# Preface

For this second volume of *Foliations*, we have selected three special topics: analysis on foliated spaces, characteristic classes of foliations, and foliated 3-manifolds. Each of these is an example of deep interaction between foliation theory and some other highly developed area of mathematics. In all cases, our aim is to give useful, in-depth introductions.

In Part 1 we treat $C^*$-algebras of foliated spaces and generalize heat flow and Brownian motion in Riemannian manifolds to such spaces. The first of these topics is essential for the "noncommutative geometry" of these spaces, a deep theory originated and pursued by A. Connes. The second is due to L. Garnett. While the heat equation varies continuously from leaf to leaf, its solutions have an essentially global character, making them hard to compare on different leaves. We will show, however, that leafwise heat diffusion defines a continuous, 1-parameter semigroup of operators on the Banach space $C(M)$ and, following Garnett [**77**], we will construct probability measures on $M$ that are invariant under this semiflow. These are called *harmonic measures*, and they lead to a powerful ergodic theory for foliated spaces. This theory has profound topological applications (*cf.* Theorem 3.1.4), but its analytic and probabilistic foundations have made access difficult for many topologists. For this reason, we have added two survey appendices, one on heat diffusion in Riemannian manifolds and one on the associated Brownian flow. For similar reasons, we have added an appendix on the basics of $C^*$-algebras. We hope that these will serve as helpful guides through the analytic foundations of Part 1.

Part 2 is devoted to characteristic classes and foliations. Following R. Bott [**9**], we give a Chern-Weil type construction of the exotic classes based on the Bott vanishing theorem (Theorem 6.1.1). The resulting theory can be viewed either as a topic in algebraic topology, motivated by foliation theory, or as a deep application of algebraic topology to the study of foliations. We take the latter viewpoint, emphasizing qualitative aspects such as G. Duminy's celebrated vanishing theorem for the Godbillon-Vey class (unpublished) and S. Hurder's analogous theorems for the generalized Godbillon-Vey classes in higher codimension [**102**]. We begin Part 2 with a chapter on the "grandfather" of all characteristic classes, the Euler class of oriented circle bundles, giving complete proofs of the applications, due to J. Milnor [**129**] and J. Wood [**189**], concerning obstructions to the existence of foliations transverse to the fibers of circle bundles over surfaces.

In Part 3, we study compact 3-manifolds foliated by surfaces, a topic that has been popular since the advent of the Reeb foliation of $S^3$. The special methods of 3-manifold topology yield existence theorems and qualitative properties unique to dimension three. The theorem of S. P. Novikov [**141**] on the existence of Reeb components has the consequence that "Reebless foliations" carry important topological information about the ambient 3-manifold. Together with a theorem of W. Thurston [**175**] on compact leaves of Reebless foliations, this led to D. Gabai's groundbreaking work in which taut foliations are used as powerful tools for studying 3-manifold topology. We develop this theory up to Gabai's constructions of taut, finite depth foliations on certain sutured 3-manifolds, giving details only in the disk decomposable case (depth one). This will bring the reader to the threshold of the "modern age" of essential laminations. These laminations are generalizations simultaneously of taut foliations and incompressible surfaces, and are the object of much current research. Essential laminations, however, need a book of their own and we hope that one or more of the specialists will provide such.

Appendix D pertains to Part 3, being a detailed account of Palmeira's theorem that the only simply connected $n$-manifold foliated by leaves diffeomorphic to $\mathbb{R}^{n-1}$ is $\mathbb{R}^n$. In fact, if $n \geq 3$, the foliated manifold is diffeomorphic to $\mathbb{R}^2 \times \mathbb{R}^{n-2}$ in such a way that the foliation is the product of a foliation of $\mathbb{R}^2$ by the space $\mathbb{R}^{n-2}$. Although valid in all dimensions $n \geq 3$, this result has important applications to Reebless-foliated 3-manifolds.

The bibliography is not intended to be a comprehensive list of all publications on these areas of foliation theory. Only references explicitly cited in the text are included, with the result that many important papers and books are omitted (with apologies to the authors).

The three parts of this book can be read independently. One minor exception to this is that certain standard properties of the Euler class, proven in Part 2, are needed in Part 3. Of course, all parts depend in various ways on material in Volume I. All references to that volume will be of the form [I,. . . ].

Finally, the first named author expresses his sincere thanks and appreciation to the second for his invitation to join in this journey through the theory of foliations, and for seeing that it got to an end.