mu)v \quad \text{for all } v \in M^\mu.$$

By Corollary 8.5.4 we have

$$(\rho(\mathfrak{b}), \beta) = 0 \quad \text{if } \beta \text{ is a simple isotropic root.} \tag{8.5.2}$$

8.6. Changing the Borel Subalgebra

Assume Hypothesis 8.3.4. We consider the behavior of highest weight modules when the Borel subalgebra is changed. For simplicity we suppose that $\mathfrak{g} = \mathfrak{g}(A, \tau)$ is a contragredient Lie superalgebra. Let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a minimal realization of A where

$$\Pi = \{\alpha_1, \ldots, \alpha_n\}, \quad \Pi^\vee = \{h_1, \ldots, h_n\}. \tag{8.6.1}$$

Then $\mathfrak{g}$ is generated by $\mathfrak{h}$ and elements e_i, f_i, for $i = 1, \ldots, n$, and relations (5.1.5)–(5.1.7) hold. Next suppose $\beta = \alpha_k \in \Pi$ is an odd isotropic root and consider the elements $e'_1, \ldots, e'_n$ and $f'_1, \ldots, f'_n$ of $\mathfrak{g}$ defined in (3.5.5). Let $\mathfrak{b}$ (resp. $\mathfrak{b}'$) be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}$ and the e_i (resp. by $\mathfrak{h}$ and the e'_i). Then $\mathfrak{b}, \mathfrak{b}'$ are adjacent Borel subalgebras, and

$$\mathfrak{g}^\beta \subset \mathfrak{b}, \quad \mathfrak{g}^{-\beta} \subset \mathfrak{b}' \tag{8.6.2}$$

for some isotropic root β. Let $\mathfrak{n}^-$ be the subalgebra of $\mathfrak{g}$ generated by the f_i, and set $Q^+ = \mathbb{N}\Pi$. Note that

$$\rho(\mathfrak{b}') = \rho(\mathfrak{b}) + \beta. \tag{8.6.3}$$

Lemma 8.6.1. *Suppose $V = U(\mathfrak{g})v_\lambda$ where v_λ is a highest weight vector for $\mathfrak{b}$ with weight λ, and set $e_{-\beta} = f_k$ and $u = e_{-\beta}v_\lambda$. Then either $u = 0$ or u is a highest weight vector of weight $\lambda - \beta$ for $\mathfrak{b}'$. Moreover one of the following holds.*

(a) $(\lambda, \beta) \neq 0$ *and* $U(\mathfrak{g})u = V$.

(b) $(\lambda, \beta) = 0$ *and u generates a proper $U(\mathfrak{g})$-submodule of V.*

Proof. If $u \neq 0$, we need to show that $e'_j u = 0$ for all j. If $j = k$, this follows since $e'_j = e_{-\beta}$ has square zero in $U(\mathfrak{g})$. If $j \neq k$ and $a_{jk} \neq 0$, then $e'_j = [e_k, e_j] \in \mathfrak{g}^{\alpha_j + \alpha_k}$ so $[e'_j, f_k] \in \mathfrak{g}^{\alpha_j}$, and

$$e'_j f_k v_\lambda = (\pm f_k e'_j + [e'_j, f_k])v_\lambda = 0.$$

Finally, if $j \neq k$ and $a_{jk} = 0$, then $e'_j = e_j$ and $[e_j, f_k] = 0$ imply that $e_j f_k v_\lambda = 0$. It follows that u is a highest weight vector for $\mathfrak{b}'$. Next observe that

$$e_\beta e_{-\beta} v_\lambda = h_\beta v_\lambda = (\lambda, \beta)v_\lambda. \tag{8.6.4}$$

This immediately gives (a). Now assume $(\lambda, \beta) = 0$. Then (b) is clear if $u = 0$. Otherwise (8.6.4) and the assumption imply that u is a highest weight vector for $\mathfrak{b}$. Hence any weight of $U(\mathfrak{g})u = U(\mathfrak{n}^-)u$ is contained in $\lambda - \beta - Q^+$. Since this set does not contain λ, (b) follows. □

Corollary 8.6.2. *Assume V as in the lemma is simple. Then one of the following holds.*

(a) $(\lambda, \beta) \neq 0$ *and V has highest weight $\lambda - \beta$ with respect to $\mathfrak{b}'$.*

(b) $(\lambda, \beta) = 0$ *and V has highest weight λ with respect to $\mathfrak{b}'$.*

Proof. Immediate. □

We say a highest weight module V is $\mathfrak{b}$-*typical* if the highest weight $\lambda(\mathfrak{b})$ satisfies $(\lambda(\mathfrak{b}) + \rho(\mathfrak{b}), \gamma) \neq 0$ for all isotropic roots γ.

