Introduction

Free or moving boundary problems appear in many areas of mathematics and science in general. Typical examples are shape optimization (least area for fixed volume, optimal insulation, minimal capacity potential at prescribed volume), phase transitions (melting of a solid, Cahn-Hilliard), fluid dynamics (incompressible or compressible flow in porous media, cavitation, flame propagation), probability and statistics (optimal stopping time, hypothesis testing, financial mathematics), among other areas.

They are also an important mathematical tool for proving the existence of solutions in nonlinear problems, homogenization limits in random and periodic media, etc.

A typical example of a free boundary problem is the evolution in time of a solid-liquid configuration: suppose that we have a container D filled with a material that is in solid state in some region $\Omega_0 \subset D$ and liquid in $\Lambda_0 = D \setminus \Omega_0$.

We know its initial temperature distribution $T_0(x)$ and we can control what happens on ∂D at all times (perfect insulation, constant temperature, etc.). Then from this knowledge we should be able to reconstruct the solid-liquid configuration, Ω_t, Λ_t, and the temperature distribution $T(x,t)$ for all times $t > 0$.

Roughly, on Ω_t, Λ_t the temperature should satisfy some type of diffusion equation, while across the transition surface, we should have some "balance" conditions that express the dynamics of the melting process.

The separation surface $\partial\Omega_t$ between solid and liquid is thus determined implicitly by these "balance conditions". In attempting to construct solutions to such a problem, one is thus confronted with a choice. We could try to build "classical solutions", that is, configurations Ω_t, Λ_t, $T(x,t)$ where the separation surface $F = \partial\Omega_t$ is smooth, the function T is smooth up to F from both sides, and the interphase conditions on T, ∇T, ... are satisfied pointwise. But this is in general not possible, except in the case of low dimensions (when F is a curve) or very special configurations.

The other option is to construct solutions of the problem by integrating the transition condition into a "weak formulation" of the equation, be it through the conservation laws that in many cases define them, or by a Perron-like supersolution method since the expectation that transition processes be "organized" and "smooth" is usually linked to some sort of "ellipticity" of the transition conditions.

The challenging issue is then, of course, to fill the bridge between weak and classical solutions

A comparison is in order with calculus of variations and the theory of minimal surfaces, one of the most beautiful and successful pursuits of the last fifty years.

In the theory of minimal surfaces one builds weak solutions as the boundary of sets of finite perimeters (weak limits of polygonals of uniformly bounded perimeter) or currents (measures supported in countable unions of Lipschitz graphs) and ends up proving that such objects are indeed smooth hypersurfaces except for some unavoidable singular set perfectly described.

This is achieved by different methods:

(i) by exploiting the invariance of minimal surface under dilations and reducing the problem of local regularity to global profiles (monotonicity formulas, classifications of minimal cones),

(ii) by exploiting the fact that the minimal surface equation linearizes into the Laplacian (improvement of flatness),

(iii) by, maybe the most versatile approach, the DeGiorgi "oscillation decay" method, which says that under very general conditions a Lipschitz surface that satisfies a "translation invariant elliptic equation" improves its Lipschitz norm as we shrink geometrically into a point.

We will see these three themes appearing time and again in these notes. In fact, we consider here a particular family of free boundary problems accessible to this approach: those problems in which the transition occurs when a "dependent variable u" (a temperature, a density, the expected value of a random variable) crosses or reaches a prescribed threshold value $\varphi(x)$.

In the same way that zero curvature forces regularity on a minimal surface, the interplay of both functions $(u-\varphi)^{\pm}$ at each side of $F = \partial\Omega$ and the transition conditions (typically relating the speed of F with $(u-\varphi)^{\pm}_{\nu}$) force regularity on F, although in a much more tenuous way.

To reproduce the general framework of the methods described above for minimal surfaces, it is then necessary to understand the interplay between harmonic and caloric measures in both sides of a domain, the Hausdorff measure of the free boundary and the growth properties of the solutions (monotonicity formulas, boundary Harnack principles).

These are important, deep tools developed in the last thirty years, which we have included in Part 3 of this book. We choose in this book to restrict ourselves to two specific free boundary problems, one elliptic and one parabolic, to present the main ideas and techniques in their simplest form.

Let us mention two other problems of interest that admit a similar treatment: the obstacle problem (see the notes [C5]) and the theory of flow through porous media.

In this book, we have restricted ourselves to the problem of going from weak solutions to classical solutions.

The issue of showing that classical solutions exhibit higher regularity has been treated extensively and forms another body of work with different techniques, more in the spirit of Schauder and other a priori estimates.

There are of course many other problems of great interest: elliptic or parabolic systems, hyperbolic equations, random perturbations of the transition surface, etc.

Although the issues become very complicated very fast, we hope that the techniques and ideas presented in this book contribute to the development of more complex methods for treating free boundary problems or, more generally, those problems where, through differential relations, manifolds and their boundaries interact.

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