Foreword

Since the appearance of the seminal monograph of Stefan Banach [**51**] on (linear) operators between normed spaces, the study of operators has been at the heart of research in analysis. For the most part, this research was confined to the topological structures of the spaces and to the topological properties of the operators. At the same time there was one more aspect of operator theory that was developed in parallel with the topological and algebraic study of operators. This natural aspect involved the order structures of the spaces between which the operators acted.

Almost every classical Banach space is equipped with a natural order that is compatible with the algebraic and topological structures of the space. Operators that preserve the order structures are known as *positive operators.* They were first introduced and studied by F. Riesz [**278, 279**], H. Freudenthal [**129**], and L. V. Kantorovich [**172, 173**] in the mid 1930's. For some peculiar reason, the investigations of the topological and order properties of operators were not connected, and for quite a while it seemed that they had little in common. The monographs written by L. V. Kantorovich, B. Z. Vulikh and A. Pinsker [**175**] in the Soviet Union and by H. Nakano [**251, 252**] in Japan contained most of the theoretical background on operators between ordered spaces accumulated by 1950. A partial list of books and monographs that deal with Banach spaces and operators between them without emphasizing in depth the order structures includes: [**56, 85, 102, 105, 106, 113, 118, 149, 152, 199, 205, 235, 250, 266, 300, 305, 306, 317**]. For a glimpse of the modern state of the art in Banach space theory and operator theory, see the special volume [**164**] edited by W. B. Johnson and J. Lindenstrauss.

From the early 1960's it was realized that the topological and order structures of operators were related and that they should be studied together. Since then much research has aimed at producing a theory that would encompass these structures in a unified manner. The "unification" of the order and topological properties of operators is due to the efforts of many mathematicians in various countries. The period of this unification can be roughly divided into two parts; the pre-1980 period and the post-1980 period. Let us describe here the basic achievements done in each period.

The pre-1980 period can be described as the period where most of the foundations of the order structure of vectors spaces and the lattice properties of operators were laid down. In this era the theory of partially ordered vector spaces was developed through the efforts of mathematicians from many countries. There were several monographs written on the subject. Those written by A. L. Peressini [**264**] and G. Jameson [**161**] and the last part of H. H. Schaefer's book [**289**] were devoted entirely to ordered vector spaces. The books by W. A. J. Luxemburg and A. C. Zaanen [**225**], B. Z. Vulikh [**328**], and S. Kaplan [**176**] studied the order structure of Riesz spaces, while D. H. Fremlin [**128**] and C. D. Aliprantis and O. Burkinshaw [**29**] studied topological properties of Riesz spaces. Monographs on Banach lattices and positive operators were written by M. A. Krasnoselsky et al. [**191**], H. J. Krieger [**194**], H. E. Lacey [**197**], J. Lindenstrauss and L. Tzafriri [**205, 206**], H. H. Schaefer [**290**], and H.-U. Schwarz [**298**].

The post-1980 period is characterized by the study of positive operators from all points of view. Special attention was paid to the properties of operators dominated by positive operators having some compactness property. We also find in this era a complete study of integral operators in terms of their lattice structure. Monographs regarding positive operators were written in this period by C. D. Aliprantis and O. Burkinshaw [**30**], W. Arendt et al. [**41**], A. V. Bukhvalov et al. [**78**], P. Meyer-Nieberg [**240**], and A. C. Zaanen [**343, 344**]. The list of very recent monographs on the subject of positive operators includes the books by Y. A. Abramovich, E. L. Arenson, and A. K. Kitover [**21**], Y. A. Abramovich and A. K. Kitover [**22**], A. G. Kusraev [**195**], and W. Wnuk [**338**].

Besides its important internal role in mathematics, the theory of Riesz spaces and positive operators has important applications to several disciplines. The areas where the theory of Riesz spaces and positive operators has been found to be useful include: game theory [**182**]; finance [**151**, Chapter 31]; economics [**26**]; nuclear reactor theory [**64, 65**]; statistical decision theory [**201**]; and structured population dynamics [**80, 248**].

The objective of this book is to present at the graduate level in a self-contained manner the theory of linear operators between Banach spaces and Banach lattices by exploiting their topological and order structures, as well as the topological and order properties of the spaces of operators. The subject matter of the book is divided into eleven chapters, which to some extent can be read independently. Each chapter is subdivided into sections. At the end of each section there is a list of exercises of varying degrees of difficulty designed to help the reader comprehend the material in the section. There are more than six hundred exercises in the book. Hints to selected exercises are also given. Our companion book *Problems in Operator Theory* contains complete solutions to all exercises.

Exercises play a considerable role in our presentation. They serve several purposes. First of all, we often relegate some technical details of proofs to the exercises. These details may not be critical for understanding the proofs but they are certainly essential for their validity, and we want to provide the student with an accurate and complete account of how such details should be presented. Second, these exercises not only allow the student to learn how the results presented in the book work out but they also offer a considerable amount of additional material and further developments. There is one more feature to these exercises. There are many useful, almost classical, results whose proofs are not readily available in the literature (for instance, the duality between the uniformly convex and uniformly smooth spaces, the equivalence between the differentiability and other properties of the norm, the explicit closed form of the resolvent of the Volterra operator, the description of the spectrum of the averaging operator $f \mapsto \frac{1}{t}\int_0^t f\,d\mu$, etc.). The reader will find detailed solutions and discussions of many of these results among our problems.