Corollary 8.6.3. *Suppose $\mathfrak{b}, \mathfrak{b}'$ are Borel subalgebras of $\mathfrak{g}$ with $\mathfrak{b}_0 = \mathfrak{b}'_0$. If V is a highest weight module for $\mathfrak{b}$ which is $\mathfrak{b}$-typical, then V is also a $\mathfrak{b}'$-typical highest weight module for $\mathfrak{b}'$, and the highest weights for $\mathfrak{b}, \mathfrak{b}'$ satisfy*

$$\lambda(\mathfrak{b}) + \rho(\mathfrak{b}) = \lambda(\mathfrak{b}') + \rho(\mathfrak{b}').$$

Proof. By Theorem 3.1.3 this reduces to the case where $\mathfrak{b}, \mathfrak{b}'$ are adjacent. Assume V is $\mathfrak{b}$-typical, and let β be as in Lemma 8.6.1. By (8.5.2), $(\rho(\mathfrak{b}), \beta) = 0$, so case (a) holds in Lemma 8.6.1, and the result follows from (8.6.3) □

This result shows that typicality does not depend on the choice of Borel subalgebra, and we say that a module V is a *typical highest weight module* if for some Borel subalgebra $\mathfrak{b}$ with even part $\mathbf{b}$, V is a highest weight module for $\mathfrak{b}$ which is $\mathfrak{b}$-typical.

8.7. Exercises

8.7.1. Prove Lemma 8.2.8

8.7.2. Prove Corollary 8.2.9

8.7.3. Suppose that (,) is a nondegenerate invariant even bilinear form on the Lie superalgebra $\mathfrak{g}$. If $\mathfrak{z}$ is the center of $\mathfrak{g}_0$, show that $([\mathfrak{g}_0, \mathfrak{g}_0], \mathfrak{z}) = 0$.

8.7.4. Prove Lemma 8.2.3(e). Hint: By Frobenius reciprocity

$$\mathrm{Hom}_{U(\mathfrak{g})}(\widetilde{M}(\lambda), \widetilde{M}(\lambda)) = \mathrm{Hom}_{U(\mathfrak{b})}(V_\lambda, \widetilde{M}(\lambda)).$$

8.7.5. Prove Corollary 8.2.11. Hint: Use Lemma 8.2.10 and Lemma 8.2.8.

8.7.6. The purpose of this exercise is to give an analog of Lemma 8.4.1 for the homogenized enveloping algebra $H(\mathfrak{n})$. First for $\eta \in \mathfrak{h}^*$ define

$$H(\mathfrak{n})^{-\eta} = \{x \in H(\mathfrak{n}) | [hT, x] = \eta(h)xT \text{ for all } h \in \mathfrak{h}\}.$$

Then for $\pi \in P(\eta)$, let

$$E_{-\pi} = e_{-\pi}T^{|\pi|} = \prod_{\alpha \in \Delta^+} E_{-\alpha}^{\pi(\alpha)}$$

where the products are taken in the same fixed order used to define the $e_{-\pi}$. Show that $H(\mathfrak{n})^{-\eta}$ is a free $K[T]$-module on the set $\{E_{-\pi} | \pi \in \mathbf{P}(\eta)\}$.

8.7.7. Suppose that the Lie superalgebra $\mathfrak{g}$ has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Show that as $K[T]$-modules

$$H(\mathfrak{g}) = H(\mathfrak{n}^-) \otimes_{K[T]} H(\mathfrak{h}) \otimes_{K[T]} H(\mathfrak{n}^+).$$