Some of the exercises are new, and some are known. Whenever possible, we indicate the sources of the known exercises. On the other hand, for the standard results or for some variations of classical results, we often omit the sources of their origins.

The prerequisites of the book are the usual graduate introductory courses in real analysis, general topology, measure theory, and functional analysis. Trying to make the book as self-contained as possible, we have provided detailed proofs to most theorems in the text. This, coupled with the inclusion of the exercises, makes the book an ideal text for a graduate or advance course in operator theory and functional analysis. Moreover, since many chapters contain the latest results (some of which have not been published before), we hope that the book will be useful not only to graduate students but also to research mathematicians in operator theory and functional analysis, as well as to scientists in other disciplines. Each of Chapters 5, 7, 9, 10, and 11 can be used for an advanced seminar and each of them will also present the students with possibilities and directions to start on independent research. A brief description of the material in each chapter should give a good idea of the topics covered in the book.

- Chapter 1 presents an introduction to Banach spaces and operator theory. Also, it introduces Banach lattices, positive operators, and vector-valued functions. The chapter culminates with a very brief but, nevertheless, rather comprehensive presentation of the fundamentals of measure theory.
- Chapter 2 reviews the basic structural properties of operators with special emphasis on positive operators. It also studies operators bounded from below and discusses the ascent and descent of an operator.
- Chapter 3 deals with AM- and AL-spaces and demonstrates their importance in operator theory. It also studies the center of a Banach lattice and presents a careful exposition of the complexification of a Banach lattice.
- Chapter 4 presents detailed accounts regarding the following classes of operators: finite-rank operators, multiplication operators, lattice and algebraic homomorphisms, Fredholm operators, and strictly singular operators.
- Chapter 5 deals exclusively with integral operators. It presents their basic properties from the measure-theoretic and abstract points of view. It also studies conditional expectation operators and demonstrates their connection with positive projections.
- Chapter 6 introduces the spectrum of a bounded operator and discusses the basic geometrical properties of the special parts of the spectrum. The study specializes in the spectral properties of positive operators. A comprehensive study of the functional calculus is also here.
- Chapter 7 describes the spectra of compact operators and lattice homomorphisms. It also contains the method of turning approximate eigenvalues to eigenvalues. In addition, this chapter introduces and studies the order spectrum of an order bounded operator as well as the essential spectrum of a bounded operator.
- Chapter 8 specializes in the study of operators to finite dimensional spaces with special emphasis on matrices with non-negative entries. It studies irreducible matrices and presents a thorough discussion on the classical Perron–Frobenius theorem.

- Chapter 9 investigates irreducible operators on Banach lattices and presents several conditions for the strict positivity of the spectral radius. This chapter also studies Krein operators between $C(\Omega)$-spaces.
- Chapter 10 deals with the famous invariant subspace problem. It starts with several classical invariant subspace theorems, but its central topic is the invariant subspace problem for positive and related operators. The results obtained for individual operators are then generalized to collections of positive operators.
- A bounded operator T on a Banach space satisfies the Daugavet equation if $\|I + T\| = 1 + \|T\|$. Chapter 11 presents a comprehensive study of operators on Banach spaces that satisfy the Daugavet equation.

We regret that the size of the book has precluded us from including a number of relevant topics, in particular, disjointness preserving operators and their spectral properties, interpolation of positive operators, and the domination problem. There is one more omission that we would like to mention. The book deals with the general theory of operators and, as a result, Hilbert space operators occur in our discussions only sporadically. There are many excellent books devoted exclusively to Hilbert spaces and operators on them, and so anyone interested in this particular subject will be able to find it easily in the literature; see for instance the monographs [**94, 148**]. At present, the realm of operator theory is so vast that any book can cover only a portion of it.

In writing the book we have received help and benefited from discussions with many colleagues and friends. We are grateful and express our sincerest thanks to all of them: O. Burkinshaw (for many fruitful conversations), W. A. J. Luxemburg (for his support and encouragement), H. Radjavi (for insightful comments on the invariant subspace problem), B. Randrianantoanina (for reading the book, correcting a number of mistakes, and providing us with many valuable suggestions), G. Sirotkin (for a number of useful remarks and problems), T. Oikhberg (for contributing several problems), K. Podgorski (for discussions concerning conditional expectations), S. Klimek and E. Mukhin (for helping us to incorporate graphics in LaTeX), S. I. Gelfand (for constant support at all stages during the production of this book), the AMS Copy Editor Arlene O'Sean (for her professional reading of the manuscript and her many recommendations for its improvement), and two anonymous referees (for their constructive criticism and comments).

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Y. A. Abramovich and C. D. Aliprantis

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